Pramāṇa, Vol. 22, No. 6, June 1984, pp. 489–496. Printed in India.

On the quantisation of dissipative systems

M C VALSAKUMAR

Materials Science Laboratory, Reactor Research Centre, Kalpakkam 603 102, India

Abstract. Two methods of quantisation of dissipative systems are considered. It is shown that the phase space description of quantum mechanics permits computational simplification, when Kanai's method is adopted. Since the Moyal Bracket is the same as the Poisson Bracket, for systems described by a most general explicitly time dependent quadratic Lagrangian, the phase space distribution can be obtained as the solution of the corresponding classical Langevin equations in canonical variables, irrespective of the statistical properties of the noise terms. This result remains true for arbitrary potentials too in an approximate sense. Also analysed are Dekker's theory of quantisation, violation of uncertainty principle in that theory and the reason for the same.

Keywords. Quantum diffusion; quantum dissipative systems; Langevin equation; stochastic Liouville equation; Wigner distribution; Moyal Bracket.

1. Introduction

The problem of quantisation of a subsystem interacting with a dissipative environment continues to be of much interest (for a detailed list of references see Louisell 1973; Agarwal 1974; Haken 1975; Hasse 1975; Dekker 1977; Yasue 1978; Greenberger 1979; Benguria and Kac 1981; Jayannavar and Kumar 1982; Gzyl 1983). There have been widely differing approaches to this issue. But for a few exceptions (*e.g.*, Senitzky 1960), one normally assumes a classical description at a phenomenological level and then tries to find a suitable quantum mechanical description. Such approximate treatments, when viewed critically, violate the algebraic requirements of quantum mechanics (Sahoo 1972, 1983). This calls for an exact treatment of the bath and the subsystem, which is impossible. Confronted with the task of explaining satisfactorily the behaviour of the perturbed subsystem (for example, how the spontaneous emission of an atom gets affected when enclosed in an electromagnetic cavity with finite loss) one is still justified in resorting to approximations. We thus view these methods as practical mathematical tools for the analysis of the physical problem and comparison with experiments as the ultimate check on the validity of any method.

In this paper we discuss the method of quantisation of dissipative systems using an explicitly time-dependent Hamiltonian (Kanai 1948) and the procedure of canonical quantisation employing complex dynamical variables (Dekker 1975). One of the aims of the present paper is to point out a computational simplification that is possible when the former method is adopted, *viz* for dissipative systems modelled by a quadratic Lagrangian, the phase space distribution function (Louisell 1975) in the Wigner representation is the same as the solution of the corresponding classical Langevin equations in canonical variables. This result is valid irrespective of the nature of the random forces. The quantum character enters the problem only through the initial condition—a point which seems to have been unrecognised and hence unexploited.

This result remains true for arbitrary potentials too in an approximate sense. The second aim of the paper is to demonstrate the violation of uncertainty principle in Dekker's method of quantisation and to trace the reason for the same.

The paper is organised as follows. In §2 we discuss the method of an explicitly time-dependent Hamiltonian. Dekker's method of quantisation is discussed in §3. Conclusions are brought out in §4 after a brief discussion.

2. The method of an explicitly time-dependent Hamiltonian

We consider quantising a dissipative system modelled by a classical phenomenological equation of motion expressed conveniently as

$$M(t)\ddot{x}(t)+\dot{M}(t)\dot{x}(t)+C(t)\frac{\partial}{\partial x}V(x,t)=F(t), \tag{1}$$

where the overdot denotes differentiation with respect to time t. The explicitly time-dependent Lagrangian L (Kanai 1948) for the above reads*

$$L(x,\dot{x},t)=M(t)\dot{x}(t)^2-C(t)V(x,t)+x(t)F(t). \tag{2}$$

In (1) and (2), the functions $C(t)$, $V(x,t)$ and $F(t)$ could be random, with arbitrary probability distributions restricted only by the physical constraints of the problem. The canonical momentum p and the Hamiltonian H are

$$p=M\dot{x} \quad \text{and} \quad H=p^2/2M+CV-xF. \tag{3}$$

Formal quantisation of the above classical system is achieved through the Schrödinger equation

$$i\hbar\dot{\psi}(x,t)=\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2}+CV-xF\right]\psi(x,t). \tag{4}$$

Rather than solving for the wavefunction $\psi(x,t)$, we will be interested in the phase space description (Louisell 1973). The phase space distribution function in the Wigner representation

$$W(x,p,t)=\int_{-\infty}^{\infty}d\alpha\exp(-i\alpha p)\psi^*\left(x-\frac{\hbar\alpha}{2},t\right)\psi\left(x+\frac{\hbar\alpha}{2},t\right) \tag{5}$$

for any system described by a Hermitian Hamiltonian H satisfies the equation (Moyal 1949)

$$\dot{W}(x,p,t)=(W,H)_M=\frac{2}{\hbar}W\sin\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}-\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}\right)\right]H. \tag{6}$$

