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Quantum chromodynamics predictions in renormalization scheme invariant perturbation theory

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Abstract. It has recently been shown that any physical quantity $\mathscr{R}$, in perturbation theory, can be obtained as a function of only the renormalization scheme (RS) invariants, $\rho_0, \rho_1, \rho_2, \ldots$ Physical predictions, to any given order, are renormalization scheme independent in this approach. Quantum chromodynamics (QCD) predictions to second order, within this RS-invariant perturbation theory, are given here for several processes. These lead to some novel relations between experimentally measurable quantities, which do not involve the unknown QCD scale parameter Λ. They can therefore be directly confronted with experiments and this has been done wherever possible. It is suggested that these relations can be used to probe the neglected higher order corrections.

Keywords. Renormalization scheme independence; perturbation theory; quantum chromodynamics.

1. Introduction

In the conventional perturbative approach to any quantum field theory, finite order predictions for physical quantities depend on the renormalization scheme (RS) used to define the renormalized parameters of the theory. This RS-dependence of the perturbative predictions is especially severe in quantum chromodynamics (QCD) where the running coupling is not so small at the currently available energies. The various schemes used (Buras 1981) give substantially different higher order corrections. The difficulty in meaningfully comparing such predictions with experiments is obvious and has been widely discussed in the literature (Buras 1981). At a deeper level, one cannot even begin to ask questions about the convergence of the perturbation series before resolving this RS-ambiguity.

The usual approach to the problem of the RS-dependence in QCD has been to look for an expansion parameter in terms of which the higher order corrections to several processes are small. This approach has been questioned by Stevenson (1981) according to whom the best scheme to use for a given physical quantity is the one in which a perturbative approximant to it is the least sensitive to small changes in the RS. The point being that in this 'optimal' scheme the perturbative approximant has to some extent the property which the actual physical quantity is known to have, namely, RS-independence. However, Stevenson's criterion for determining the 'optimal' scheme does not make *full* use of the RS-independence of the physical quantities.

The RS-dependence problem has recently been solved (Dhar 1982, hereafter referred to as I) by making use of the full renormalization group invariance of the physical

quantities. It has been shown that, for a massless field theory with a single coupling constant, any physical quantity R in perturbation theory is determined as a function of only the RS-invariants, ρ_0, ρ_1, ρ_2, ... (see § 2 for definition), through the transcendental equation

$$\rho_0 = \frac{1}{R} - \rho_1 \ln\left(1 + \frac{1}{\rho_1 R}\right) + \int_0^R dx \left(\frac{1}{x^2(1+\rho_1 x)} + \frac{1}{\rho(x)}\right), \tag{1}$$

$$\rho(x) = -x^2(1 + \rho_1 x + \rho_2 x^2 + \ldots). \tag{2}$$

The purpose of this paper is to cast QCD predictions for a number of processes in this RS-invariant form and to confront them with experiments wherever possible. In § 2 we first give a brief summary of I and establish the notation. This section also includes a general discussion of (1). In § 3 general tests, based on the second order approximation to (1) are given. These tests remarkably involve only experimental quantities and can be directly confronted with the data. Theoretical estimates of the third order corrections are discussed in § 4. In § 5 second order RS-invariant QCD predictions for several physical processes are given and confronted with the data wherever possible. Finally, § 6 contains a discussion and summary of the results.

2. Renormalization scheme invariant perturbation theory (RESIPE)

Consider a physical quantity $\mathcal{R}$, which has the perturbation expansion

$$\mathcal{R} = a^p \sum_{n=0}^{\infty} R_n a^n, \quad R_0 = 1. \tag{3}$$

Here $a = a(\mu) \equiv \alpha(\mu)/\pi$ is the couplant ($\alpha(\mu)$ being the renormalized coupling constant) and μ is the renormalization point. The power p can be non-integral or even negative. The couplant a satisfies the β-function equation

$$\mu \frac{\partial a}{\partial \mu} = \beta(a) = -ba^2 \sum_{n=0}^{\infty} c_n a^n, \quad c_0 = 1. \tag{4}$$

The first two coefficients b and c_1 are RS-invariant. Their values in QCD are $b = (33 - 2f)/6$, $c_1 = (153 - 19f)/2(33 - 2f)$, where f is the number of fermion flavours. The other coefficients $c_n (n \geqslant 2)$ are RS-dependent and only c_2 has been calculated (Tarasov *et al* 1980). Its value in the $\overline{\text{MS}}$ scheme is $(c_2)_{\overline{\text{MS}}} = 3(2857/2 - 5033f/18 + 325f^2/54)/16(33 - 2f)$. The RS in which a and the coefficients R_n are evaluated in (3) can be labelled by the set of parameters $\{\tau \equiv b \ln \mu/\Lambda, c_2, c_3, \ldots\}$ where Λ is the scale parameter which determines the boundary condition for the solution to (4).

The derivation of (1) proceeds as follows. Corresponding to $\mathcal{R}$ we first construct the physical quantity $(\mathcal{R})^{1/p}$ since it is this which satisfies (1). We write

$$(\mathcal{R})^{1/p} \equiv R = a \sum_{n=0}^{\infty} r_n a^n, \tag{5}$$

where
$$r_0 = 1;\ r_n = \sum_{m=0}^{n-1} \frac{1}{(n-m)!} \frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right).\ldots.$$

$$\left(\frac{1}{p} - n + m + 1\right)\tilde{F}_m^{(n-m)}, \quad n \geqslant 1; \tag{6}$$

and $\tilde{F}$'s satisfy the recursion relation

$$\tilde{F}_l^{(n+1)} = \sum_{m=0}^{l} R_{m+1}\, \tilde{F}_{l-m}^{(n)},\ n \geqslant 1; \tag{7}$$

$$\tilde{F}_l^{(1)} = R_{l+1}. \tag{8}$$

we give below explicit expressions for the first few r_n 's:

$$r_1 = R_1/p, \tag{9}$$

$$r_2 = R_2/p + (1-p)\, R_1^2/2p^2, \tag{10}$$

$$r_3 = R_3/p + (1-p)\, R_1\, R_2/p^2 + (1-p)(1-2p)\, R_1^3/6\, p^3, \tag{11}$$

etc. The RS-invariants, $\rho_0, \rho_1, \rho_2, \ldots$, are simple polynomial functions of the c_n 's and r_n ' s. The first few are

$$\rho_0 = \tau - r_1, \tag{12}$$

$$\rho_1 = c_1, \tag{13}$$

$$\rho_2 = c_2 + r_2 - \rho_1\, r_1 - r_1^2, \tag{14}$$

$$\rho_3 = c_3 + 2r_3 - 4r_1\, r_2 - 2r_1\, \rho_2 - r_1^2\, \rho_1 + 2r_1^3, \tag{15}$$

etc. For a proof of the RS-invariance of the ρ_n 's we refer the reader to I. The derivation of (1), given in I, is then based on the following observation: In the RS in which all r_n 's vanish* ρ_0 equals τ and ρ_n 's ($n \geqslant 1$) are equal to the β-function co-

*It is always possible to choose this RS because r_n's are in one-to-one correspondence with c_n's and furthermore r_n depends linearly on c_n (see equations (12) to (15)).

efficients, the c_n 's. Moreover, in this RS the value of the couplant a is numerically equal to the value of the physical quantity R. Using this information* in the solution to (4) we get (1) which determines R implicitly as a function of only the RS-invariants, $\rho_0, \rho_1, \rho_2, \ldots$. The nth order approximation to R in RESIPE corresponds to setting $\rho_i = 0$ for $i \geqslant n$.

