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Quark potentials for mesons in the Klein-Gordon equation

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Abstract. Two relativistic potential models are applied to describe meson spectroscopy in a unified way, encompassing both light and heavy quark systems. A combination of linear and coulomb potentials has been investigated for Klein-Gordon equation using the WKB approximation. A power-like phenomenological potential model has also been studied in the Klein-Gordon framework. Meson masses calculated for both the potentials give a good agreement with the corresponding experimental values.

Keywords. Relativistic quark models; quark; confinement; Klein-Gordon spectroscopy.

1. Introduction

The recent discovery of Υ (Herb *et al* 1977), and its subsequent resolutions in three peaks at 9·4, 10 and 10·4 GeV (Innes *et al* 1977) strongly suggests the opening of a new charmonium-like picture called the "bottomium". This new observation has stimulated a great deal of interest in the role which quark-antiquark ($q\bar{q}$) forces play in explaining the spectra of heavier mesons. Many attempts have since been made to obtain an adequate phenomenological fit to the ψ and Υ spectra using different potential models. A simple minded static noncoulombic power law potential of the form

$$V(r) = g_1 r^{\nu} - V_0 \, (g_1, \nu > 0) \tag{1}$$

has been successful in explaining the spectra of ψ and Υ systems with nonrelativistic approach (Martin 1980; Barik and Jena 1980). Other potentials with an emphasis on employing minimum number of adjustable parameters were also proposed by various authors (Crater 1977; Bhanot and Rudaz 1978; Lichtenberg and Wills 1979; Richardson 1979; Crater and Alstine 1981). The combination of a linear plus coulomb potential has, however, been the most successful one in explaining the meson spectroscopy (Bevis *et al* 1979; Eichten *et al* 1980; Pignon and Piketty 1978).

Despite the success of these potential models in Schrödinger equation, it is always tempting to be more realistic by considering the problem of quark confinement within the relativistic framework. A complete treatment of the problem should, therefore, incorporate both relativistic and quantum effect. This, in addition, requires a fuller understanding of the underlying dynamics of the quark and the exact Lorentz character of the potentials to be used. In this respect the relativistic bound state problem of the $q\bar{q}$ mesons has been studied both for the Klein-Gordon (Kang and Schnitzer 1975; Ram and Halasa 1979; Sharma and Iyer 1982; Iyer and Sharma 1982) and the Dirac (Gunion and Li 1975; Critchfield 1975; Rein 1977; Ferreira 1977; Ferreira and Zagury 1977; Ferreira *et al* 1980; Sharma *et al* 1982) equations. A power law potential which is an equal admixture of scalar and vector potentials has been successfully used (Barik and Barik 1981) recently in generating the Dirac bound states of $c\bar{c}$ and $b\bar{b}$ systems.

Recalling that relativistic corrections are not negligible (Schnitzer 1978), we extend the following linear plus coulomb potential model

$$V(r) = g_1 r + (g_2/r) - V_0, \tag{2}$$

and a simple power law potential (1) (with $v = 0{\cdot}1$) to the relativistic domain using the Klein-Gordon equation. The model given by (2) is more suited to explain the Klein-Gordon bound states for lighter mesons (including the $c\bar{c}$ system). The power law model does not possess certain behaviours expected from the theoretical approach (non-coulombic short range behaviour of this potential is in apparent contradiction with the predictions of QCD). However, it is capable of giving good experimental fit to the systems consisting of $c\bar{c}$ and $b\bar{b}$ quark-antiquark pairs. To give a firm stand to the above potential model we study it within the framework of Klein-Gordon equation. Thus the motivation behind the present investigation is to give further support to the phenomenological power law potential in explaining the spectra of ψ, Υ charmed ($c\bar{q}$) and bottomed ($b\bar{q}$) mesons. The present study also reveals the effect of relativistic corrections to the spectra of lighter quarks using a combination of linear and coulomb potentials.

