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Factorisation in large-N limit of lattice gauge theories revisited

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Abstract. We prove using the Schwinger-Dyson equations that the factorisation property holds for all gauge-invariant Green's function in the large-N limit of a Wilson-Polyakov lattice gauge theory.

Keywords. Large-N limit; factorisation; Schwinger-Dyson equations; lattice gauge theory.

1. Introduction

The dynamics of a gauge theory simplifies to a great extent in the limit of large number of gauge degrees of freedom. This simplification leads to some of the recent remarkable results like the closed form of equations for the Wilson loop average (Makeenko and Migdal 1979; Wadia 1981) and the reduction in the number of effective space-time degrees of freedom (Eguchi and Kawai 1982; Gross and Kitazawa 1982; Das and Wadia 1982; Parisi 1982; Bhanot *et al* 1982).

As in spin models where it results in complete solution, this simplification in gauge theories is ultimately related to the factorisation property exhibited by the gauge-invariant Green's functions. This implies that the fluctuations in the gauge-invariant observables become negligible and the Hartree-Fock approximation is exact.

However, this factorisation property though a *fait accompli*, we feel, is not all that obvious. In the present paper we make a modest investigation of the factorisation property in the $U(N)$ Wilson-Polyakov lattice gauge theories based on the study of the Schwinger-Dyson equations and prove that it indeed is satisfied by all gauge invariant Green's functions in the large-N limit.

In § 2 we introduce the loop space Green's functions. In § 3 we review the derivation of Schwinger-Dyson (SD) equation and write down its general form for $U(N)$ lattice gauge theory. In § 4 we discuss the single loop Green's function and write down the corresponding SD equation. In § 5 we derive the SD equation satisfied by the multiloop Green's functions and prove the factorisation property.

2. Loop space Green's functions

The gauge-invariant physical content of a lattice gauge theory is described by the multiloop Green's functions

$$W(C_1, C_2, \ldots) = \left\langle \frac{1}{N} \mathrm{Tr}\, U(C_1) \frac{1}{N} \mathrm{Tr}\, U(C_2) \ldots \right\rangle, \tag{1}$$

where $C_1, C_2, \ldots$ are closed contours on the lattice and

$$U(C) = \prod_{l \in c} U_l, \tag{2}$$

with the product being path-ordered and U_l the $U(N)$ (or $SU(N)$) matrix in the fundamental representation corresponding to the link l. N is the number of internal (" colour ") degrees of freedom.

In particular for a single closed loop, the Wilson loop average

$$W(C) = \left\langle \frac{1}{N} \operatorname{Tr} U(C) \right\rangle,$$

determines the static inter-quark potential.

The aim of the present paper is to show that in the limit $N \to \infty$, with $g^2 N$ fixed where g is a coupling constant, the multiloop Green's functions defined above (equation (1)) factorise *i.e.*:

$$W(C_1, C_2, \ldots) \simeq W(C_1)\, W(C_2) \ldots + O(N^{-2}). \tag{3}$$

Thus, for $N \to \infty$ we have a class of non-fluctuating operators characterising the configuration space of the theory. One then has the appealing interpretation: large N gauge theory is a string theory where the relevant operators are the non-fluctuating loop operators that create bare strings of (" colour ") flux from the vacuum.

3. Schwinger-Dyson equations

We first briefly review the derivation of the SD equations for $U(N)$ gauge theory defined on a hypercubical lattice. The action is

$$S = \frac{1}{g^2} \sum_p \operatorname{Tr} U(p), \tag{4}$$

with the sum over all oriented plaquettes p; g is the coupling and the plaquette variable

$$U(p) = \prod_{l \in p} U_l, \tag{5}$$

where U_l is the $U(N)$ matrix in the fundamental representation ($N \times N$) corresponding to the link l. Explicitly

$$U_l = U_{x,\mu} = U^{-1}_{x+\hat{\mu},\,-\mu} = U^{+}_{x+\hat{\mu},\,-\mu}, \tag{6}$$

x denoting a site and μ the direction.

