

Thermodynamic and magnetic properties of the electron gas in the inhomogeneous magnetic field Hr^{-1}

P ACHUTHAN and K VENKATESAN

Department of Mathematics, Indian Institute of Technology, Madras 600036, India

MS received 7 September 1983; revised 12 March 1984

Abstract. The nonrelativistic and relativistic energy eigenvalues of the electron in the inhomogeneous magnetic field $H_z = Hr^{-1}$, $r = (x^2 + y^2)^{1/2}$ are derived in a form displaying the explicit spin dependence. The possibility of magnetic hydrogen atom formation and the spontaneous e^+e^- pair creation following from these eigenvalues is mentioned. The expressions for pressure, energy, particle number and magnetic moment of an electron gas in this IMF are calculated in the degenerate limit. The possibility of spontaneous magnetisation, *i.e.*, ferromagnetic behaviour, is established. Further, the pressure of the electron gas in the same type of fields is an order of magnitude higher than those in a homogeneous magnetic field and crossed homogeneous electric and magnetic fields for comparable field strengths.

Keywords. Inhomogeneous magnetic field; electron gas; thermodynamic properties; magnetic properties.

1. Introduction

It is well-known that generally the magnetic fields in nature, both terrestrial as well as celestial, are spatially inhomogeneous in character, but theoretically it is the homogeneous magnetic field (HMF) which has been used to study the magnetic properties at the microscopic and macroscopic levels. In HMF, quantum effects start appearing only when the field strength is of the order of 10^9 G or more (*e.g.*, Canuto and Kelly 1972; Ruderman 1974, 1977). The question naturally arises whether the inclusion of inhomogeneity, *i.e.*, space-dependence of the field will circumvent the (at present) impossibility of producing such strong HMF's. Recently (Achuthan *et al* 1982) a study of the behaviour of electron gas in the inhomogeneous magnetic field (IMF) $H_z = H \operatorname{sech}^2(ay)$, where a is the inhomogeneity parameter, was started with the above question in mind. But it so happened that the field existed over reasonable spatial dimensions only for negligibly small values of a and hence the energy eigenvalues and the statistical degeneracy turned out to be the same for both IMF and HMF. The thermodynamic properties are therefore the same for the two cases. The fact that there were deviations in the magnetic behaviour was due to the explicit spatial dependence factor arising from the definition of the magnetic moment as the negative field derivative of the thermodynamic potential. Further, for this IMF the possibility of spontaneous magnetisation was claimed only for field strengths of the order of 10^{12} G, which are supposed to exist only in neutron stars.

An examination of the energy eigenvalues of the electron in the magnetic field

$$H_x = H_y = 0, \quad H_z = Hr^{-1}, \quad r = (x^2 + y^2)^{1/2} \quad (1)$$

as done below shows that there is the possibility of spontaneous pair creation but with an added advantage that the field exists over macroscopic distances, *i.e.*, distances large compared to the atomic or interatomic distances. So this particular field seems to be a suitable candidate for probing further into other properties of the single electron as well as the electron gas in an IMF. Apart from noticing, in parentheses, the possibility of formation of the magnetic hydrogen atom and of the spontaneous creation of electron-positron pairs we consider in this paper the thermodynamic and magnetic properties of an electron gas in this IMF.

In §2, we derive the energy eigenvalues for both the nonrelativistic and relativistic electrons in the magnetic field Hr^{-1} which display their dependence on the spin of the electron explicitly. Two side remarks on the possibility of forming the magnetic hydrogen atom and spontaneous e^+e^- creation are also made. In §3, expressions for the thermodynamic properties, *viz* the pressure, energy density and number density are derived. In §4, expressions for the magnetic moment density as well as the total magnetic moment are given. Finally, in §5 some numerical values for the thermodynamic quantities and the magnetic moment in respect of the degenerate electron gas are presented, followed by concluding remarks.

