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Meson spectra with a power-law potential in the Dirac equation

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Abstract. A power law potential which is an equal admixture of scalar and vector parts with effective power $\nu \propto 1/m_q$, is proposed as a quark confining potential in the Dirac equation. The model is capable of predicting the meson spectroscopy encompassing both light and heavy quark-antiquark systems in a unified way.

Keywords. Relativistic quark model; quark confinement; meson spectroscopy; light mesons; heavy mesons.

1. Introduction

The power-law potential models have been quite successful in explaining the meson spectroscopy (Kang and Schnitzer 1975; Ram and Halasa 1979; Martin 1980; Barik and Jena 1980). Kang and Schnitzer (1975) using linear potential and Ram and Halasa (1979) using harmonic oscillator potential calculated the bound-state energies of ψ, ϕ and ρ mesons. The spectra of Υ and ψ mesons were explained by Martin (1980) and Barik and Jena (1980) using a fractional power potential ($V(r) = Ar^{0.1} + V_0$). In their non-relativistic study of power-law potential, Quigg and Rosner (1979) observed that the weak dependence of level spacing $(E_2 - E_1)$ upon quark masses for ψ and Υ mesons requires that the effective power ν is given by $\nu = 0{\cdot}08 \pm 0{\cdot}05$. Further from their study of level density of the bound systems of ψ and Υ mesons they proposed $\nu = 0{\cdot}20 \pm 0{\cdot}06$, for the ψ family, while for Υ family they found $\nu = 0{\cdot}33 \pm 0{\cdot}23$. On the other hand based on the field-theoretic arguments (Nielsen and Oelsen 1973), linear and oscillator potentials have been quite successful in explaining the spectra of ψ and other lighter mesons.

The present investigation is motivated with a desire to predict the entire meson spectroscopy (including the leptonic decay widths), using only a single power-law potential. Taking a clue from the values that power ν can possibly assume for the correct prediction of various spectra, we propose a power-law potential with effective power given by $\nu = m_0/2m_q$. Here m_q is the mass of the constituent quarks (in GeV) of the mesons under consideration. The reason for introducing the parameter m_0 in the power ν is to make it dimensionless. This is done by setting the parameter $m_0 = 1$ GeV. We choose a potential which is an equal admixture of scalar and vector parts as suggested by the phenomenology of fine-hyperfine splitting of heavy quarkonium system in a non-relativistic approach (Magyari 1980a, b).

2. Calculations of Dirac bound states

Based on the simple minded phenomenological viewpoint as mentioned in §1 our potential has the form

$$V(r) = g_1 r^{(m_0/2m_q)} - V_0. \tag{1}$$

($V(r)$ is in GeV and r in GeV^{-1}).

We use the Dirac equation written in the independent particle model of quarks. Following the prescription of Magyari (1980a, b) the potential $V(r)$ can be written as

$$V'(r) = \tfrac{1}{2} V(r) = V_s(r) + V_v(r). \tag{2}$$

With the choice of the vector fraction $g_v = 1/2$, the potential for the independent particle model of quarks would approximately read as

$$V'(r) = \tfrac{1}{2} V(r) = V_s(r) + V_v(r).$$

For this case then each of the scalar and vector parts would be equal to $1/4\, V(r)$. Now the Dirac equation can be written as (with $\hbar = c = 1$)

$$(\alpha \cdot P + m_q \beta)\psi(r) = [E' - V_v(r) - V_s(r)\beta]\,\psi(r). \tag{3}$$

Since $V_v(r)$ and $V_s(r)$ are spherically symmetric, (3) can be separated in a system of the following coupled equations for the radial wave functions (Schiff 1968) $\phi(r)$ and $\chi(r)$:

$$\left.\begin{aligned} &(E' - V_v - V_s - m_q)\phi(r) + \left(\frac{k+1}{r} + \frac{\mathrm{d}}{\mathrm{d}r}\right)\chi(r) = 0 \\ &(E' - V_v + V_s + m_q)\chi(r) + \left(\frac{k-1}{r} - \frac{\mathrm{d}}{\mathrm{d}r}\right)\phi(r) = 0 \end{aligned}\right\} \tag{4}$$

Here $k = l+1$ if the total angular momentum of the quark is $j = l + \frac{1}{2}$, and $k = -l$; if $j = l - \frac{1}{2}$.