The average of any function f of the U's is defined by

$$\langle f \rangle = Z^{-1} \int \prod_l \mathrm{d}U_l \, e^S f, \tag{7}$$

where
$$Z = \int \prod_l \mathrm{d}U_l \, e^S, \tag{8}$$

is the partition function and $\mathrm{d}U_l$ is the invariant Haar measure over $U(N)$.

We now make an infinitesimal transformation:

$$U_{x,\mu} \to \exp(i\epsilon T_j)\, U_{x,\mu} = (1 + i\epsilon T_j)\, U_{x,\mu} + O(\epsilon^2), \tag{9}$$

in the numerator of (7) and compute the corresponding change in $\langle f \rangle$. This being a unitary transformation the measure $\mathrm{d}U_{x,\mu}$ is invariant and f changes according to

$$f \to f + \epsilon\, D^{x,\mu} f + O(\epsilon^2), \tag{10}$$

where the linear differential operator $D_j^{x,\mu}$ at the link (x, μ) is defined by

$$D_j^{x,\mu} f = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [f(\ldots, (1 + i\,\epsilon\, T)\, U_{x,\mu}, \ldots) - f(\ldots, U_{x,\mu}, \ldots)], \tag{11}$$

T_j's are hermitian generators of $U(N)$ in the fundamental representation normalised according to

$$\sum_j (T_j)_{ab} (T_j)_{cd} = \delta_{ad}\, \delta_{bc}. \tag{12}$$

Equating the first order change in $\langle f \rangle$ to zero we obtain

$$\langle D_j^{x,\mu} f \rangle + \langle f (D_j^{x,\mu} S) \rangle = 0. \tag{13}$$

Now replacing f by f_j and summing over the possible values of j corresponding to the mutually orthogonal directions of a local tangent plane in the group manifold we get

$$\sum_j \langle D_j^{x,\mu} f_j \rangle + \sum_j \langle f_j (D^{x,\mu} S) \rangle = 0. \tag{14}$$

The action S according to (4) is given by a sum over oriented plaquettes and the differential operator $D_j^{x,\mu}$ acts only on the plaquettes containing either the link (x, μ) or the oppositely oriented link $(x + \hat{\mu}, -\mu)$.

Thus

$$D_j^{x,\mu} S = \frac{1}{g^2} \Big\{ \sum_{\substack{p \\ (x,\mu) \in \partial p}} \mathrm{Tr}\, \{i\, T_j\, U(p)\} - \sum_{\substack{p \\ (x,\mu) \in \partial p^{-1}}} \mathrm{Tr}\, \{i\, T_j\, U(p)\} \Big\},$$

where the first (second) sum is over all plaquettes p such that the link (x, μ) is contained in the boundary ∂p of p (the boundary ∂p^{-1} of the oppositely oriented plaquette p^{-1}).

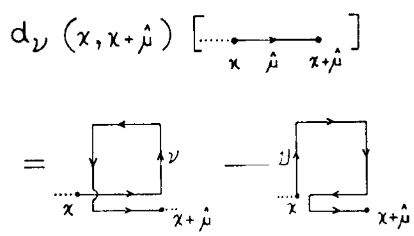
Finally from (14)

$$\sum_j \langle D_j^{x,\mu} f_j \rangle + \frac{1}{g^2} \sum_j \Big\{ \sum_{\substack{p \\ (x,\mu)\in\partial p}} \langle f_j \operatorname{Tr}(i T_j U(p))\rangle - \sum_{\substack{p \\ (x,\mu)\in\partial p^{-1}}} \langle f_j \operatorname{Tr}(i T_j U(p))\rangle \Big\} = 0. \tag{15}$$

This is the general form of the SD equations. In what follows we apply it to the case of single loop as well as multiloop Greens' functions.