2. The energy eigenvalues of the Schrödinger and Dirac electrons in the IMF

$$H_z = Hr^{-1}$$

Hautot (1972, 1974) has derived the energy eigenvalues for the nonrelativistic electron (without spin) and the Dirac electron in the IMF, equation (1), generated from the vector potential

$$\mathbf{A} = (-Hy, r^{-1}, H_x r^{-1}, 0). \quad (2)$$

We include spin in the first case and give an alternative derivation in the second case.

2.1 Nonrelativistic case with spin

Using the quaternion formalism, Hautot (1974) has shown that the Schrödinger equation without spin

$$\nabla^2 \psi + \frac{2ei}{c\hbar} \mathbf{A} \cdot \nabla \psi - \frac{e^2}{c^2 \hbar^2} |\mathbf{A}|^2 \psi + \frac{2me}{\hbar^2} E \psi = 0, \quad (3)$$

where $i = \sqrt{-1}$ can be separated in the cylindrical polar coordinates, if and only if the vector potential is of the form

$$\mathbf{A} = [-yg(r)r^{-1}, xg(r)r^{-1}, h(r)], \quad (4)$$

where $g(r)$ and $h(r)$ are any two functions of r . The separated wavefunction looks like

$$\psi = \exp(im\phi) \exp[(i/\hbar) p_z z] u(r), \quad (5)$$

with u obeying the radial equation

$$r^2 u'' + ru' + \left\{ \frac{2m_e E - p_z^2}{\hbar^2} r^2 - m^2 - \frac{2em}{c\hbar} g(r)r - \frac{e^2}{c^2 \hbar^2} [g^2(r) + h^2(r)] r^2 - \frac{2ep_z}{c\hbar^2} r^2 h(r) \right\} u = 0, \quad (6)$$

where m is the usual azimuthal quantum number and the particle is free in the z -direction. He has further shown that the Schrödinger equation can be solved for eigenvalues in two cases:

$$(i) \quad g(r) = Hr/2, \quad h(r) = 0, \tag{7}$$

corresponding to the homogeneous magnetic field

$$\mathbf{B} = (0, 0, H), \tag{8}$$

$$(ii) \quad g(r) = H, \quad h(r) = Kr^{-1} \tag{9}$$

corresponding to the magnetic field

$$\mathbf{B} = (-Kyr^{-3}, Kxr^{-3}, Hr^{-1}). \tag{10}$$

Here H and K are constants.

Putting $K = 0$ we introduce the spin part proportional to $\sigma_z B_z$ into the radial equation (6). The resulting equation for the product of the radial, $[u_s(r)]$ and spin, (χ_s) wave functions can be written explicitly as

$$\left\{ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{\left[2m_e E - p_z^2 - \frac{e^2 H^2}{c^2} \right]}{\hbar^2} - \frac{m^2}{r^2} - \frac{2emH}{\hbar r} - \frac{e\sigma_z B_z}{\hbar} \right\} u_s(r) \chi_s = 0. \tag{11}$$

Since σ_z acting on χ_s gives the eigenvalues ± 1 , which we can abbreviate as s , the above equation can be written, dropping the spin wave function χ_s , as

$$u''(r) + \frac{u'(r)}{r} + \left\{ \frac{\left[2m_e E - p_z^2 - \frac{e^2 H^2}{c^2} \right]}{\hbar^2} - \frac{m^2}{r^2} - \frac{(2m+s)eH}{\hbar r} \right\} u(r) = 0. \tag{12}$$

The transformation $u = (1/\sqrt{r}) v$ followed by the scale transformation $R = 2\sqrt{\mathcal{N}}r$, where

$$\mathcal{N} = \frac{\frac{e^2 H^2}{2} + p_z^2 - 2m_e E}{\hbar^2}, \tag{13}$$

reduces (12) to the Whittaker equation (Magnus *et al* 1966):

$$v''(R) + \left(-\frac{1}{4} + \frac{d}{R} + \frac{\frac{1}{4} - m^2}{R^2} \right) v(r) = 0 \tag{14}$$

with
$$d = -(2m+2) \frac{eH}{\hbar} \cdot \frac{1}{2\sqrt{\mathcal{N}}}. \tag{15}$$