In (6), $(\ ,\ )_M$ stands for the Moyal Bracket. For the present problem (6) becomes

$$\dot{W}=-\frac{\partial}{\partial x}\left[\frac{p}{M}W\right]-\frac{\partial}{\partial p}\left[\left(F-C\frac{\partial V}{\partial x}\right)W\right]+C\sum_{r=1}^{\infty}\left(\frac{\hbar}{2}\right)^{2r}\frac{(-1)^r}{(2r+1)!}\left(\frac{\partial^{2r+1}V}{\partial x^{2r+1}}\right)\left(\frac{\partial^{2r+1}W}{\partial p^{2r+1}}\right). \tag{7}$$

* Though, for clarity of presentation, we have adopted a one dimensional treatment, the method is applicable in arbitrary dimensions. Similarly, there is no loss of generality on account of not considering a general linearly damped system which will also have an $x\dot{x}$ term in the Lagrangian.

We give a simple alternative derivation of (7) in the Appendix (for an algebraic proof see Sahoo 1972).

The quantity of our interest is not the Wigner distribution W itself but rather its stochastic average $\langle W \rangle$ (over the ensembles of the random variables $C(t)$, $V(x, t)$ and $F(t)$), which satisfies the equation

$$\langle \dot{W} \rangle = -\frac{\partial}{\partial x}\left[\frac{p}{M}\langle W\rangle\right] - \frac{\partial}{\partial p}\left\langle\left(F - C\frac{\partial V}{\partial x}\right)W\right\rangle + \left\langle C\sum_{r=1}^{\infty}(\hbar/2)^{2r}\frac{(-1)^r}{(2r+1)!}\left(\frac{\partial^{2r+1}V}{\partial x^{2r+1}}\right)\left(\frac{\partial^{2r+1}W}{\partial p^{2r+1}}\right)\right\rangle. \tag{8}$$

Incidentally, (8) can be simplified for Gaussian noises using Novikov's theorem (Novikov 1965).

Several results follow trivially from (7) and (8). Firstly, we note that the Moyal Bracket is the same as the Poisson Bracket for a quadratic Hamiltonian. Hence, for such a system,

$$\dot{W} = -\frac{\partial}{\partial x}\left(\frac{p}{M}W\right) - \frac{\partial}{\partial p}\left[\left(F - C\frac{\partial V}{\partial x}\right)W\right], \tag{9}$$

and

$$\langle \dot{W} \rangle = -\frac{\partial}{\partial x}\left[\frac{p}{M}\langle W\rangle\right] - \frac{\partial}{\partial p}\left\langle\left(F - C\frac{\partial V}{\partial x}\right)W\right\rangle. \tag{10}$$

It is interesting to note that Planck's constant $\hbar$ does not appear in (9) and (10). Equation (10) can now be identified to be the stochastic Liouville equation (van Kampen 1976) for the classical Langevin equations (in *canonical variables*)

$$\dot{x} = p/M \quad \text{and} \quad \dot{p} = -C(\partial V/\partial x) + F. \tag{11}$$

The quantal character of the system enters the analysis only through the initial condition*. To the best of our knowledge, this result has not been recognised and used. For example, it is clear from this finding that we do not expect any difference, in the asymptotic behaviour in the classical and quantum treatments of 'diffusion' in a randomly fluctuating medium. The results of the quantum continuum should agree with those of the classical continuum and likewise for the lattice problem. The confusion which arose out of a wrong comparison of the quantum lattice problem with the classical continuum problem, which was resolved only recently (Jayannavar and Kumar 1982), would have been a non-problem if the present result had been recognised. Similarly the quantum treatments seem to have been restricted to the case of Gaussian white noise only—presumably because one wants to get average density matrix or $\langle W \rangle$—the evolution equation of which can be simplified only when the noise is Gaussian and white. We mention in passing that there exists a path integral solution for a general quadratic Lagrangian (Papadopoulos 1976). But all these can be done

* By this we do not mean that quantisation *per se* is an initial value problem. However, the problem at hand is really so. We are interested in the time evolution of a quantal subsystem (initially prepared in some state) which is subjected to the perturbation at time $t = 0$. Since the Hamiltonian is explicitly time dependent, the question of finding the stationary states does not arise.

straightaway using the present method. The prescription is

$$\langle W(x,p,t)\rangle = \langle \int\int dx_0\, dp_0 W_0(x_0,p_0) \delta[x - x(x_0,p_0,t)]\,\delta[p - p(x_0,p_0,t)]\rangle, \tag{12}$$

where $W_0(x_0, p_0)$ is the initial Wigner distribution and $x(x_0, p_0, t)$ and $p(x_0, p_0, t)$ are the solutions of (11).