2. WKB approximation

The Klein-Gordon equation for the system of a quark and an antiquark interacting *via* the potential $V(r)$ (Kang and Schnitzer 1975) is

$$[\nabla^2 + \tfrac{1}{4}(E - V)^2 - m^2]\psi(r) = 0, \tag{3}$$

$$(c = \hbar = 1)$$

where m is the mass of the quark or antiquark. Equation (3) when reduced to the Schrödinger form, becomes

$$\left[-\frac{1}{2\mu}\nabla^2 + V_{\text{eff}}(r)\right]\psi(r) = \bar{E}\psi(r), \tag{4}$$

where the reduced mass $\mu = 1/2m$,

$$V_{\text{eff}} = -\frac{1}{2\mu}\left(\frac{1}{4}V^2 - \frac{1}{2}EV\right), \tag{5}$$

and

$$\bar{E} = (\tfrac{1}{4}E^2 - m^2)/2\mu.$$

With

$$\psi(r) = R(r)\, Y_l^m(\theta, \phi),$$

and

$$U_{n,l}(r) = rR(r),$$

the radial part of (4) takes the form

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} - \frac{1}{4}(E - V)^2 + m^2\right]U(r) = 0. \tag{6}$$

In the WKB approximation the bound state energy eigenvalues are obtained from the following quantization conditions:

For the fractional power potential (1) the s-wave energies are obtained by the following well-known relation

$$\int_0^{r_1} k(r)\,\mathrm{d}r = (n+\tfrac{3}{4})\pi, \tag{7}$$

where

$$k(r) = [\tfrac{1}{4}(E+V_0-g_1 r^{0.1})^2 - m^2]^{1/2}$$

and r_1 is the root of $k(r) = 0$. The WKB quantization condition appropriate to all l values for potential (2) and for $l \neq 0$ for the potential (1) is

$$\int_{r_1}^{r_2} p(r)\,\mathrm{d}r = (n+\tfrac{1}{2})\pi, \tag{8}$$

where

$$p(r) = \left[\frac{1}{4}(E-V)^2 - \frac{l(l+1)}{r^2} - m^2\right]^{1/2}$$

and r_1 and r_2 are the roots of $p(r) = 0$. The integrals appearing in (7) and (8) have been evaluated numerically using the Newton-Cotes four-point composite formula. Predictions for various meson mass spectra ensuing from the use of the potential (2) along with the respective values of the parameters m, g_1, g_2 and V_0 are given in tables 1, 3 and 4, where they have been compared with those resulting from the linear potential (Kang and Schnitzer 1975) and oscillator potential (Ram and Halasa 1979). For potential (1), the bound state mass spectra for $b\bar{b}$ and $c\bar{c}$ systems have been displayed in table 5. These results, along with the corresponding values of parameters m, V_0 and g_1, have been compared with those used for the Schrödinger bound state masses (Martin 1980; Barik and Barik 1981) and Dirac bound state masses (Barik and Barik 1981).

3. Mesons with unequal quark masses

For a classical system consisting of a quark with mass m_q and an antiquark with mass m_a, the total energy E is given by

$$E - V = (\mathbf{p}_q^2 + m_q^2)^{1/2} + (\mathbf{p}_a^2 + m_a^2)^{1/2} \tag{9}$$

where $\mathbf{p}_q$ and $\mathbf{p}_a$ are the three-momentum of quark and antiquark. On the basis of the string model the momentum of the quark and antiquark are assumed to be oppositely directed. Thus writing m_a and p_a in terms of m_q and p_q respectively we have

$$m_a/m_q = \mathbf{p}_a/\mathbf{p}_q = |\lambda| \text{ say} \tag{10}$$

equation (9) now reduces to

$$\frac{1}{(1+|\lambda|)^2}(E-V)^2 = \mathbf{p}_q^2 + m_q^2. \tag{11}$$

Making the usual quantum identifications, we obtain the following Klein-Gordon

Table 1. Predicted masses (GeV) for $I = 0$ mesons with $(c\bar{c})$ charmed quark pairs with linear plus coulomb potential (LP + CP).

l	0			1			2			3			4		
n	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP
0	3·096	3·105	3·179	3·423	3·456	3·443	3·68	3·761	3·702	3·915	4·03	3·957	4·15	4·286	4·207
1	3·65	3·696	3·695	3·91	3·964	3·946	4·105	4·215	4·193	4·305	4·45	4·436	4·47	4·673	4·675
2	4·11	4·169	4·186	4·257	4·397	4·426	4·44	4·616	4·662	4·62	4·826	4·895	—	—	—
3	4·42	4·58	4·656	4·606	4·783	4·886	4·77	4·980	5·114	—	—	—	—	—	—