Apart from a normalization factor, the solution of (14) can be written as

$$v(R) = R^{m+1/2} \exp(-R/2) {}_1F_1(a, b; R), \quad (16)$$

where ${}_1F_1$ is the confluent hypergeometric function and

$$a = \frac{1}{2} \cdot \frac{-2d + \frac{2}{\hbar} \left[\frac{e^2 H^2}{c^2} + p_z^2 - 2m_e E(1 + 2m) \right]^{1/2}}{2 \left[\frac{e^2 H^2}{c^2} + p_z^2 - 2m_e E \right]^{1/2}},$$

$$b = 1 + 2m. \quad (17)$$

The eigenvalue condition $a = -n$, where $n = 0, 1, 2, \dots$ gives the eigenvalues explicitly as

$$E_{n,m}^{\text{nonrel}} = \frac{p_z^2 + \frac{e^2 H^2}{c^2}}{2m_e} - \frac{e^2 H^2}{2m_e c^2} \frac{(2m+s)^2}{(2n+2m+1)^2}. \quad (18)$$

Note that the principal quantum number n and the azimuthal quantum number m vary independently of each other in their ranges of validity. All the states are doubly degenerate except the $n = 0$ state as in HMF. The energy eigenvalues given by (18) are used in deriving the density of state factor (equation (31) below).

Before going to the relativistic case we would like to comment on the energy eigenvalues, equation (18), in the context of the possibility of forming a magnetic hydrogen atom, *i.e.* an electron bound in the combined coulomb and magnetic fields, the magnetic binding energy being comparable to or greater than the coulomb binding energy. In an HMF in the z -direction the motion of the electron is in the xy -plane so that only the coulomb field can restrain its motion in the z -direction. The quantized part of the energy eigenvalue in this case, *viz.*,

$$\left(n + \frac{1}{2} \right) \frac{e\hbar H}{m_e c}, \quad n = 0, 1, 2, \dots$$

(see Mathews and Venkatesan 1983), attains the value comparable to the ionization potential 13.6 eV of the normal hydrogen atom only when the field strength H is of the order 10^9 G (if n is of order 1) which is presently unattainable terrestrially. For terrestrial field strengths the magnetic contribution will be negligible compared to the coulomb contribution. So we have the ordinary hydrogen atom perturbed by a magnetic field. Only for fields of strength comparable to 10^9 G can a magnetic hydrogen atom be thought of at all. The motion of electron in our IMF is also free in the z -direction and quantized in the xy plane. If we examine the quantized part

$$\frac{e^2 H^2}{2m_e c^2} \left[1 - \frac{(2m+s)^2}{(2n+2m+1)^2} \right] \quad (19)$$

of (18) we get by taking $n = 0, m = 1, s = -1$ for instance, the value $4e^2 H^2 / 9m_e c^2 \approx 8.7 \times 10^{-2} H^2$ eV. Thus even for a value $H = 10$ G-cm the energy is already comparable to the ground state energy of coulomb hydrogen atom.

2.2 Relativistic case

The Feynman-Gell-Mann (1958) formalism for the Dirac electron in an electro-

magnetic field has been most fruitful in obtaining solutions of the Dirac equation containing static magnetic fields (Stanciu 1967; Vasudevan *et al* 1967; Canuto and Chiuderi 1969; Achuthan and Benjamin 1983).