Secondly we note that (7) is a perturbation series in the small parameter $\hbar$. Since the higher order terms in the series are of order $\hbar^2$, the above conclusion is correct up to second order. This may be of immense value in analysing problems such as diffusion in a bistable potential (Valsakumar 1983) at low temperatures, where the quantum tunnelling competes with the classical escape over the potential barrier (this, in fact, was one of the motivations for carrying out this work).

Another area where this result may be of use is in answering the question "what should be the statistical property of $F(t)$ (we assume that C and V are deterministic) so that the asymptotic solution is the quantum mechanical canonical distribution?" (Benguria and Kac 1981). We know that the condition for the corresponding classical problem is that $F(t)$ be Gaussian and white. Then the condition for systems described by quadratic Lagrangians should be the same. In the light of this investigation, we may safely conclude that the necessary condition for arbitrary potentials also should be the same (we cannot expect that higher order terms will so conspire as to make the asymptotic distribution canonical for arbitrary $F(t)$).

3. Dekker's method of quantisation

There have been criticisms (Dekker 1975; Greenberger 1979) against the method employing an explicitly time-dependent Lagrangian. Dekker proposed the method of quantisation of complex dynamical variables (Dekker 1975, 1977) to surmount those problems. This method proceeds with a description in terms of complex variables* for the classical problem. He then finds a complex Hamiltonian which generates classical dynamics. The intimate relation between the imaginary part of the Hamiltonian and dissipation is demonstrated. Physically, this analysis is equivalent to considering both the system and its mirror image simultaneously. Transition to quantum mechanics is achieved by constructing an equivalent non-Hermitian Hamiltonian operator for generating an infinitesimal time translation.

Even though the basic premise sounds physical, as we shall demonstrate, the uncertainty principle is violated at the stage of application of the theory and our aim is to trace the reason for the same. Let us start our analysis by considering a subsystem interacting with a bath kept at zero temperature described classically by the equation

$$\ddot{x} + 2\lambda\dot{x} + (\partial V/\partial x) = 0. \tag{13}$$

Now, Dekker's procedure is operationally equivalent to discovering a non-Hermitian Hamiltonian operator $\hat{\mathscr{H}} = \hat{H} + i\hat{\Gamma}$ such that the Heisenberg equations of motions for

* The existence of the complex dynamical variables is not always guaranteed. For example, one cannot find them for an overcritically damped harmonic oscillator.

the operators $\hat{x}$ and $\hat{p}$ are

$$\dot{\hat{x}} = \frac{1}{i\hbar}[\hat{x}, \hat{H}]_- + \frac{1}{\hbar}[\hat{x}, \hat{\Gamma}]_+ = \hat{p}, \tag{14}$$

and

$$\dot{\hat{p}} = \frac{1}{i\hbar}[\hat{p}, \hat{H}]_- + \frac{1}{\hbar}[\hat{p}, \hat{\Gamma}]_+ = -2\lambda\hat{p} - (\partial\hat{V}/\partial x). \tag{15}$$

In (14) and (15), $[\ ,\ ]_-$ and $[\ ,\ ]_+$ are the usual commutator and anticommutator respectively. It is easy to show that

$$\hat{H} = \frac{\hat{p}^2}{2} + \hat{V}(x) + \frac{\lambda}{2}[\hat{x}, \hat{p}]_+ \quad \text{and} \quad \hat{\Gamma} = -\hbar\lambda/2 \tag{16}$$

satisfy the above requirement, though by no means is the choice unique. Incidentally, in this approach the question of the existence of complex variables is irrelevant and hence it is applicable under general circumstances.