The ρ_n 's contain all the information coming from the Feynman diagram calculations. They are different for different physical quantities and, in general, depend on the external kinematic variables on which the physical quantity itself depends. However, for an R that depends on only one external energy scale Q, the ρ_n 's ($n \geqslant 1$) are constants independent of Q. To see this, note that, in this case, the r_n 's are functions of the dimensionless variable μ/Q only. Thus, in the scheme in which $\mu = Q$, they are given by

$$\bar{r}_n = r_n(\mu/Q)\,|_{\mu = Q}, \tag{16}$$

which are constants independent of Q. Therefore the ρ_n 's will not depend on Q, except for ρ_0 which is given by**

$$\rho_0 = \frac{b}{2} \ln \frac{Q^2}{\Lambda^2} - \bar{r}_1 \equiv t - \bar{r}_1. \tag{17}$$

For more than one energy scale, $Q, Q', Q'', \ldots$, the $\bar{r}_n$'s will be functions of $\eta' = Q'/Q$, $\eta'' = Q''/Q$, etc. and so will be the ρ_n 's.

A striking property of (1) is that it is not a perturbation expansion for R in the conventional sense since there is *no* expansion parameter in it. One can, however, recover the usual perturbative results from it, under certain approximations. For example, the second order approximation to (1) is

$$\rho_0 = \frac{1}{R} - \rho_1 \ln (1 + 1/\rho_1 R). \tag{18}$$

Then, for fixed η', η'', etc., in the limit $t \to \infty$, one gets from (17) and (18).

$$R \underset{t\to\infty}{\sim} a(t)\,[1 + \bar{r}_1\, a(t) + \ldots], \tag{19}$$

$$a(t) = \frac{1}{t} - \rho_1 \frac{\ln t}{t^2}, \tag{20}$$

which is the usual renormalization group improved, but RS-dependent, second order perturbation approximation for R.

One can meaningfully ask questions about the convergence of a perturbation series in RESIPE. It is clear from (1) that for good convergence R should be small, which

*Note that this scheme is being used only as a mathematical device to arrive at the final result. Equation (1) is independent of any RS because of its manifest RS invariant form.

**In this case one can differentiate (1) w.r.t. t to obtain $\mathrm{d}R/\mathrm{d}t = \rho(R)$, which is analogous to (4).

implies that ρ_0 should be large. In addition, we must also require the ρ_n's ($n \geqslant 1$) to be small. More precisely, the following convergence criterion can be set:

$$\rho_0 R \sim 1, \tag{21}$$

$$|\rho_n R^n| \ll 1, \text{ i.e. } |\rho_n| \ll \rho_0^n, \; n \geqslant 1. \tag{22}$$

Clearly for large ρ_0 (22) is quite a weak condition on the ρ_n's ($n \geqslant 1$). If these criteria are satisfied then one can confidently hope to use low order approximations to (1) to make predictions for R.

The above discussion and what follows demonstrate the central role that ρ_0 plays in RESIPE. In a sense, one can compare ρ_0^{-1} with the running coupling of the conventional perturbation theory. There is, however, a crucial difference between the two, namely, that ρ_0 is RS-invariant but process-dependent while just the opposite is true for the running coupling. An important consequence of this property of ρ_0 is that, in RESIPE, it automatically picks up a scheme independent scale, Λ_{eff}, for each process.

$$\Lambda_{\text{eff}} = \Lambda \exp(\bar{r}_1/b). \tag{23}$$

In general, Λ_{eff} will depend on η', η'', etc. through $\bar{r}_1$. For example, Λ_{eff} will be a function of the other variable x for the deep inelastic structure functions and the photon structure function. For a given process perturbation theory will make sense only if $Q \gg \Lambda_{\text{eff}}$ for it. Perturbation theory will not apply for those values of the kinematic variables η', η'', etc. for which Λ_{eff} becomes large. This happens for example in the case of the deep inelastic structure functions where, in the limit $x \to 1$,

$$\Lambda_{\text{eff}}^2(x) \sim \frac{\Lambda^2}{(1-x)}.$$

Comparison of RESIPE predictions with experiments proceeds as usual. Massless field theories with a single coupling constant have only one free parameter, the scale Λ. In principle this can be determined by a single experimental input. Everything else can then be predicted. In practice it may be better to determine Λ separately from experimental data for each process and then compare the values of Λ thus obtained. Of course, Λ is scheme-dependent and so its value must be accompanied by a scheme label. It is, however, entirely meaningful to compare the values of Λ *in a given scheme* for different processes to see how well perturbation theory works. The important point to keep in mind is that in RESIPE physical predictions are *order-by-order* RS-invariant.

3. The second-order formula

Since a third order calculation does not exist for any QCD process we shall restrict our discussion to the second order formula given in (18). From this equation

physical predictions are made most conveniently in a graphical form. To that end we define

$$\hat{\rho}_0 \equiv \rho_0/\rho_1, \quad \hat{R} \equiv \rho_1 R, \tag{24}$$

and rewrite (18) as follows:

$$\hat{\rho}_0 = \hat{R}^{-1} - \ln(1 + \hat{R}^{-1}). \tag{25}$$

Figure 1 shows the graph of $\hat{R}$ vs. $\hat{\rho}_0$. From this figure one can immediately read off $\hat{R}$ for any given value of $\hat{\rho}_0$ and vice versa. Note the strong dependence of $\hat{R}$ on $\hat{\rho}_0$ for small values of $\hat{\rho}_0$ ($\lesssim 1$). Thus predictions for $\hat{R}$ are not expected to be reliable for such values of $\hat{\rho}_0$. For large values of $\hat{\rho}_0$ ($\gtrsim 5$), $\hat{R}$ is sufficiently small for one to be able to use (25) with some measure of confidence.

The conventional method of testing QCD predictions is to fit the theoretical expressions to the data and compare the values of Λ so obtained from different processes. However, one cannot assess the reliability of the values of Λ so extracted without actually doing higher order calculations. It is therefore highly desirable to devise tests through which one can directly check with the data the reliability of a perturbative approximant for a process without having to know the value of Λ. RESIPE provides us precisely such tests. We discuss these below.