The two-component equation for our case is given by*

$$\left\{ c^2 \left[\left(p_x + \frac{eA_x}{c} \right)^2 + \left(p_y + \frac{eA_y}{c} \right)^2 + p_z^2 \right] + m_e^2 c^4 + e\hbar \sigma_z B_z \right\} \psi_s \chi_s = W^2 \psi_s \chi_s \tag{20}$$

or equivalently

$$\left\{ c^2 \left[\hbar^2 \nabla^2 + \frac{2e\hbar}{c} \mathbf{A} \cdot \nabla - \frac{e^2}{c^2} (A_x^2 + A_y^2) \right] + W^2 - m^2 c^4 - e\hbar s \frac{H}{r} \right\} \psi_s \chi_s = 0, \tag{21}$$

where the components of vector potentials A_x and A_y are given by (2). Other quantities are as in the nonrelativistic case. This equation again splits up into two Schrödinger-like equations for the two values $s = \pm 1$. We notice that p_z is a constant of motion. The four-component spinor ψ_D which is the solution of the Dirac equation is obtained from the solutions $\psi_s \chi_s$ by the association

$$\psi_D = \begin{pmatrix} \left[c \boldsymbol{\sigma} \cdot \left(\mathbf{p} + \frac{e\mathbf{A}}{c} \right) + W + mc^2 \right] \psi_s \chi_s \\ \left[c \boldsymbol{\sigma} \cdot \left(\mathbf{p} + \frac{e\mathbf{A}}{c} \right) + W - mc^2 \right] \psi_s \chi_s \end{pmatrix}. \tag{22}$$

Now if we add the spin term $-(es/c\hbar)(H/r)$ to the nonrelativistic nonspin equation (3) (of course, now with the wavefunction $\psi_s \chi_s$), multiply it by $c^2 \hbar^2$ and make a term-by-term comparison with (21), we find that the two equations are identical, if the eigenvalue W is connected with the eigenvalue E through

$$W^2 - m^2 c^4 = 2m_e c^2 E. \tag{23}$$

Hence from (18) follows the relativistic energy eigenvalue

$$W^2(n, m, s) = m_e^2 c^4 + c^2 p_z^2 + e^2 H^2 \left[1 - \frac{(2m + s)^2}{(2n + 2m + 1)^2} \right]. \tag{24}$$

A comparison with the eigenvalues for the electron in the same field given by Hautot (1972) shows that the two results agree for $s = +1$.

The fact that Hautot's energy eigenvalues correspond to the $s = +1$ case is due to his choice of a part of the ϕ -dependent eigenfunction as $\exp[-\sqrt{1} (m + 1/2)\phi]$ (see Hautot 1972 equation (2)). This in turn was necessitated by single-valuedness of the wavefunction (Hautot 1970). We notice that the choice of $m - \frac{1}{2}$ instead of $m + \frac{1}{2}$ in the exponent of the above function will still preserve single-valuedness and with this change we recover the missing eigenvalues corresponding to our $s = -1$ case.

* We use the symbol W for the relativistic energy eigenvalues and E for the nonrelativistic values for future convenience.

Here again we make an aside with regard to the eigenvalues (24) before employing them and the nonrelativistic eigenvalues (18) for the main purpose of this paper, namely, the study of the thermodynamic and magnetic properties of the electron gas in the magnetic field Hr^{-1} . For the choice $p_z = 0, n = 1, m = -1, s = -1$ we see from (24) that the energy vanishes for a field strength $H = m_e c^2 / (2 \sqrt{2} e) \simeq 1700$ G-cm. This may be contrasted with the value $H = 10^{16}$ G for the phenomenon to happen in the IMF, $H_z = H \operatorname{sech}^2(ay)$ (Achuthan *et al* 1979) and its utter impossibility in an HMF with or without an additional anomalous magnetic moment term (Jancovici 1969, 1969a).