One of the conditions for the quantisation rule $[\hat{x}, \hat{p}]_- = i\hbar$ to be a consistent scheme is that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\hat{x}, \hat{p}]_- = 0. \tag{17}$$

But from (14) and (15),

$$\frac{\mathrm{d}}{\mathrm{d}t}[\hat{x}, \hat{p}]_- = -2\lambda[\hat{x}, \hat{p}]_-. \tag{18}$$

The right side of (18) can be zero if $\lambda = 0$ (no dissipation). This implies that the vacuum quantal fluctuations are essential for the preservation of the fundamental commutator. Or physically this means that any coupling always acts both ways. This aspect is taken care of by adding quantal terms to the Heisenberg equations of motion such that, in the quantum-classical correspondence limit, we get back the usual classical equations of motion. The modified equations of motion read

$$\dot{\hat{x}} = \hat{p} + \hat{\eta}_x \quad \text{and} \quad \dot{\hat{p}} = -2\lambda\hat{p} - (\partial\hat{V}/\partial x) + \hat{\eta}_p, \tag{19}$$

and the consistency condition (17) becomes

$$-2\lambda[\hat{x}, \hat{p}]_- + [\hat{x}, \hat{\eta}_p]_- + [\hat{\eta}_x, \hat{p}]_- = 0. \tag{20}$$

In (19) and (20) the operators $\hat{\eta}_x$ and $\hat{\eta}_p$ involve the coordinates of both the system and the bath. It is obvious that (20) can be satisfied *exactly* iff both the system and the bath are treated dynamically, which of course is well-known. Dekker's method of constructing an effective Hamiltonian corresponds to satisfying (20) in an *average* sense (treating $\hat{\eta}_x$ and $\hat{\eta}_p$ as random variables to give $\langle [\hat{x}, \hat{p}]_- \rangle = i\hbar$) and hence is not consistent with quantum algebra (Sahoo 1972, 1983). Thus, at the phenomenological level, the commutator $[\hat{x}, \hat{p}]_-$ is generally a complex stochastic process with nonzero real and imaginary parts. Hence, in our opinion, just satisfying (20) in an average sense is not *sufficient*. We must also make sure that $|\langle [\hat{x}, \hat{p}]_-^2 \rangle| > \hbar^2$ (which is not automatically satisfied), in order that the uncertainty product $\sigma_x^2\sigma_p^2$ be $\geqslant$ to $\hbar^2/4$.

Just to demonstrate this, let us consider Dekker's solution (Dekker 1977) to the

damped harmonic oscillator problem ($V(x) = \Omega^2 x^2/2$)

$$\dot{W} = -\frac{\partial}{\partial x}(pW) + \frac{\partial}{\partial p}[(2\lambda p + \Omega^2 x)W] + \hbar\lambda\Omega\frac{\partial^2 W}{\partial p^2}. \tag{21}$$

From (21), we get the rate of change of the uncertainty product $\sigma_x^2\sigma_p^2$ to be

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_x^2\sigma_p^2) = -4\lambda\sigma_x^2\sigma_p^2 + 2\hbar\lambda\Omega\sigma_x^2 + 2\sigma_{xp}(\sigma_p^2 - \Omega^2\sigma_x^2), \tag{22}$$

where σ_x^2 and σ_p^2 are the variances of x and p respectively. Let us take the initial wavefunction to be a Gaussian wave packet so that the initial Wigner distribution $W(x, p, 0)$ is

$$W(x, p, 0) = 1/\pi\hbar \exp -\left(\frac{x^2}{2a} + \frac{2ap^2}{\hbar^2}\right), \tag{23}$$

where a is the initial variance of x. In the present case $\sigma_x^2\sigma_p^2$ at zero time is the minimum permissible $\hbar^2/4$. It is easy to see that the right side of (22) becomes negative in the vicinity of $t = 0$ when $a < \hbar/2\Omega$, thereby allowing $\sigma_x^2\sigma_p^2 < \hbar^2/4$. We must add that this violation of the uncertainty principle is only transient—the asymptotic results are correct.

4. Discussion and conclusions

Before summing up let us compare the methods described here. We note that the approaches are phenomenological: while the subsystem is treated exactly, no attempt is made to treat the loss mechanism dynamically. Both the Hamiltonians are non-unique: each of them is a member of a set of functions consistent with the given equation of motion. In the limit of no damping, they reduce to the usual Hamiltonian.

A serious objection raised against Kanai's method is that the commutator $[\hat{x}, \hat{p}_k]_-$ ($\hat{p}_k$ is the kinetic momentum) vanishes asymptotically (Dekker 1975; Greenberger 1979). We would like to remark that in this theory, the above commutator is *not* the fundamental one. Quantisation is done in canonical variables and it is the commutator $[\hat{x}, \hat{p}_k]_-$ which should be preserved Further the vanishing of the commutator $[\hat{x}, \hat{p}_k]_-$ has nothing to do with quantum mechanics, for it is the same story with the corresponding Poisson Bracket in classical mechanics. Since the Poisson Bracket $\{x, p\}$ and the commutator $[\hat{x}, \hat{p}]_-$ are preserved, we think that this really provides a consistent picture. Physically this means that the kinetic momentum ceases to be the generator of infinitesimal translations. The question, as to which of the methods is correct, should be resolved by comparison with experiments only.