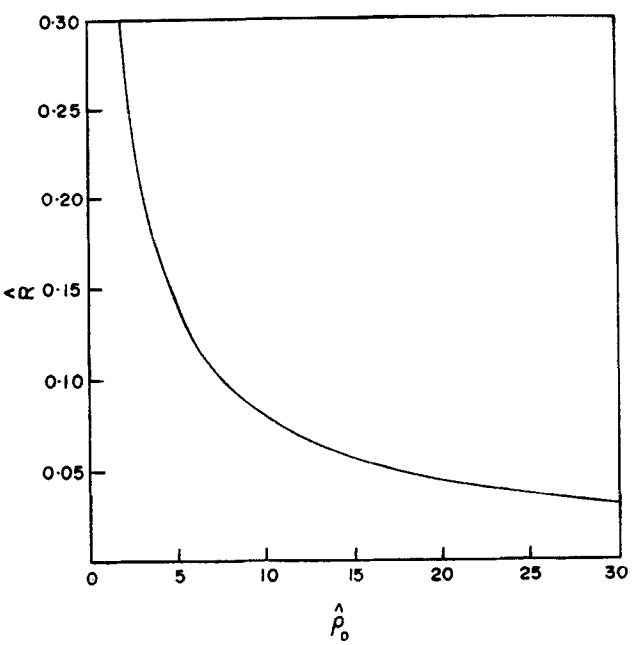

Figure 1. $\hat{R}$ as a function of $\hat{\rho}_0$ as given by equation (25).

3.1 *The geometric test*

Consider a process with only one external scale Q. For such a process we have from (17)

$$\hat{\rho}_0(Q^2) = \frac{b}{2\rho_1} \ln \frac{Q^2}{\Lambda^2} - \bar{r}_1/\rho_1. \tag{26}$$

The geometric test follows from the identity

$$\hat{\rho}_0(Q_1^2) + \hat{\rho}_0(Q_2^2) = 2\hat{\rho}_0\,[(Q_1^2\,Q_2^2)^{1/2}] \tag{27}$$

which together with (25) gives

$$\hat{R}^{-1}(Q_1^2) + \hat{R}^{-1}(Q_2^2) - \ln\,[(1 + \hat{R}^{-1}(Q_1^2))\,(1 + \hat{R}^{-1}(Q_2^2))]$$

$$= 2\hat{R}^{-1}\,[(Q_1^2\,Q_2^2)^{1/2}] - 2\ln\{[1 + \hat{R}^{-1}\,[(Q_1^2\,Q_2^2)]^{1/2}]\} \tag{28}$$

Equation (28) relates the values of the physical quantity $\hat{R}(Q^2)$ at two different values of Q^2 to its value at their geometric mean. A simple way of testing (28) would be to vary Q_1^2 and Q_2^2 holding $Q_1^2\,Q_2^2 = Q_0^4$ fixed. This equation then simplifies to

$$\hat{R}^{-1}(Q^2) + \hat{R}^{-1}(Q_0^4/Q^2) = \ln\,[(1+\hat{R}^{-1}(Q^2))\,(1+\hat{R}^{-1}(Q_0^4/Q^2))] + c, \tag{29}$$

where $\frac{1}{2}\,c = \hat{R}^{-1}(Q_0^2) - \ln(1 + \hat{R}^{-1}(Q_0^2))$ is independent of Q^2. Any deviations of the data from (29) would indicate the magnitude of the higher order corrections.

3.2 *The scaling test*

This test follows from the identity

$$\hat{\rho}_0(Q_2^2) - \hat{\rho}_0(Q_1^2) = \frac{b}{2\rho_1} \ln(Q_2^2/Q_1^2). \tag{30}$$

If we keep $Q_2^2/Q_1^2 = \eta$ fixed then (30) implies the following relation between the values of $\hat{R}$ at Q^2 and the scaled-up point ηQ^2:

$$\hat{R}^{-1}(\eta Q^2) - \ln(1+\hat{R}^{-1}(\eta Q^2)) = \hat{R}^{-1}(Q^2) - \ln(1+\hat{R}^{-1}(Q^2)) + c', \tag{31}$$

where $c' = b/2\rho_1 \ln \eta$ is independent of Q^2. Equation (31) can be tested by varying Q^2 for different fixed values of η. Any deviations of the data from it would again indicate the magnitude of the higher order corrections.

3.3 *The average value test*

Often it is convenient to consider the average of a physical quantity over a range of Q^2. Since the natural variable here is $\ln Q^2$, we calculate the quantity

$$\langle \mathscr{R} \rangle \equiv \frac{1}{\ln (Q_2^2/Q_1^2)} \int_{\ln Q_1^2}^{\ln Q_2^2} \mathrm{d} \ln Q^2 \, \mathscr{R}(Q^2), \tag{32}$$

where, in general, $\mathscr{R} = (\hat{R}/\rho_1)^p$ (see equation (5) and (24)).

To evaluate the integral in (32) we note that (25) can be written in the following equivalent form

$$\hat{R} = 1/[\hat{\rho}_0 + F(\hat{\rho}_0)], \tag{33}$$

where $F(\hat{\rho}_0)$ satisfies the transcendental equation

$$e^F - F - 1 = \hat{\rho}_0 \tag{34}$$

Since,

$$\mathrm{d} \ln Q^2 = \frac{2\rho_1}{b} \frac{d\hat{\rho}_0}{\mathrm{d}F} \mathrm{d}F = \frac{2\rho_1}{b} (e^F - 1) \, \mathrm{d}F,$$

we get, with $\mathscr{R}_i^{-p} = \mathscr{R}(Q_i^2)$,

$$\langle \mathscr{R} \rangle = \frac{2}{b \ln (Q_2^2/Q_1^2)} \int_{\mathscr{R}_1}^{\mathscr{R}_2} \mathrm{d}z \, \frac{z^{1-p}}{(\rho_1 + z)} \tag{35}$$

Of special interest are the cases with $p = -1$ (the photon structure function) and $p = +1$ (the $e^+ e^-$ ratio R). We shall discuss these in detail in § 5.2 and § 5.3 respectively.

Some remarkable features of the above tests need to be emphasized. The geometric and the scaling tests exploit the $\ln Q^2$ dependence of $\hat{\rho}_0$ directly. It is this which also helps us to calculate the average $\langle \mathscr{R} \rangle$. All the tests give simple relations between experimentally measurable quantities. Further, they do not require any knowledge of Λ or even of the second order coefficient $\bar{r}_1$ and as such can be directly confronted with the data. This makes them ideal for probing the neglected higher order corrections. Once the region(s) of the external kinematic variables where higher order corrections are small is (are) determined one can fit (25) to the data and determine Λ. An example of this procedure is given in § 5.1 where we discuss the moments of the non-singlet structure functions.