3. Thermodynamic properties

The expressions for the pressures P_x, P_y, P_z and the energy density of an electron gas in a magnetic field represented by the vector potential \mathbf{A} can be derived explicitly from quantum mechanical and statistical averaging of the xx, yy, zz and 00 components of the matter part of the gauge invariant energy momentum tensor (Korneev and Starostive 1973)

$$T_{\mu\nu} = \bar{\psi} (\gamma_\nu \partial_\mu - ie \gamma_\nu A_\mu) \psi. \tag{25}$$

Here ψ 's are the Dirac fields, which are the solutions of the Dirac equation in the presence of the magnetic field, γ_ν 's are the usual γ -matrices and A_μ is the electromagnetic potential of which we are interested only in the space part \mathbf{A} . The number density is obtained by a similar averaging over the number operator (Canuto and Chiu 1968). But here we follow a simpler procedure due to Canuto *et al* (1969) for deriving these quantities. In this procedure the pressures P_x, P_y and P_z are defined by

$$P_{x, y, z} = \langle m v_{x, y, z} v_{x, y, z} \rangle \tag{26}$$

in which the bracket stands for a statistical average over the \mathcal{N}^2 -particle system. We can identify the corresponding expressions in the bracket of (26) using the energy eigenvalues (24). Explicitly

$$\begin{aligned} P_x = P_y &= \left\langle \frac{c^2 p_{x,y}^2}{W(n, m, s)} \right\rangle \\ &= \frac{1}{2} e^2 H^2 \left\langle \frac{4n\beta}{\gamma^2 W(n, m, s)} \right\rangle \\ &= \frac{1}{2\pi\hbar} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \omega_j \int_{-\infty}^{\infty} \frac{dp_z}{W(n, m, s)} \left[\frac{1/2 e^2 H^2 4n\beta}{2} \right] F \end{aligned} \tag{27}$$

and

$$\begin{aligned} P_z &= \left\langle \frac{c^2 p_z^2}{W(n, m, s)} \right\rangle \\ &= \frac{1}{2\pi\hbar} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \omega_j \int_{-\infty}^{\infty} \frac{dp_z}{W(n, m, s)} c^2 p_z^2 F. \end{aligned} \tag{28}$$

Further the energy

$$U = \frac{1}{2\pi\hbar} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \omega_j \int_{-\infty}^{\infty} W(n, m, s) dp_z F \tag{29}$$

and the particle number

$$N = \frac{1}{2\pi\hbar} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \omega_j \int_{-\infty}^{\infty} dp_z F \tag{30}$$

In the above equations ω_j represents the density of state factor which, using the normal procedure (Huang 1963), can be computed to be

$$\omega_j = \iint dx dy \frac{2e^2 H^2 \alpha (2m+s)^2}{\pi \hbar^2 c^2 \gamma^2 \delta^2} \tag{31}$$

and F is the Fermi function. In (27) through (31), $j =$ the set of quantum numbers (n, m, s) ; $\alpha = n + m + 1$; $\beta = n + 2m + 1$; $\gamma = 2n + 2m + 1$; $\delta = 2n + 2m + 3$.

In the above expressions, corresponding to each n value, the summation over m extends from $-\infty$ to $+\infty$. We introduce two physical requirements which would reduce this range: (i) the lower limit of the summation is restricted by the requirement that ω_j is positive definite, *i.e.*, $n + m + 1$ is positive definite. From this condition we see that the lowest nontrivial value of m is $-n$; (ii) the pressure components should be nonnegative, which constraint leads from the definition of P_x to the minimum value $-\frac{1}{2}[n + 1]$ for m , where $[x]$ denotes the highest integer $\leq x$.

For the degenerate electron gas, for which $F = 1$, $\mu > W(n, m, s)$, and $T \rightarrow 0$, the p_z -integration is restricted to the finite interval, $-p_z^{\text{fermi}}$ to p_z^{fermi} , where

$$p_z^{\text{fermi}} = m_e c \left\{ \tilde{\mu}^2 - 1 - \frac{e^2 H^2}{m_e^2 c^4} \left[1 - \frac{(2m+s)^2}{\gamma^2} \right] \right\}^{\frac{1}{2}}, \tag{32}$$

with the chemical potential $\tilde{\mu} = \mu/m_e c^2$.