Parameters used for various types of potentials are: (LP + CP) $- m = 20$ GeV; $g_1 = 0{\cdot}25$ GeV^2; $g_2 = 0{\cdot}20$; $V_0 = 1{\cdot}7$ GeV. Linear potential (LP) $- m = 2$ GeV; $a = 0{\cdot}30$ GeV^2; $b = 1{\cdot}72$ GeV. Oscillator potential (OP) $- m = 1{\cdot}566$ GeV; $\alpha = 0{\cdot}0303$ GeV^3; $\beta = -0{\cdot}361$ GeV.

Table 2. All parameters same as in table 1 but with $a = 3$ GeV.

State	Mass (GeV)	
	calculated	observed
1^3P_2	3·47	3·55
1^3P_1	3·375	3·51
1^3P_0	3·29	3·42

equation for unequal quark and antiquark masses

$$\left[\nabla^2 + \frac{1}{(1+|\lambda|)^2}(E-V)^2 - m^2\right]\psi(r) = 0. \tag{12}$$

replacing $1/4$ with $1/(1+|\lambda|)^2$ in (3) and proceeding as before, energy eigenvalues for charmed ($c\bar{q}$) and bottomed ($b\bar{q}$) mesons are obtained. Our predictions along with the corresponding results for these mesons obtained by Crater and Alstine (1981) have been displayed in table 6.

4. *P*-state splitting

Spin-dependent forces *viz* spin-orbit, spin-spin and tensor forces are generally taken over from the corresponding work on positronium (Müller-Kirsten *et al* 1979). We write the spin-dependent $q\bar{q}$ interactions, to leading order in $(v/c)^2$ as

$$\begin{aligned} V_c(r) = {} & \frac{3}{2m^2}\cdot\frac{1}{r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}V_{\mathrm{eff}}(r)\,\mathbf{L}\cdot\mathbf{S} \\ & + \frac{1}{6m^2}\nabla^2 V_{\mathrm{eff}}(r)\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ & - \frac{1}{12m^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}V_{\mathrm{eff}}(r) - \frac{1}{r}\cdot\frac{\mathrm{d}}{\mathrm{d}r}V_{\mathrm{eff}}(r)\right)S_{12}, \end{aligned} \tag{13}$$

where V_{eff} is the effective potential in the Schrödinger form (given (5)) with V substituted from (2). Here $\mathbf{L}$ is the orbital angular momentum operator, $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, $\mathbf{S}_i = \frac{1}{2}\boldsymbol{\sigma}_i$ and S_{12} is the standard tensor operator, *i.e.*

$$S_{12} = 3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\hat{r}) - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2.$$

The correction to be added to the potential $V(r) = g_1 r + (g_2/r) - V_0$ is obtained by (13) with V_{eff} replaced from (5) and (2). With this substitution we obtain terms which are singular at $r = 0$ (*i.e.* more divergent than $1/r^2$). For such singular potentials there are no acceptable bound-state solution. We regularize the singularity by introducing a cut-off parameter a into the potential. We choose this parameter by using the following replacement in the singular terms (Wills *et al* 1977)

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{1}{r}\right) \to \frac{1}{r}\cdot\frac{1}{(r^2+a^2)}. \tag{14}$$

Table 3. Predicted masses (GeV) for $I = 0$ mesons with λ-type quark pairs in a triplet state with the linear plus coulomb potential (LP + CP).

l	0			1			2			3			4		
n	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP
0	1·02	1·019	1·022	1·56	1·516	1·449	1·94	1·935	1·843	2·27	2·304	2·212	2·56	2·637	2·56
1	1·765	1·806	1·80	2·15	2·165	2·157	2·445	2·498	2·499	2·72	2·807	2·827	2·97	3·095	3·14
2	2·345	2·410	2·468	2·65	2·704	2·785	2·95	2·986	3·093	3·15	3·256	3·393	—	—	—
3	2·83	2·920	3·069	3·12	3·175	3·359	—	—	—	—	—	—	—	—	—

Parameters used for various types of potentials are: (LP + CP) $-m = 0{\cdot}474$ GeV; $g_1 = 0{\cdot}25$ GeV^2; $g_2 = 0{\cdot}2$; $V_0 = 0{\cdot}99$ GeV. (LP) $-m = 0{\cdot}475$ GeV; $a = 0{\cdot}30$ GeV^2; $b = -1{\cdot}16$ GeV. (OP) $-m = 0{\cdot}365$ GeV; $\alpha = 0{\cdot}0303$ GeV^3; $\beta = -0{\cdot}448$ GeV.