4. Single loop Green function

In writing down the SD equations the local variations of the contour are of paramount importance as stressed by Makeenko and Migdal (1979) and Wadia (1981). In this context we introduce the loop deformation operator (Wadia 1981) for the unitary group in loop space, in the $\nu \neq \mu$ direction at the link $(x, x+\hat{\mu})$ as:



Defining

$$\sum_{\nu \neq \mu} d_\nu (x, x+\hat{\mu}) \equiv d(x, x+\mu),$$

one has

$$\mathrm{d}(x, x+\mu)\, U(C) = \sum_{\substack{p \\ (x,\mu)\in\partial p}} U(C \cup \partial p) - \sum_{\substack{p \\ (x,\mu)\in\partial p^{-1}}} U(C \cup \partial p), \tag{16}$$

where $C \cup \partial p$ is the loop obtained by adding the boundary ∂p of the plaquette p to C.

The SD equation corresponding to a single loop is obtained from (15) by making the choice

$$f_j = \frac{1}{N} \operatorname{Tr}[i T_j U(C)],$$

and using the identities

$$\mathrm{Tr}\, \mathrm{I} = N, \tag{17}$$

$$\sum_j \mathrm{Tr}\,(T_j A)\,\mathrm{Tr}\,(T_j B) = \mathrm{Tr}\,(AB), \tag{18}$$

$$\sum_j \mathrm{Tr}\,(T_j A\ T_j B) = \mathrm{Tr}\, A\ \mathrm{Tr}\, B, \tag{19}$$

which follow from (12). Since this equation is already available in the literature (Foerster 1979; Eguchi 1979; Weingarten 1979; Wadia 1981) we omit details and write down its final form

$$\frac{1}{g^2 N} \mathrm{d}\,(x, x+\mu)\, W(C) + \sum_{(y,\nu)\in C} [\delta\,(x, x+\mu \,|\, y, y+\nu)\, W(C_{xy}, C_{yx}) - \delta\,(x, x+\mu \,|\, y+\nu, y) \times W(C_{x,y+\nu}, C_{y+\nu,x})] = 0. \tag{20}$$

Here C_{xy} represents part of the loop C from the point x to the point y. The first δ-function always contributes when $C_{xy} = C$. It also contributes when there is multiple traversal of the link $(x, x+\hat{\mu})$ in the same direction leading to string rearrangement. The second δ-function contributes only when the link $(x, x+\hat{\mu})$ is multiply-traversed in opposite directions and leads to string splitting.

5. Multiloop Green functions

We first consider the case of two loops C_1 and C_2 and make the choice

$$f_j = \frac{1}{N} \mathrm{Tr}\,[iT_j\, U(C_1)]\, \frac{1}{N} \mathrm{Tr}\, U(C_2)$$

in (15).

If C_1 and C_2 are completely disjoint *i.e.* have no link in common the SD equation is obtained by simply replacing $W(C)$ in (20) by $W(C_1, C_2)$.

Next we consider the case where the loops have a single link $(x, x+\hat{\mu})$ in common and the common link is traversed only once in both C_1 and C_2.

In this case a new term appears in the SD equation because the operator $D_j^{x,\mu}$ can now act on C_2 producing a new term in $D_j^{x,\mu} f_j$:

$$\frac{1}{N} \mathrm{Tr}\,[iT_j\, U(C_1)]\, \frac{1}{N} \mathrm{Tr}\,[iT_j\, U(C_2)] = -\frac{1}{N^2} \mathrm{Tr}\, U(C_1 \cup C_2),$$

where we have used the identity (18). Thus the SD equation in this case is

$$\frac{1}{g^2N}\Big[\sum_{\substack{p\\(x,\mu)\in\partial p}} W(C_1\cup\partial p, C_2) - \sum_{\substack{p\\(x,\mu)\in\partial p^{-1}}} W(C_1\cup\partial p, C_2)\Big]$$