The integration in (27)–(31) can be performed and the final expressions are

$$P_x = P_y = \frac{V e^4 H^4}{n_0 m^3 c^6} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\frac{1}{2}[n+1]}^{\infty} \frac{4n(2m+s)^2}{\gamma^4 \delta^2} \ln \left[\frac{\tilde{\mu} + K}{\tilde{\mu} - K} \right] \tag{33}$$

where

$$K m_e c^2 = [(\tilde{\mu}^2 - 1)m_e^2 c^4 \gamma^2 - 4n\beta e^2 H^2 - 4m(1-s)e^2 H^2]^{\frac{1}{2}} \tag{34}$$

and

$$P_z = \frac{V 2e^2 H^2}{n_0 m_e c^2} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\frac{1}{2}[n+1]}^{\infty} \frac{(2m+s)^2 \alpha}{\gamma^2 \delta^2} \left\{ \tilde{\mu} K - \frac{(\tilde{\mu}^2 - K^2)}{2} \ln \left[\frac{\tilde{\mu} + K}{\tilde{\mu} - K} \right] \right\} \tag{35}$$

$$U = \text{expression for } P_z \text{ above with } -(\tilde{\mu}^2 - K^2)/2 \text{ replaced by } +(\tilde{\mu}^2 - K^2)/2. \tag{36}$$

$$N = \frac{V 2e^2 H^2}{n_0 m_e^2 c^4} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\frac{1}{2}[n+1]}^{\infty} (2K) \tag{37}$$

In the above $n_0 = 2\pi^2 \tilde{\lambda}^3$, where $\tilde{\lambda} (= \hbar/m_e c)$ is the electron Compton wavelength and $V =$ total volume of the gas.

As can be seen from (33), both P_x and P_y vanish in the extreme quantum limit, *i.e.*, when $n = 0$. This feature of the system becoming a one-dimensional gas is common to the HMF also in the degenerate limit (Canuto and Chiu 1968a).

4. Magnetic moment

The thermodynamic potential from which the magnetic moment of the electron gas is derived is defined by (Landau and Lifshitz 1975)

$$\Omega = -kT \sum_j \log \left[1 + \exp \frac{(\mu - W_j)}{kT} \right], \quad (38)$$

where j refers to the discrete quantum numbers n , m and s and the summation includes an integration over P_z , k is the Boltzmann constant and T the absolute temperature, μ the chemical potential plus electron rest mass energy. The summand in Ω includes the density of states factor ω_j introduced in the previous section.

The magnetic moment is defined in terms of the thermodynamic potential, as usual, by

$$M_z = -\partial\Omega/\partial H_z. \quad (39)$$

As noted earlier (Achuthan *et al* 1982), for the inhomogeneous case this definition cannot be carried over as such; we must instead have the notion of the thermodynamic potential density

$$\sigma(x, y, z) = -I_z(x, y, z) H_z(x, y, z), \quad (40)$$

where the magnetic moment density I_z is given by

$$I_z(x, y, z) = \frac{-\partial\sigma(x, y, z)}{\partial H_z(x, y, z)}. \quad (41)$$

The total magnetic moment M of the gas is obviously given by

$$M = \int_V I_z(x, y, z) dV. \quad (42)$$

Using (24), (31) and (38), we get for the magnetic moment density I_z for the non-degenerate gas

$$\begin{aligned} I_z(x, y) &= \frac{kT}{2\pi\hbar} \int_{-\infty}^{\infty} dp_z \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\frac{1}{2}[n+1]}^{\infty} \\ &\left\{ \omega_j \frac{\exp(\mu - W(n, m, s))}{1 + \exp(\mu - W(n, m, s))} \frac{-1/kT}{W(n, m, s)} \frac{2He^2}{m_e^2 c^4} \frac{4n\beta}{\gamma^2} \right. \\ &\quad \left. + \log[1 + \exp(\mu - W(n, m, s))] \right. \\ &\quad \left. \frac{4e^2 H}{\pi \hbar^2 c^2} \frac{(2m+s)^2}{\gamma^2 \delta^2} \right\} (x^2 + y^2)^{1/2}. \end{aligned} \quad (43)$$