Table 4. Predicted masses (GeV) for $I = 0$ and $I = 1$ mesons with π- and p-type quark pairs in a triplet state with the linear plus coulomb potential (LP + CP).

l	0			1			2			3			4		
n	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP	LP + CP	LP	OP
0	0·777	0·77	0·776	1·405	1·310	1·242	1·805	1·758	1·662	2·15	2·147	2·049	2·46	2·496	2·409
1	1·59	1·60	1·603	2·01	1·981	1·979	2·32	2·333	2·336	2·61	2·658	2·676	2·88	2·959	3·001
2	2·2	2·228	2·295	2·53	2·536	2·625	2·785	2·832	2·943	3·035	3·114	3·252	—	—	—
3	2·71	2·754	2·911	2·985	3·019	3·210	—	—	—	—	—	—	—	—	—

Parameters used for various types of potentials are: (LP + CP) $-m = 0{\cdot}25$ GeV; $g_1 = 0{\cdot}25$ GeV^2; $g_2 = 0{\cdot}20$; $V_0 = 0{\cdot}911$; (LP) $-m = 0{\cdot}26$ GeV; $a = 0{\cdot}30$ GeV^2; $b = -1{\cdot}13$ GeV; (OP) $-m = 0{\cdot}206$ GeV; $\alpha = 0{\cdot}0303$ GeV^3; $\beta = -0{\cdot}505$ GeV.

Table 5. Klein-Gordon bound state spectrum of the $c\bar{c}$ and $b\bar{b}$ system.

		$M_{nl}(c\bar{c})$ (GeV)					$M_{nl}(b\bar{b})$ (GeV)				
n	l	Theory	Expt.[a]	Martin (1980)	Sch. bound states[b]	Dirac bound states[b]	Theory	Expt.[a]	Martin (1980)	Sch. bound states[b]	Dirac bound states[b]
1	S	3·095	3·095	3·095	3·0969	3·1175	9·46	9·46	9·46	9·43393	9·4462
2	S	3·683	3·684	3.687	3·6859	3·6789	10·02	10·02	10·025	9·9939	9·9977
3	S	3·99	4·03	4·032	4·02	3·9966	10·34	10·35	10·36	10·3116	10·3101
4	S	4·22	—	—	4·25789	4·2222	10·55	10·58	10·60	10·5377	10·5321
5	S	4·39	4·414	4·280	4·44389	4·3985	10·72	—	10·76	10·7145	10·7057
1	P	3·555	3·520	3·502	3·44632	3·4508	9·93	—	9·861	9·76612	9·7735
2	P	3·91	—	—	3·86996	3·8539	10·27	—	10·242	10·1689	10·1698
3	P	4·15	—	—	4·1473	4.1173	10·50	—	—	10·4325	10·4289

[a] Berkelmann (1980); [b] Barik and Barik (1981). Parameters used are: Theory—$g_1 = 5{\cdot}996$ (GeV)$^{1{\cdot}1}$; $V_0 = 7{\cdot}01$ GeV; $m_c = 1{\cdot}806$ GeV; $m_b = 5{\cdot}15$ GeV. Martin (1980)—$g_1 = 6{\cdot}8698$ (GeV)$^{1{\cdot}1}$; $V_0 = 8{\cdot}064$ GeV; $m_c = 1{\cdot}8$ GeV; $m_b = 5{\cdot}174$ GeV. Schrödinger and Dirac bound states (Barik and Barik 1981)—$g_1 = 5{\cdot}4072$ (GeV)$^{1{\cdot}1}$; $V_0 = 7{\cdot}452$ GeV; $m_c = 1{\cdot}77$ GeV; $m_b = 5{\cdot}11$ GeV.