$$+ W(C_1, C_2) - \frac{1}{N^2} W(C_1\cup C_2) = 0. \tag{21}$$

If the common link $(x, x+\hat{\mu})$ is traversed more than once in one of the loops say C_1, then we get the usual string rearrangement and splitting terms and the SD equation becomes

$$\frac{1}{g^2N} W[d(x, x+\mu)\, C_1, C_2]$$

$$+ \sum_{(y,\nu)\in C_1} [\delta(x, x+\mu\,|\, y, y+\nu)\, W(C^1_{xy}, C^1_{yx}; C_2)$$

$$- \delta(x, x+\mu\,|\, y+\nu, y)\, W(C^1_{x,\, y+\nu}, C^1_{y+\nu,x}; C_2)]$$

$$- \frac{1}{N^2} W(C_1\cup C_2) = 0, \tag{22}$$

where

$$d(x, x+\mu)\, C_1 = \sum_{\substack{p\\(x,\mu)\in\partial p}} (C_1\cup\partial p) - \sum_{\substack{p\\(x,\mu)\in\partial p^{-1}}} (C_1\cup\partial p),$$

gives the effect of loop deformation.

If in the second loop C_2, the common link $(x, x+\hat{\mu})$ occurs more than once then C_1 may be connected to it in various ways and $W(C_1\cup C_2)$ would be replaced by an algebraic sum over the various possibilities.

The SD equation for a general n-loop Green function is obtained by an obvious generalisation of the process outlined in the two loop case and some straightforward algebra. Thus we take

$$f_j = \frac{1}{N}\mathrm{Tr}\,[i\, T_j\, U(C_1)]\, \frac{1}{N}\mathrm{Tr}\, U(C_2) \ldots \frac{1}{N}\mathrm{Tr}\, U(C_n)$$

in (15) and the SD equation becomes

$$\frac{1}{g^2 N} W[\mathrm{d}(x, x+\mu)\, C_1, C_2, \ldots, C_n]$$

$$+ \sum_{(y,\nu)\in C_1} [\delta(x, x+\mu\,|\, y, y+\nu)\, W(C^1_{xy}, C^1_{yx}; C_2; \ldots, C_n)$$

$$- \delta (x, x+\mu \,|\, y+\nu, y)\, W(C^1_{x,y+\nu}, C^1_{y+\nu,x};\, C_2, \ldots\, C_n)]$$

$$-\frac{1}{N^2}\Big\{ \sum_{\substack{k \neq 1 \\ (x,\mu) \in C_k}} W(C_1 \cup C_k, C_2, \ldots, C_{k-1}, C_{k+1}, \ldots\, C_n)$$

$$- \sum_{\substack{k \neq 1 \\ (x,\mu) \in C_k^{-1}}} W(C_1 \cup C_k, C_2, \ldots, C_{k-1}, C_{k+1}, \ldots\, C_n)\Big\} = 0. \quad (23)$$

We are now ready to prove the factorisation property stated in (3).

We note that because of (20)

$$W(C_1, C_2, \ldots, C_n) = W(C_1)\, W(C_2) \ldots\, W(C_n)$$

satisfies (23) with the last $O(N^{-2})$ term on the left side omitted. However this $O(N^{-2})$ term is negligible in the limit $N \to \infty$ with g^2N fixed. Thus assuming the uniqueness of the solution of the SD equation (23) we immediately arrive at the factorisation property (3).

6. Conclusion

In this paper we have proved that the factorisation property holds for all gauge invariant Green's functions in the large-N limit of a Wilson-Polyakov lattive gauge theory. Even though our discussions are based on the gauge group $U(N)$ it may be easily extended to other gauge groups like $SU(N)$ and $O(N)$. In our proof the lattice with its•intrinsic short distance cut-off provides a convenient framework and we believe that the same method would work in the continuum case as well, though some of the steps may require careful definition.

Before we end, we hasten to add that the correctness of our proof depends of course on the validity of the currently accepted philosophy that the Schwinger-Dyson equations describe a theory completely.

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