In the degeneracy limit this expression becomes

$$\begin{aligned} I_z(xy) &= \frac{1}{n_0 H_z(x, y)} \frac{1}{m_e^2 c^4} \sum_{s=-1, +1} \sum_{n=0}^{\infty} \sum_{m=-\frac{1}{2}[n+1]}^{\infty} \\ &\left\{ \frac{4m_e c^2 e^2 H^2 (2m+s)^2}{\gamma^2 \delta^2} \left[\bar{\mu} K - \frac{(\bar{\mu}^2 - K^2)}{2} \log \left[\frac{\bar{\mu} + K}{\bar{\mu} - K} \right] \right] \right. \\ &\quad \left. - \frac{2e^4 H^4}{m_e c^2} \frac{(2m+s)^2 4n\alpha\beta}{\gamma^4 \delta^2} \log \left[\frac{\bar{\mu} + K}{\bar{\mu} - K} \right] \right\} (x^2 + y^2)^{1/2}. \end{aligned} \quad (44)$$

The magnetic moment density I_z has the correct dimension of energy density per unit field strength H_z , which follows from eH having the dimension of energy and n_0 that of volume. The total magnetic moment M is given from its definition, equation (42), by

$$M = \frac{2}{3}(x^2 + y^2) L_z I_z, \tag{45}$$

where L_z is the range of z -integration.

We end this section by giving the expression for I_z in the quantum limit, $n = 0$ and also $m = 0$:

$$I_z = \frac{4}{9}(x^2 + y^2)^{\frac{1}{2}} \left\{ \bar{\mu} K - \frac{1}{2} \log \left[\frac{\bar{\mu} + K}{\bar{\mu} - K} \right] \right\} \tag{46}$$

with $K = (\bar{\mu}^2 - 1)^{\frac{1}{2}}$ and $1 < \bar{\mu} < \left[1 + \frac{8 e^2 H^2}{9 m_e^2 c^4} \right]^{\frac{1}{2}}$. (47)

5. Numerical results

In this section we present the numerical values of the quantities P_z , U , N and I_z for certain ranges of values of the chemical potential $\bar{\mu}$ (table 1). In the ranges chosen, P_x and P_y turned out to be almost zero making the electron gas effectively one-dimensional even for $n > 0$ values. The H value was taken, by way of illustration, as 1.5 kG-cm corresponding to a field strength $H_z = 1.5 \times 10^{11}$ G at atomic distances. All the quantities are rounded off to integers and are expressed in units of 10^{-7}

As discussed in Achuthan *et al* (1982) the possibility of spontaneous magnetisation arises if the induced magnetic moment is greater than or equal to the field producing it, *i.e.*, $I_z(x, y) \geq H_z(x, y)$. That this can happen in our case is evident from the values given in the last column of table 1. For instance, for $\bar{\mu} = 5$ we have

$$I_z \simeq 10^{22} (x^2 + y^2)^{1/2} \simeq 10^{19} (x^2 + y^2) H_z \tag{48}$$

the second step following from our choice of H . Here we have in addition to an intrinsic factor of 10^{19} between the magnetic moment density and field strength, also a purely space-dependent factor $(x^2 + y^2)$ which can further increase the I_z value. It may be pointed out that this space factor will not increase indefinitely because of the finite size of the systems of interest.

In the quantum limit equation (47), the numerical I_z value will be of the order $10^{20}(x^2 + y^2)^{1/2}$ which shows that up to the occupied levels n , corresponding to the

Table 1. Numerical values in units of 10^{-7} (cgs) for the thermodynamic quantities and magnetic moment density for the electron gas in the magnetic field $H_z = Hr^{-1}$ with $H = 1.5$ kG-cm

Chemical potential $\bar{\mu}$	$P_z n_0 / V$	$N n_0 / V$	$U n_0 / V$	$I_z H_z n_0$
2	61	9	81	122
3	116	10	135	231
4	188	13	219	377
5	285	15	318	570

chemical potential $\bar{\mu} = 5$, there is only a paramagnetic contribution and hence the magnetic moment increases.