4. Third order corrections

To make theoretical estimates of the higher order corrections to the second-order approximant $\hat{R}$, we consider the third order approximation to (1):

$$\hat{\rho}_0 = \hat{R}_3^{-1} - \ln(1+\hat{R}_3^{-1}) + \int_0^{\hat{R}_3} dx \left[\frac{1}{x^2(1+x)} - \frac{1}{x^2(1+x+\hat{\rho}_2 x^2)}\right], \tag{36}$$

where $\hat{\rho}_2 = \rho_2/\rho_1^2$ and we have attached the subscript '3' to $\hat{R}_3$ to distinguish it from the second order approximant $\hat{R}$. The integral in (36) can be done analytically and is equal to

$$\frac{1}{2}\ln\frac{(1+\hat{R}_3)^2}{(1+\hat{R}_3+\hat{\rho}_2\hat{R}_3^2)} + \frac{(2\hat{\rho}_2-1)}{\sqrt{(4\hat{\rho}_2-1)}}$$

$$\times\left[\tan^{-1}\frac{(1+2\hat{\rho}_2\hat{R}_3)}{\sqrt{(4\hat{\rho}_2-1)}} - \tan^{-1}\frac{1}{\sqrt{(4\hat{\rho}_2-1)}}\right]$$

for $\hat{\rho}_2 > 1/4$ and

$$\frac{1}{2}\ln\frac{(1+\hat{R}_3)^2}{(1+\hat{R}_3+\hat{\rho}_2\hat{R}_3^2)} + \frac{1}{2}\frac{(2\hat{\rho}_2-1)}{\sqrt{(1-4\hat{\rho}_2)}}\ln\left[\frac{2+\hat{R}_3(1+\sqrt{(1-4\hat{\rho}_2)})}{2+\hat{R}_3(1-\sqrt{(1-4\hat{\rho}_2)})}\right]$$

for $\hat{\rho}_2 < 1/4$.

In figure 2 $\hat{R}_3$ *vs* $\hat{\rho}_0$ is plotted for $\hat{\rho}_2 = \pm 5$, ± 10 and ± 15, where for comparison the curve for $\hat{\rho}_2 = 0$ has also been included. We see that $\hat{R}_3$ depends sensitively on $\hat{\rho}_2$ for small values of $\hat{\rho}_0$. On the other hand $\hat{R}_3$ varies little with $\hat{\rho}_2$ for large values of $\hat{\rho}_0$. The second order formula, equation (25), will therefore break down for sufficiently small $\hat{\rho}_0$. The precise value of $\hat{\rho}_0$ for which this happens will depend on the magnitude of $\hat{\rho}_2$. It has been suggested (Duke and Kimel 1982; Abbot 1980; Blumenfeld and Moshe 1982) that the general trend of the existing higher order calculations is consistent with taking $r_2 \sim r_1$ and $r_2 \sim r_1^2$. With these guesses for r_2 and a typical value of $r_1 = 4$ we find that a safe estimate for $\hat{\rho}_2$ is $\hat{\rho}_2 \sim \pm 5$. Using these values in figure 2 we see that the error in the predicted value of $\hat{R}$ (*i.e.* of $\hat{R}_3$ for $\hat{\rho}_2 = 0$) is reasonably small ($\lesssim 20\%$) for $\hat{\rho}_0 \gtrsim 4$. Of course, in particular cases the actual value of $\hat{\rho}_2$ could be much smaller than the above estimate and so (25) could be valid down to much smaller values of $\hat{\rho}_0$. It is here that the tests mentioned in § 3.1 to 3.3 play a crucial role. Thus, while in general one should restrict the use of (25) only to value of $\hat{\rho}_0 \gtrsim 4$, one may use it confidently even for lower values of $\hat{\rho}_0$ if the data so indicate.

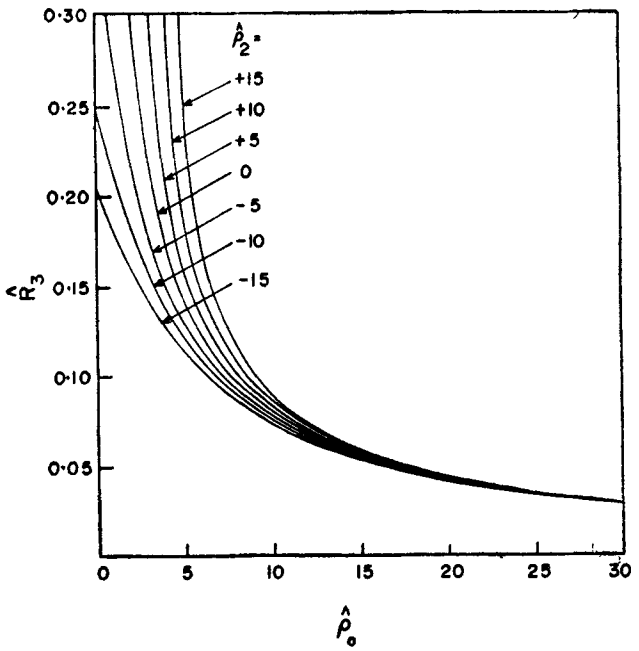

Figure 2. $\hat{R}_3$ as a function of $\hat{\rho}_0$ (equation 36) for various values of $\hat{\rho}_2$. For comparison, the curve for $\hat{\rho}_2 = 0$ is also shown.

5. Applications to QCD

We now consider the predictions of the second order RESIPE for some QCD processes in detail. Throughout the following we shall use the $\overline{\text{MS}}$ scheme as the base scheme. However, for ease of notation we shall not use this label explicitly, it being understood that all the scheme-dependent quantities are given in the $\overline{\text{MS}}$ scheme.