Substituting $\phi(r) = U'(r)/r$ and $V_s(r) = V_v(r) = \frac{1}{4}(g_1 r^{(m_0/2m_q)} - V_0)$, the equation for large component $\phi(r)$ takes the form (Magyari 1980a).

$$\frac{\mathrm{d}^2 U'(r)}{\mathrm{d}r^2} + \left[(E' + m_q)(E' - m_q - 2V_s(r)) - \frac{l(l+1)}{r^2}\right] U'(r) = 0. \tag{5}$$

Choosing $\rho = (r/r_0)$ and $r_0 = [(m_q + E')g_1/2]^{-1/[2+(m_0/2m_q)]}$, (5) reduces to a Schrödinger equation with a power-law potential. This Schrödinger-like equation is solved using the semiclassical solution of Quigg and Rosner (1979). Our procedure is similar to that used by Barik and Barik (1981) except that v is replaced by $m_0/2m_q$. Thus the equation for Dirac bound state masses can be written as (Barik and Barik 1981)

$$M_{nl}(Q\bar{Q}) = (2ax_{nl} + 2m_q - V_0), \tag{6}$$

where $a = (g_1)^{1/[1+(m_0/2m_q)]}$ and x_{nl} is the positive root of the following equation:

$$x_{nl}^{[1+(4m_q/m_0)]}(x_{nl} + b) = 2^{-4m_q/m_0}(\varepsilon)^{[1+(4m_q/m_0)]}$$

where

$$\varepsilon_{nl} = \left[\left(n + \frac{l}{2} - \frac{1}{4}\right)\frac{2(\pi)^{1/2}\,\Gamma\left(\dfrac{3}{2} + \dfrac{2m_q}{m_0}\right)}{\Gamma\left(1 + \dfrac{2m_q}{m_0}\right)}\right]^{2/[1+(4m_q/m_0)]}$$

and

$$b = (2m_q - \tfrac{1}{2} V_0)/a. \tag{7}$$

Values of the various parameters g_1, V_0 and m_q and the bound state masses of Υ, ψ, ϕ and ρ mesons are displayed in tables 1 and 2 respectively.

Following the method used by Quigg and Rosner (1979) and Barik and Barik (1981) the following expression for the ratio of leptonic decay widths is obtained

$$R_{ns} = \left[\frac{M_{1s}(Q\bar{Q})}{M_{ns}(Q\bar{Q})}\right]^2 \frac{|\psi_{ns}(0)|^2}{|\psi_{1s}(0)|^2}, \tag{8}$$

where

$$|\psi_{ns}(0)|^2 = \frac{(m_q)^{3/2}}{4\pi^2} \frac{a^{3/2}\, 2^{[(5/2)-(4m_q/m_0)]}\left[\left(n-\frac{1}{4}\right)G(\nu)\right]}{x_{ns}^{[(4m_q/m_0)-(1/2)]}\left[2\left(1+\frac{2m_q}{m_0}\right)x_{ns}+\left(1+\frac{4m_q}{m_0}\right)b\right]}$$

Table 1. Values of the parameters m_0, m_q, g_1 and V_0.

Meson	Mass of the constituent quark (GeV) m_q	m_0	g_1 $[(\mathrm{GeV})^{(2m+1)/2m}]$	V_0 (GeV)
Υ	5	1	6·550	7·3662
ψ	2	1	2·365	3·8833
ϕ	0·5	1	0·2725	1·089
ρ	0·25	1	0·0372	0·761

Table 2. Dirac bound state masses (GeV) for Υ, ψ, ϕ and ρ mesons.