Table 6. Klein-Gordon bound state spectrum of charmed ($c\bar{q}$) and bottomed ($b\bar{q}$) system.

Meson	Theory (GeV)	Richardson's potential (GeV)	Expt. (GeV)
$D^*(1s)$	2	1.99	2.01
$D^*(2s)$	2·575	2·575	—
$F^*(1s)$	2·14	2·102	2·14
$F^*(2s)$	2·72	2·685	—
$b\bar{u}(1s)$	5·055	5·311	5·165–5·315
$b\bar{u}(2s)$	5·6	5·830	—
$b\bar{s}(1s)$	5·2	5·414	—
$b\bar{s}(2s)$	5·72	5·939	—
$b\bar{c}(1s)$	6·535	6·337	—
$b\bar{c}(2s)$	7·09	6·879	—

Parameters used are: $g_1 = 5{\cdot}996\ (\mathrm{GeV})^{1\cdot 1}$; $V_0 = 6{\cdot}46$ GeV; $m_b = 5{\cdot}15$ GeV; $m_c = 1{\cdot}806$ GeV; $m_s = 0{\cdot}52$ GeV; $m_d = 0{\cdot}391$ GeV; $m_u = 0{\cdot}39$ GeV.

We further assume that the average separation of quark and antiquark in the meson is such that we neglect all powers of r/a higher than the second power (Müller-Kirsten *et al* 1979). Under these conditions we have to add to $V(r)$ the contribution $V_c(r)$ such that

$$\begin{aligned} V_{\text{total}} &= V(r) + V_c(r) \\ &= \left\{ g_1 - \frac{3g_2(E+V_0)}{4m^3a^4}\mathbf{L}\cdot\mathbf{S} - \frac{g_2(E+V_0)}{6m^3a^2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \right. \\ &\quad \left. + \frac{g_2(E+V_0)}{12m^3a^2}\left(1-\frac{1}{2a^2}\right)S_{12}\right\} r \\ &\quad + \left\{ g_2 + \frac{3(E+V_0)}{4m^3}\left(g_1+\frac{g_2}{a^2}\right)\mathbf{L}\cdot\mathbf{S} + \frac{(E+V_0)}{24m^3}\left(g_1+\frac{g_2}{a^2}\right)S_{12}\right\}\frac{1}{r} \\ &\quad + \left\{ \frac{-3g_2^2}{4m^3a^2}\mathbf{L}\cdot\mathbf{S} + \frac{g_2^2}{12m^3a^2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \frac{g_2^2}{12m^3a^2}S_{12}\right\}\frac{1}{r^2} \\ &\quad - V_0 + \left(\frac{3}{2m^3}\mathbf{L}\cdot\mathbf{S} + \frac{1}{6m^3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)\left(\frac{g_2^2}{2a^4} - \frac{g_1^2}{2}\right). \end{aligned} \tag{15}$$

With this potential we obtain the energy eigenvalues of ${}^{2s+1}P_j$ states of charmonium. In table 2, we give the masses of ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$ states.

5. Conclusion

From the results obtained in tables 1–4, it is easy to see that the meson masses predicted by the combination of linear and coulomb potentials (2) are not significantly different from those obtained with linear and harmonic oscillator potentials considered separately. However some of our results with potential (2) give better agreement with the experimental results. This is particularly true for ψ_{2s} and ψ_{3s} states. Further the model reasonably predicts the P-state splitting of $c\bar{c}$ system. From this work it can also be conjectured that power law potential (1) is capable of generating the bound state mass spectra of $b\bar{b}$, $c\bar{c}$, $b\bar{q}$ and $c\bar{q}$ systems not only in the Schrödinger and Dirac equations but in the Klein-Gordon equation as well. Relativistic effect, assumed to be non-negligible in the $b\bar{b}$ sepctra, does not spoil the agreement of the comparatively lighter quark-antiquark ($c\bar{c}$) system. Further a comparison of the parameter used in table 5 reveals that the extension of a purely phenomenological non-relativistic potential model to the relativistic domain does not change considerably the parameters involved in the model.

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