5.1 *Deep inelastic scattering: moments of nonsinglet (structure functions)*

The physical quantities which are perturbatively calculable here are not the moments but their Q^2 variation (Duke and Kimel 1982)

$$\hat{R}_i^{(n)}(Q^2) = -\frac{2\rho_1}{b\mathrm{d}_i^{(n)}} \frac{\mathrm{d} \ln M_i^{(n)}(Q^2)}{\mathrm{d} \ln Q^2}, \quad i = 2, 3, L, \tag{37}$$

$$\mathrm{d}_{2,3}^{(n)} = \mathrm{d}_n, \quad \mathrm{d}_L^{(n)} = (\mathrm{d}_n + 1), \tag{38}$$

where $$M_i^{(n)}(Q^2) \equiv \int_0^1 \mathrm{d}x\, x^{n-2} F_i^{NS}(x, Q^2), \quad n \geqslant 2, \tag{39}$$

and $i = 2$, 3 and L refer to moments of the structure functions F_2^{NS}, xF_3 and F_L respectively. Also, we have (Duke and Kimel 1982)

$$\hat{\rho}_{0,i}^{(n)}(Q^2) = \frac{b}{2\rho_1}\ln\frac{Q^2}{\Lambda^2} - \frac{\alpha_i^{(n)}}{\rho_1}. \tag{40}$$

The quantities d_n and $\alpha_i^{(n)}$* have been calculated (Bardeen *et al* 1978; Buras 1980; Duke *et al* 1981) and are listed for 4 flavours for $i = 2$ and 3 for the first few moments in table 1. The effective scale in this case is

$$\Lambda_{\text{eff},i}^{(n)} = \Lambda \exp(\alpha_i^{(n)}/b), \tag{41}$$

which increases with n for all the structure functions. Thus the uncertainty in the second-order predictions will increase with n for all the structure functions.

To second order $\hat{R}_i^{(n)}$ and $\hat{\rho}_{0,i}^{(n)}$ satisfy (25). This equation can be easily integrated (see § 3.3) to get the following prediction for the moments:

$$M_i^{(n)}(Q^2) = M_i^{(n)}(Q_0^2)\exp[-\mathrm{d}_i^{(n)}\{F_i^{(n)}(Q^2) - F_i^{(n)}(Q_0^2)\}], \tag{42}$$

where $F_i^{(n)}$ satisfies the transcendental equation

$$\exp(F_i^{(n)}) - F_i^{(n)} - 1 = \hat{\rho}_{0,i}^{(n)}. \tag{43}$$

From (42) and (43) we find that the moments satisfy the equation

$$\frac{\ln[M_i^{(n)}(Q_0^2)/M_i^{(n)}(Q^2)]^{1/d_i^{(n)}} + \dfrac{b}{2\rho_1}\ln(Q^2/Q_0^2)}{[M_i^{(n)}(Q_0^2)/M_i^{(n)}(Q^2)]^{1/d_i^{(n)}} - 1} = K_i^{(n)} \tag{44}$$

where $K_i^{(n)}$ is constant independent of Q.

Table 1. Numerical values of the quantities d_n and $\alpha_i^{(n)}$ $(i = 2, 3)$ for the first few moments.

n	d_n	$\alpha_2^{(n)}$	$\alpha_3^{(n)}$
2	0·427	2·768	1·467
3	0·667	3·476	2·892
4	0·837	3·964	3·606
5	0·970	4·334	4·082
6	1·080	4·628	4·438
7	1·173	4·875	4·722
8	1·255	5·082	4·956
9	1·327	5·265	5·159
10	1·392	5·425	5·333
11	1·451	5·570	5·490
12	1·505	5·700	5·629

*$\alpha_{2,3}^{(n)}$ can be obtained from $\overline{R}_{i,n}^{NS}$'s given in table 1 of Buras (1980) by using $\alpha_2^{(n)} = (R_{2,n}^{NS}/4d_n) + c_1$ and $\alpha_3^{(n)} = (\overline{R}_{3,n}^{NS}/4d_n) + c_1$.

Equation (44) involves only experimentally measurable quantities and can be tested directly using the data on the moments. Knowledge of Λ or the second-order coefficients, $a_i^{(n)}$, is not required to carry out this test. Deviation of $K_i^{(n)}$ from a constant value, as Q^2 is varied, is a measure of the higher order corrections. Provided (44) is satisfied one can obtain the value of Λ from a measurement of $K_i^{(n)}$ by using the following relation

$$\Lambda^2 = Q_0^2 \exp\left(-\frac{2\rho_1}{b}(K_i^{(n)} - \ln K_i^{(n)} - 1 + a_i^{(n)}/\rho_1)\right). \tag{45}$$

We have carried out this novel test on the moments of xF_3 and F_2^{ep-en} using the moments computed by Duke *et al* (1982) from the CDHS and SLAC data*. Tables 2 and 3 give the values of $K_i^{(n)}$ for different Q^2, using $n = 4$, 5 and $Q_0^2 = 90$ (GeV)^2 for xF_3 and $n = 5$, 6 and $Q_0^2 = 22{\cdot}5$ (GeV)^2 for F_2^{ep-en}. We see that the data satisfies (44) fairly well down to low Q^2 values. There seems to be no indication of large higher order corrections or any other effects like higher twist. This fair constancy of $K_i^{(n)}$ with respect to Q^2 encourages us to evaluate Λ using (45). In table 4 we have listed tthe best values of $K_i^{(n)}$ and the corresponding values of Λ for the four cases studied here. All the data are consistent with a value of Λ of 450–500 MeV. A more detailed and complete analysis of the data using (44) is in progress and will be reported elsewhere.

Table 2. Values of $K_i^{(n)}$ for the fourth and fifth moments of the structure function xF_3 for $Q^2 \leqslant 28{\cdot}4$ (GeV)^2, with $Q_0^2 = 90$ (GeV)^2, obtained from the moments of the CDHS data computed by Duke *et al* (1982). Data at higher Q^2 were not used because the errors on $K_i^{(n)}$ becomes too large as Q^2 approaches Q_0^2.

Q^2 (GeV)^2	$K_3^{(4)}$	$K_3^{(5)}$
5·7	$9{\cdot}037 \begin{smallmatrix}+0{\cdot}882\\-0{\cdot}573\end{smallmatrix}$	$8{\cdot}620 \begin{smallmatrix}+0{\cdot}833\\-0{\cdot}566\end{smallmatrix}$
7·0	$9{\cdot}579 \begin{smallmatrix}+1{\cdot}981\\-1{\cdot}121\end{smallmatrix}$	$8{\cdot}873 \begin{smallmatrix}+0{\cdot}994\\-0{\cdot}684\end{smallmatrix}$
9·0	$9{\cdot}709 \begin{smallmatrix}+2{\cdot}161\\-1{\cdot}243\end{smallmatrix}$	$8{\cdot}741 \begin{smallmatrix}+1{\cdot}058\\-0{\cdot}733\end{smallmatrix}$
11·3	$8{\cdot}996 \begin{smallmatrix}+1{\cdot}894\\-1{\cdot}115\end{smallmatrix}$	$8{\cdot}392 \begin{smallmatrix}+1{\cdot}168\\-0{\cdot}780\end{smallmatrix}$
14·2	$9{\cdot}540 \begin{smallmatrix}+2{\cdot}470\\-1{\cdot}402\end{smallmatrix}$	$8{\cdot}492 \begin{smallmatrix}+1{\cdot}290\\-0{\cdot}869\end{smallmatrix}$
18·0	$8{\cdot}646 \begin{smallmatrix}+2{\cdot}144\\-1{\cdot}236\end{smallmatrix}$	$7{\cdot}713 \begin{smallmatrix}+1{\cdot}153\\-0{\cdot}779\end{smallmatrix}$
22·5	$7{\cdot}362 \begin{smallmatrix}+1{\cdot}510\\-0{\cdot}923\end{smallmatrix}$	$6{\cdot}745 \begin{smallmatrix}+0{\cdot}898\\-0{\cdot}621\end{smallmatrix}$
28·4	$8{\cdot}490 \begin{smallmatrix}+3{\cdot}570\\-1{\cdot}675\end{smallmatrix}$	$7{\cdot}319 \begin{smallmatrix}+1{\cdot}801\\-1{\cdot}048\end{smallmatrix}$

*We are grateful to D P Roy for getting the moments computed by Duke *et al* (1982) for us.