Meson	Theory	Richardson potential[a]	Experiment	Meson	Theory	Richardson potential[a]
$\gamma(1S)$	9·46	9·46	9·46	$\Upsilon(1P)$	9·7945	9·935
$\gamma(2S)$	10·0235	10·0207	10·0207 ± 0·0008	$\Upsilon(2P)$	10·1993	—
$\gamma(3S)$	10·3426	10·3487	10·3511 ± 0·0007	$\Upsilon(3P)$	10·464	—
$\gamma(4S)$	10·5695	10·604	10·5725 ± 0·005	—	—	—
$\gamma(5S)$	10·7468	10·824	—	—	—	—
$\psi(1S)$	3·097	3·097	3·097 ± 0·002	$\psi(1P)$	3·4377	3·556
$\psi(2S)$	3·6863	3·661	3·686 ± 0·003	$\psi(2P)$	3·8773	—
$\psi(3S)$	4·0398	4·055	4·03 ± 0·01	$\psi(3P)$	4·1804	—
$\psi(4S)$	4·3047	4·383	—	—	—	—
$\psi(5S)$	4·5181	4·673	4·417 ± 0·01	—	—	—
$\phi(1S)$	1·020	1·020	1·020	$\phi(1P)$	1·5119	1·564
$\phi(2S)$	1·8352	1·706	—	$\phi(2P)$	2·0726	—
$\phi(3S)$	2·3669	—	—	$\phi(3P)$	2·5966	—
$\rho(1S)$	0·767	0·759	0·767	$\rho(1P)$	1·2127	1·353
$\rho(2S)$	1·60	1·509	1·60	$\rho(2P)$	1·9547	—
$\rho(3S)$	2·2825	—	—	$\rho(3P)$	2·5910	—

[a] Crater and Alstine (1981)

with

$$G(v) = 2(\pi)^{1/2}\Gamma\left(\frac{3}{2}+\frac{2m_q}{m_0}\right)\Big/\Gamma\left(1+\frac{2m_q}{m_0}\right). \tag{9}$$

In (9) the x_{ns} value for a particular meson state can be obtained with the help of (6). Equation (8) can be further simplified to

$$R_{ns} = \left[\frac{M_{1s}(Q\bar{Q})}{M_{ns}(Q\bar{Q})}\right]^2 \times \frac{(4n-1)\, x_{1s}^{[(4m_q/m_0)-(1/2)]}\left[2\left(1+\frac{2m_q}{m_0}\right)x_{1s}+\left(1+\frac{4m_q}{m_0}\right)b\right]}{3x_{ns}^{[(4m_q/m_0)-(1/2)]}\left[2\left(1+\frac{2m_q}{m_0}\right)x_{ns}+\left(1+\frac{4m_q}{m_0}\right)b\right]}. \tag{10}$$

The predicted results for the ratio of leptonic decay widths for ψ and Υ mesons are displayed in table 3.

3. Conclusion

The present results are compared with the corresponding results obtained by Crater and Alstine (1981) using the Richardson potential. A look at table 2 shows that the masses predicted by potential (1) are in very good agreement with the experimental results. The prediction for the leptonic decay widths is also comparable with their corresponding experimental results.

Thus in spite of certain discomforting features (non-coulombic short range behaviour of this potential is in apparent contradiction with the prediction of QCD) the simple power-law potential proposed in this paper is capable of reproducing the spectra of all the mesons simultaneously. So we conclude with the remark that our potential model fairly represents the actual interaction between $Q\bar{Q}$ for a wide range of quark-antiquark separation distance probed in the study of the entire meson spectra. It is also of interest to note that the coupling constant g_1 exhibits the following linear relationship (except for ρ meson) with the quark mass m:

$$g_1 = 1{\cdot}395\, m - 0{\cdot}4250.$$

Table 3. Ratio of decay width $R_{ns} = [\Gamma(ns \to e^+e^-)]/[\Gamma(1s \to e^+e^-)]$ for $c\bar{c}$ and $b\bar{b}$ systems. Parameters are same as those used in table 1.

		$R_{ns} = \Gamma(ns \to e^+e^-)/\Gamma(1s \to e^+e^-)$			
		$c\bar{c}$ system		$b\bar{b}$ system	
n	ρ	Theory	Experiment	Theory	Experiment
1	s	1	1	1	1
2	s	0·391	0·45 ± 0·09	0·433	0·44 ± 0·06
3	s	0·240	0·156	0·276	0·32 ± 0·04
4	s	0·171	0·102	0·202	0·2 ± 0·06

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