For comparison it may be noted that for the electron gas in the HMF the relation between the magnetic moment and field is $M = 10^{-3} H$ where $H \simeq 10^{13}$ G (Canuto and Chiu 1968b). This clearly indicates that for HMF there is no possibility of spontaneous magnetisation, which, however, becomes possible if there is, in addition, a crossed homogeneous electric field E , but then only for the values $E/H = 0.9$, $H = 2.21 \times 10^{13}$ G and $\bar{\mu} = 5$ (Achuthan *et al* 1983), (which is again only an extra-terrestrial possibility).

A similar comparison can be made for the values of the z -component of pressure P_z . For $\bar{\mu} = 5$, for example, column 2 of table 1 gives a value $\simeq 3 \times 10^{-5}$ for the quantity $p_z n_0 / V$ whereas the corresponding values for the homogeneous and the CHEM field cases (see table 1 of Achuthan *et al* 1984) are of the order 10^{-6} for $H \simeq 10^{13}$ G.

The present results suggest that a study of other properties of the charged particles in this magnetic field is likely to be rewarding and this is being done.

Acknowledgement

The authors thank the referee for valuable comments and suggestions.

References

- Achuthan P, Chandramohan T and Venkatesan K 1979 *J. Phys. A* **12** 2521
 Achuthan P, Benjamin S and Venkatesan K 1982 *J. Phys. A* **15** 3607
 Achuthan P, Benjamin S and Venkatesan K 1983 *J. Phys. A* **16** 1307
 Achuthan P and Benjamin S 1983 *Lett. Nuovo Cimento* **36** 417
 Achuthan P, Benjamin S and Venkatesan K 1984 *J. Math. Phys.* **18** S169
 Canuto V and Chiu H Y 1968 *Phys. Rev.* **173** 1210
 Canuto V and Chiu H Y 1968a *Phys. Rev.* **173** 1220
 Canuto V and Chiu H Y 1968b *Phys. Rev.* **173** 1229
 Canuto V, Chiu H Y and Canuto L F 1969 *Astr. Sp. Sci.* **3** 258
 Canuto V and Chiuderi C 1969 *Lett. Nuovo Cimento* **11** 223
 Canuto V and Kelly D C 1972 *Astr. Sp. Sci.* **17** 277
 Canuto V and Tsiang E 1976 *Phys. Lett.* **B62** 366
 Feynman R P and Gell-Mann M 1958 *Phys. Rev.* **109** 193
 Hautot A 1970 *Physica* **48** 609
 Hautot A 1972 *J. Math. Phys.* **13** 710
 Hautot A 1974 *Bull. Soc. R. Sci. Liege* **43** 348
 Huang K 1963 *Statistical mechanics* (New York: Wiley)
 Jancovici B 1969 *Phys. Rev.* **187** 2275
 Jancovici B 1969a *J. Phys.* **30** C3
 Korneev V V and Starostive A N 1973 *Sov. Phys. JETP* **36** 487
 Landau L D and Lifshitz E M 1975 *The classical theory of fields* (Oxford: Pergamon Press)
 Magnus W, Oberhettinger F and Soni R P 1966 *Formulas and theorems for the special functions of mathematical physics* (Berlin: Springer-Verlag) p 296
 Mathews P M and Venkatesan K 1983 *Quantum mechanics* (New Delhi: McGraw-Hill Co.) (equation (4.212)) p 145
 Ruderman M 1974 in *Physics of dense matter*, IAO, p 177
 Ruderman M 1977 in *Statistical mechanics and statistical methods in theory and applications* (ed) V Landman (New York: Plenum Press)
 Stanciu G N 1967 *J. Math. Phys.* **8** 2043
 Vasudevan R, Venkatesan K and Ranganathan N R 1967 *J. Math. Phys. Sci.* **1** 159