Table 3. Values of $K_i^{(n)}$ for the fifth and sixth moments of the structure function $F_2^{ep\text{-}en}$ for $Q^2 \leqslant 9{\cdot}0$ (GeV)^2, with $Q_0^2 = 22{\cdot}5$ (GeV)^2 obtained from the moments of the SLAC data computed by Duke *et al* (1982). Data at $Q^2 = 12{\cdot}5$ (GeV)^2 were not used for the same reason as in table 2.

Q^2 (GeV)^2	$K_2^{(5)}$	$K_2^{(6)}$
2·5	$6{\cdot}430 \begin{smallmatrix}+0{\cdot}479\\-0{\cdot}327\end{smallmatrix}$	$6{\cdot}313 \begin{smallmatrix}+0{\cdot}298\\-0{\cdot}220\end{smallmatrix}$
3·5	$6{\cdot}301 \begin{smallmatrix}+0{\cdot}743\\-0{\cdot}491\end{smallmatrix}$	$6{\cdot}169 \begin{smallmatrix}+0{\cdot}449\\-0{\cdot}333\end{smallmatrix}$
4·5	$5{\cdot}924 \begin{smallmatrix}+0{\cdot}870\\-0{\cdot}539\end{smallmatrix}$	$5{\cdot}778 \begin{smallmatrix}+0{\cdot}504\\-0{\cdot}360\end{smallmatrix}$
5·5	$6{\cdot}250 \begin{smallmatrix}+3{\cdot}167\\-1{\cdot}091\end{smallmatrix}$	$6{\cdot}178 \begin{smallmatrix}+2{\cdot}221\\-0{\cdot}909\end{smallmatrix}$
6·5	$5{\cdot}644 \begin{smallmatrix}+1{\cdot}522\\-0{\cdot}772\end{smallmatrix}$	$5{\cdot}595 \begin{smallmatrix}+0{\cdot}902\\-0{\cdot}566\end{smallmatrix}$
7·5	$5{\cdot}668 \begin{smallmatrix}+2{\cdot}401\\-0{\cdot}988\end{smallmatrix}$	$5{\cdot}466 \begin{smallmatrix}+1{\cdot}164\\-0{\cdot}663\end{smallmatrix}$
9·0	$5{\cdot}548 \begin{smallmatrix}+3{\cdot}215\\-1{\cdot}141\end{smallmatrix}$	$5{\cdot}378 \begin{smallmatrix}+1{\cdot}437\\-0{\cdot}774\end{smallmatrix}$

Table 4. The best values of $K_i^{(n)}$ obtained from tables 2 and 3, and the corresponding values of Λ obtained using (45).

$K_i^{(n)}$	Best value of $K_i^{(n)}$	Λ (MeV)
$K_3^{(4)}$	8·866 ± 0·470	$489 \begin{smallmatrix}+81\\-70\end{smallmatrix}$
$K_3^{(5)}$	8·146 ± 0·318	$551 \begin{smallmatrix}+60\\-54\end{smallmatrix}$
$K_2^{(5)}$	6·232 ± 0·285	$477 \begin{smallmatrix}+44\\-41\end{smallmatrix}$
$K_2^{(6)}$	6·090 ± 0·177	$464 \begin{smallmatrix}+26\\-25\end{smallmatrix}$

5.2 *The photon structure function*

The point-like contribution to the photon structure function, $F_2^\gamma(x, Q^2)$, is calculable in perturbative QCD. The second order calculations give (Bardeen and Buras 1979):

$$\frac{1}{\alpha^2} F_2^\gamma(x, Q^2) = \frac{2h(x)}{b} \frac{1}{a(Q)} \left(1 - \frac{bg(x)}{2} a(Q)\right). \tag{46}$$

Here α is the fine structure constant and the functions $h(x)$ and $g(x)$ are listed for 4 flavours in table 5 for $0{\cdot}4 \leqslant x \leqslant 0{\cdot}9$.

Table 5. Numerical values of $h(x)$ and $g(x)$ for $0{\cdot}4 \leqslant x \leqslant 0{\cdot}9$. The approximate analytic forms given by Bardeen and Buras (1979) have been used.

x	$h(x)$	$g(x)$
0·40	1·249	1·186
0·45	1·342	1·087
0·50	1·422	1·047
0·55	1·490	1·061
0·60	1·545	1·127
0·65	1·588	1·240
0·70	1·618	1·401
0·75	1·636	1·607
0·80	1·641	1·861
0·85	1·634	2·163
0·90	1·613	2·523

This is an example of the case with $p = -1$. So, to second order, the physical quantity

$$\hat{R}(x, Q^2) = \left[\frac{b}{2\rho_1 h(x)} \frac{1}{\alpha^2} F_2^\gamma (x, Q^2)\right]^{-1}, \tag{47}$$

with $$\hat{\rho}_0 (x, Q^2) = \frac{b}{2\rho_1}\left(\ln \frac{Q^2}{\Lambda^2} - g(x)\right). \tag{48}$$

satisfies (25).

Tests mentioned in § 3.1 to 3.3 can be carried out in this case. Particularly interesting from a practical point of view is the average value test. In this case, (35) gives

$$\left\langle \frac{1}{\alpha^2} F_2^\gamma (x, Q^2) \right\rangle = [2h(x) \ln (Q_2^2/Q_1^2)]^{-1}\left(\left\{ \frac{1}{\alpha^2} F_2^\gamma (x, Q_2^2) \right\}^2 - \left\{ \frac{1}{\alpha^2} F_2^\gamma (x, Q_1^2) \right\}^2 \right) - \frac{2\rho_1 h(x)}{b}. \tag{49}$$

A simple way of testing the above prediction would be to study its x dependence. If the data satisfy (49) then a value of Λ can be obtained by using the experimental data in (25), (47) and (48). Alternatively, in the absence of precise data, given Λ one can theoretically calculate the right side of (49) using (25). As an illustration the prediction for $\left\langle \frac{1}{\alpha^2} F_2^\gamma \right\rangle$ as a function of x is given in figure 3 or $\Lambda = 0{\cdot}2$ and $0{\cdot}4$ GeV with $Q_1^2 = 5$ (GeV)^2 and $Q_2^2 = 20$ (GeV)^2. Clearly the curve for $\Lambda = 0{\cdot}2$ GeV lies much above the curve for $\Lambda = 0{\cdot}4$ GeV for all x. Thus a precise measurement of $\left\langle \frac{1}{\alpha^2} F_2^\gamma \right\rangle$ may provide a good indication of the magnitude of Λ.