To conclude, computational simplification is possible by a phase space description of quantum mechanics in case we adopt Kanai's method of quantisation. Dekker's theory violates uncertainty principle which can be rectified by one more constraint on the quantum fluctuations. Both the methods are phenomenological and hence experiments should provide the justification for one method or the other.

Acknowledgements

The author thanks Debendranath Sahoo for a number of useful discussions. Thanks are also due to N Kumar for a series of lectures on the topic which provided the impetus for this work.

Appendix

From the definition given in (5) we identify the Wigner distribution $W(x, p, t)$ to be the inverse Fourier transform of the density matrix

$$\rho(y, z, t) = \psi^*(y, t)\psi(z, t), \quad y = x - \frac{\hbar\alpha}{2}, \quad z = x + \frac{\hbar\alpha}{2}. \tag{A1}$$

The time evolution of the density matrix can be obtained using the Schrödinger equation (4) as

$$\frac{\partial}{\partial t}\rho = \frac{i\hbar}{2M}\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2}\right)\rho + \frac{i}{\hbar}[C\{V(y) - V(z)\} + (z - y)F]\rho. \tag{A2}$$

Going back to the variables $x = (z + y)/2$ and $\alpha = (z - y)/\hbar$, we get

$$\frac{\partial}{\partial t}\rho = \frac{i}{M}\frac{\partial^2}{\partial\alpha\partial x}\rho + \frac{i}{\hbar}C\left[V\left(x - \frac{\hbar\alpha}{2}\right) - V\left(x + \frac{\hbar\alpha}{2}\right)\right]\rho + i\alpha F\rho. \tag{A3}$$

If $V(x)$ is analytic, we have

$$V\left(x - \frac{\hbar\alpha}{2}\right) - V\left(x + \frac{\hbar\alpha}{2}\right) = -2\sum_{r=0}^{\infty}(\hbar/2)^{2r+1}\frac{1}{(2r+1)!}\left(\frac{\partial^{2r+1}V}{\partial x^{2r+1}}\right)\alpha^{2r+1}. \tag{A4}$$

Substituting (A4) in (A3) and taking inverse Fourier transform with respect to α, we obtain

$$\dot{W} = -\frac{\partial}{\partial x}\left(\frac{p}{M}W\right) - \frac{\partial}{\partial p}\left[\left(F - C\frac{\partial V}{\partial x}\right)W\right] + C\sum_{r=1}^{\infty}(\hbar/2)^{2r}\frac{(-1)^r}{(2r+1)!} \times\left(\frac{\partial^{2r+1}V}{\partial x^{2r+1}}\right)\left(\frac{\partial^{2r+1}W}{\partial p^{2r+1}}\right), \tag{A5}$$

which is the same as (7).

References

Agarwal G S 1974 in *Springer tracts in modern physics* (ed) G Hohler (New York: Springer Verlag) Vol 70
Benguria R and Kac M 1981 *Phys. Rev. Lett.* **46** 1
Dekker H 1975 *Z. Phys.* **B21** 295
Dekker H 1977 *Phys. Rev.* **A16** 2126
Greenberger D M 1979 *J. Math. Phys.* **20** 762
Gzyl H 1983 *Phys. Rev.* **A27** 2297
Haken H 1975 *Rev. Mod. Phys.* **47** 67
Hasse R W 1975 *J. Math. Phys.* **16** 2005

Jayannavar A M and Kumar N 1982 *Phys. Rev. Lett.* **48** 553
van Kampen 1976 *Phys. Rep.* **C24** 171
Kanai E 1948 *Prog. Theor. Phys.* **8** 440
Louisell W H 1973 in *Quantum statistical properties of radiation* (New York: Wiley)
Moyal J E 1949 *Proc. Camb. Philos. Soc.* **45** 99
Novikov E A 1965 *Sov. Phys. JETP* **20** 1290
Papadopoulos G J 1976 *Phys. Rev.* **D11** 2870
Sahoo D 1972 in unpublished thesis (New York: Yeshiva University)
Sahoo D 1983 Private communication
Senitzky I R 1960 *Phys. Rev.* **119** 670
Valsakumar M C 1983 *J. Stat. Phys.* **32** 545
Yasue K 1978 *Ann. Phys.* **114** 479

Note added in proof

A general quantum mechanical master equation for the damped oscillator has subsequently been investigated with respect to the uncertainty principle (Dekker and Valsakumar, 1984: to be published). This treatment leads to a condition to be imposed on the diffusion coefficients which is unique in Dekker's theory.