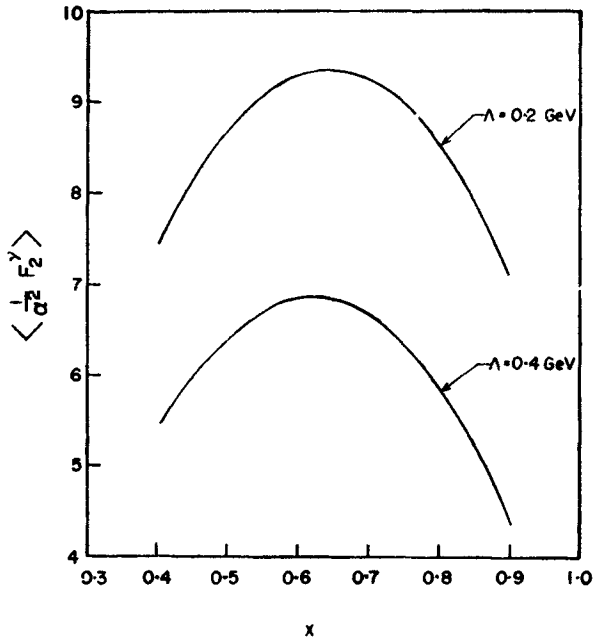

Figure 3. The x dependence of $\left\langle \frac{1}{\alpha^2} F_2^\gamma (x, Q^2) \right\rangle$ (equation (49)) for $\Lambda = 0{\cdot}2$ and 0·4 GeV. The average is taken for $Q_1^2 = 5$ (GeV)^2 and $Q_2^2 = 20$ (GeV)^2.

In the present case the effective scale picked up by $\hat{\rho}_0$ is a function of x and is given by

$$\Lambda_{\text{eff}}(x) = \Lambda \exp [g(x)/2]. \tag{50}$$

This decreases from 1·81 Λ to 1·69 Λ as x goes from 0·4 to 0·5 and then increases to 3·53 Λ at $x = 0{\cdot}9$.

5.3 $e^+ e^-$ *Annihilation*

We write the experimentally-measured cross-section ratio as

$$\frac{\sigma(e^+ e^- \to \text{hadrons})}{\sigma(e^+ e^- \to \mu^+ \mu^-)} \equiv \mathcal{R}(Q^2) = R_0 [1 + R(Q^2)], \tag{51}$$

where Q is the total c.m. energy and

$$R_0 = 3 \sum_{i=1}^{f} e_i^2$$

is the parton model value, independent of Q. So what will enter (1) is the quantity $[\mathcal{R}(Q^2)/R_0 - 1] = R(Q^2)$ which has a perturbative expansion in the form of (5).

The second-order QCD calculations give (Dine and Sapirstein 1979; Chetyrkin *et al* 1979; Celmaster and Gonsalves 1980)

$$R(Q^2) = a(Q)\,[1 + (1{\cdot}986 - 0{\cdot}115f)\,a(Q) + \ldots]. \tag{52}$$

To this order then

$$\hat{R}(Q^2) = \rho_1\, R(Q^2), \tag{53}$$

and

$$\hat{\rho}_0(Q^2) = \frac{b}{2\rho_1} \ln \frac{Q^2}{\Lambda^2} - (1{\cdot}986 - 0{\cdot}115f)/\rho_1, \tag{54}$$

satisfy (25).

In the present case $p = 1$. So we get from (35)

$$\langle \mathscr{R}(Q^2) \rangle = \frac{2\,R_0^2}{b\,\rho_1 \ln (Q_2^2/Q_1^2)} \frac{[\mathscr{R}(Q_1^2) - \mathscr{R}(Q_2^2)]}{[\mathscr{R}(Q_1^2) - R_0]\,[\mathscr{R}(Q_2^2) - R_0]}$$

$$+ \left(1 - \frac{1}{\rho_1}\right) R_0. \tag{55}$$

Experimentally, $\mathscr{R}$ is found to be constant within errors for c.m. energies between 14 and 37 GeV with an average value of $\mathscr{R} = 4{\cdot}01 \pm 0{\cdot}03 \pm 0{\cdot}20$ (Lüke 1982). The difference $[\mathscr{R}(Q_1^2) - \mathscr{R}(Q_2^2)]$ in this range of Q is less than the systematic uncertainty of about 5%. Consequently it is not possible to test (55) meaningfully at present.

5.4 *Heavy quark-antiquark systems*

Calculations upto second order now exist for a number of processes in the J/ψ and Υ systems. We shall consider here only two examples, namely, Υ-decay and the hyperfine splitting in the J/ψ system.

5.4a *Υ-decay:* The widths Γ_g, Γ_γ and Γ_μ for Υ decaying into gluons, a direct photon plus gluons and $\mu^+\mu^-$ pair respectively have been calculated upto the second order (Mackenzie and Lepage 1981). The quantities predictable in perturbative QCD are the ratios

$$\Gamma_g/\Gamma_\mu = \frac{10\,(\pi^2 - 9)\,\pi^2}{9\alpha^2}\, a^3(M_\Upsilon)\,[1 + 9{\cdot}1a(M_\Upsilon) + \ldots], \tag{56}$$

and

$$\Gamma_\gamma/\Gamma_\mu = \frac{8\,(\pi^2 - 9)\,\pi}{9\alpha}\, a^2(M_\Upsilon)\,[1 + 3{\cdot}7a(M_\Upsilon) + \ldots]. \tag{57}$$

Here α is the fine structure constant. We have here examples of physical quantities with $p=3$ and $p=2$ respectively. Thus the quantities which should satisfy (25) are

$$\hat{R}_g = \rho_1 \left(\frac{9\alpha^2}{10\,(\pi^2 - 9)\,\pi^2}\,\Gamma_g/\Gamma_\mu\right)^{1/3},\ \hat{\rho}_{0,g} = \frac{b}{\rho_1} \ln \frac{M_\Upsilon}{\Lambda} - \frac{3{\cdot}03}{\rho_1}; \tag{58}$$

and $$\hat{R}_\gamma = \rho_1 \left[\frac{9\alpha}{8(\pi^2-9)\pi}\Gamma_\gamma/\Gamma_\mu\right]^{1/2}, \hat{\rho}_{0,\gamma} = \frac{b}{\rho_1}\ln\frac{M_\Upsilon}{\Lambda} - \frac{1{\cdot}85}{\rho_1}. \tag{59}$$

The $\hat{\rho}_0$'s are given for 4 flavours.

Experimentally, the sum $\Gamma_g/\Gamma_\mu + \Gamma_\gamma/\Gamma_\mu$ can be determined by using the measured values for the leptonic branching ratio B_μ for the Υ-decay and the $e^+ e^-$ ratio $\mathscr{R}$ in the relation

$$\Gamma_g/\Gamma_\mu + \Gamma_\gamma/\Gamma_\mu = B_\mu^{-1} - (3 + \mathscr{R}). \tag{60}$$

Taking $B_\mu = (3{\cdot}3 \pm 0{\cdot}05)$ % (Alam *et al* 1981; Mueller *et al* 1981), $\mathscr{R} = 3{\cdot}5$ for 4 light quarks and using (58) and (59) in (60) we get

$$23{\cdot}8^{+5{\cdot}4}_{-4{\cdot}0} = \frac{10(\pi^2-9)\pi^2}{9\alpha^2\rho_1^3}\hat{R}_g^3 + \frac{8(\pi^2-9)\pi}{9\alpha\rho_1^2}R_\gamma^2. \tag{61}$$

Equation (61) involves only one unknown, the scale Λ. Given a value of Λ, one can obtain $\hat{R}_g$ and $\hat{R}_\gamma$ from (25), (58) and (59). The right side of (61) can be then evaluated. In this way one can fit a value of Λ to (61). We find for $M_\Upsilon = 9{\cdot}46$ GeV and 4 flavours that

$$\Lambda = 105^{+35}_{-26}\,\text{MeV}. \tag{62}$$

For this value of Λ, $\hat{\rho}_{0,g} = 10{\cdot}2 \pm 0{\cdot}8$ ($\Gamma_g/\Gamma_\mu = 23{\cdot}2$) and $\hat{\rho}_{0,\Upsilon} = 11{\cdot}0 \pm 0{\cdot}8$ ($\Gamma_\gamma/\Gamma_\mu = 0.6$) are both large. Higher order corrections are therefore not expected to change it appreciably.

5.4b. *Hyperfine splitting in the J/ψ system:* The physical observable calculable in perturbative QCD in this case is the ratio of the hyperfine splitting to the leptonic width (Buchmuller *et al* 1981)

$$\frac{\Delta E(\psi)}{\Gamma(\psi \to \mu^+\mu^-)} = \frac{2\pi}{\alpha^2}a(m_c)\,[1 + 6{\cdot}1\,a(m_c) + \ldots]. \tag{63}$$

This calculation has been done for 3 flavours. The quantities that should satisfy (25) in this case are

$$\hat{R} = \frac{\alpha^2}{2\pi}\rho_1\left(\frac{\Delta E(\psi)}{\Gamma(\psi\to\mu^+\mu^-)}\right), \quad \hat{\rho}_0 = \frac{b}{\rho_1}\ln\frac{m_c}{\Lambda} - \frac{6{\cdot}1}{\rho_1}. \tag{64}$$

The experimental value for the ratio $\Delta E/\Gamma$ is $0{\cdot}21 \pm 0{\cdot}03$ (Berkelman 1981). Therefore, $\hat{R} = 0{\cdot}37 \pm 0{\cdot}05$ and the corresponding value of $\hat{\rho}_0$ from figure 1 is

$$\hat{\rho}_0 = 1{\cdot}39^{+0{\cdot}32}_{-0{\cdot}23}. \tag{65}$$

For $m_c = M_\psi/2 = 1{\cdot}55$ GeV and 3 flavours this implies

$$\Lambda = 232^{+22}_{-28}\,\text{MeV}. \tag{66}$$

Because of the very small value of $\hat{\rho}_0$ in this case, higher order corrections could appreciably change the value of Λ obtained above. This number is therefore not expected to be very reliable.

6. Discussion and summary

In this paper we have given second-order RESIPE predictions for (i) moments of the nonsinglet structure functions, (ii) photon structure function, (iii) cross-section ratio in e^+e^- annihilation, (iv) gluonic width of Υ and (v) hyperfine splitting in the J/ψ system. Some of these predictions remarkably do not involve the unknown scheme-dependent QCD scale parameter Λ. Consequently they can be directly confronted with experiments. Any deviations from them would indicate the magnitude of the neglected higher order corrections which can thus be probed by these tests. We have carried out such a test, equation (44), for the moments of the nonsinglet structure functions xF_3 and $F_2^{\text{ep-en}}$. The data satisfy (44) fairly well within errors, indicating that higher order corrections are small. One can therefore consider the value of $\Lambda = 450$–500 MeV obtained in this case as quite reliable. Such tests can be carried out for the cases (ii) and (iii) also when more accurate data, then at present, become available. In the case of (iv) and (v) the data consist of only one number each. As such, one can only extract Λ in these cases. The values we obtain are $\Lambda = 105^{+35}_{-26}$ MeV from the gluonic width of the Υ and $\Lambda = 232^{+22}_{-28}$ MeV from the hyperfine splitting in the J/ψ system. In the former case ρ_0 is large and so higher order corrections are not expected to appreciably change the value of Λ obtained. However, in the latter case the extracted value of Λ is not expected to be reliable because of the smallness of ρ_0. The values of Λ quoted above are in the $\overline{\text{MS}}$-scheme which was used as the base scheme throughout this work. It is clear that Λ is much smaller for the heavy quark systems than for the moments of the structure functions. At the level of the present analysis it does not seem possible to reconcile the two values and give a definite value of Λ. This is a serious problem, in our view, which deserves further careful and detailed study.

Finally, a word about RESIPE and the central role of ρ_0 in it. The basic equation of RESIPE, equation (1), can be approximated to any given order to get a scheme-independent approximation for the physical quantity R which satisfies it. The magnitude of ρ_0 for the process then provides a guide to the convergence of such a perturbation series for R. Through the $\ln Q^2$ term, ρ_0 gives the basic energy dependence for R. Actually, for processes with only one external energy scale, Q, ρ_0 alone depends on Q, all the other invariants being constants. The new second order tests (§ 3) are a consequence of the $\ln Q^2$ dependence of ρ_0 and essentially test this. Moreover, ρ_0 is instrumental in automatically defining a scheme-independent scale Λ_{eff} (equation (23)) for each process. This Λ_{eff} depends on the other dimensionless kinematic variables (e.g. x for the structure functions) in precisely the way phenomenology and other theoretical considerations seem to require.

Clearly further work is required to extend the RESIPE ideas to massive theories and also theories with more than one coupling constant. This is necessary for treating quark mass effects in QCD. It would also open up applications to electroweak gauge theories and grand unified theories.

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