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A new approach to charged-particle scattering in the presence of laser plus Coulomb-field

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Abstract. A new approach to charge-particle scattering in the presence of laser plus coulomb-field by using Fourier analysis technique is described. Explicit expressions for positive energy states and their asymptotic limits for the zero, one and two photon processes are evaluated exactly.

Keywords. Scattering; laser; Coulomb field; Fourier analysis; photon

1. Introduction

Recently extensive studies have been carried out in the field of atomic and molecular collision processes in the presence of EM field (Levine and Bernstein 1974; Walther 1976; Hertal *et al* 1980; Mohan and Chand 1979; Mohan 1981) due to its importance in laser-induced chemistry, working of different type of lasers, laser-induced gas-breakdown, plasma-heating by laser etc. The understanding of the laser-plasma interaction related to the laser-fusion reactions requires a knowledge of the collision process in the presence of EM field occurring under various conditions among atoms, molecules, neutrals and charged particles.

This paper investigates the positive energy states in the presence of laser and strong Coulomb field which are very important in the study of various physical processes like (a) free-free transition process in laser plus strong Coulomb field (Kroll and Watson 1973; Burkin and Fedorov 1965, 1966; Mohan 1974; Henneberger 1968; Rosenberg 1979), (b) electron impact ionization of an atom or a molecule in the presence of laser beam etc (Gavrila 1978; Gavrila and van der Wiel 1978). In our analysis we use the Fourier analysis technique as introduced by Karplus and Kolker (1963), Dalgarno (1966) and others for treating the time-dependent problem. We have tried to explain the salient features involved in the above mentioned physical processes. In §§ 2 and 3, the theory from the first-principle is developed and the one-photon process discussed. In § 4 we deal with the two-photon process and the corresponding positive energy state with the asymptotic states is evaluated. In § 5, the elastic scattering in the presence of EM field plus strong coulomb-field is discussed using partial wave analysis technique. The results thus obtained are discussed.

2. Theory

The Schrödinger equation for the system consisting of a charged particle (*e.g.* electron) moving in the presence of an EM field plus a Coulomb field (*e.g.* a proton) can be written as

$$i\hbar \frac{\partial \psi_k(\mathbf{r}, t)}{\partial t} = \{H(\mathbf{r}) + v(\mathbf{r}, t)\} \psi_k(\mathbf{r}, t) \tag{1}$$

where $H(\mathbf{r}) = H_0(\mathbf{r}) + v(\mathbf{r})$; $H_0(\mathbf{r})$ is the free electron Hamiltonian and $v(\mathbf{r})$ is the Coulomb potential, $v(\mathbf{r}, t) = - e\, \mathbf{E}\cdot\mathbf{r} \cos \omega t$ is the interaction Hamiltonian representing the interaction between the electron and EM field (E) with frequency ω.

From (1), the time dependent solution of the (time dependent) unperturbed Schrödinger equation (where $v(\mathbf{r}, t)$ is the perturbation) i.e.

$$i\hbar \frac{\partial \psi_k^0(\mathbf{r}, t)}{\partial t} = H(\mathbf{r})\, \psi_k^0(\mathbf{r}, t), \tag{2}$$

can be written as:

$$\psi_k^0(\mathbf{r}, t) = X_k^0(\mathbf{r}, 0)\, [\exp(-\, iE_k\, t)]/\hbar \tag{3}$$

where $X_k^0(\mathbf{r}, 0)$ represents the first term for the zero photon process, which will be clear in the next sections.

Expanding the solution of (1) *i.e.* $\psi_k(\mathbf{r}, t)$ as (Mohan 1981)

$$\psi_k(\mathbf{r}, t) = \psi_k^0(\mathbf{r}, t) + \sum_{s=1}^{\infty} \psi_k^{(s)}(\mathbf{r}, t), \tag{4}$$

and putting in (1) we obtain

$$\left(H - i\hbar \frac{\partial}{\partial t}\right) \psi_k^0(\mathbf{r}, t) = 0, \tag{5}$$

$$\left(H - i\hbar \frac{\partial}{\partial t}\right) \psi_k^1(\mathbf{r}, t) + v(\mathbf{r}, t)\, \psi_k^0(\mathbf{r}, t) = 0, \tag{6}$$

$$\left(H - i\hbar \frac{\partial}{\partial t}\right) \psi_k^2(\mathbf{r}, t) + v(\mathbf{r}, t)\, \psi_k^1(\mathbf{r}, t) = 0, \tag{7}$$

$$\left(H - i\hbar \frac{\partial}{\partial t}\right) \psi_k^s(\mathbf{r}, t) + v(\mathbf{r}, t)\, \psi_k^{s-1}(\mathbf{r}, t) = 0. \tag{8}$$

For finding the solutions of (5) to (8) the time-dependent equations are made into time-independent ones by using the Fourier analysis technique (Mohan 1981).

As the perturbation in (1) is harmonic *i.e.* $v(\mathbf{r}, t) = - e\, \mathbf{E}\cdot\mathbf{r} \cos\omega t$, solution to (6) is written as

$$\psi_k^1(\mathbf{r}, t) = X_k^1(\mathbf{r}, \omega) \exp[i(\omega - \omega_k)t] + X_k^1(\mathbf{r}, -\omega) \exp[-i(\omega + \omega_k)t] \tag{9}$$

where the first term represents one photon emission and the second term represents one photon absorption process.
The second term of (6) can be written by using (3) as

$$v(\mathbf{r}, t)\, \psi_k^0 (\mathbf{r}, t) = -\frac{e\, \mathbf{E}\cdot\mathbf{r}}{2} \times \{\exp\,[+\, i\, (\omega - \omega_k)\, t]$$

$$+ \exp\,[-\, i\, (\omega + \omega_k)\, t]\}\, X_k^0 (\mathbf{r}) \tag{10}$$

Here $\omega_k = E_k/\hbar$ and ω is the frequency of the radiation.
Substituting (3) into (5), (9) and (10) into (6) and equating the coefficients of $\exp\,[-\, i\, (\omega - \omega_k)\, t]$ and $\exp\,[-\, i\, (\omega + \omega_k)\, t]$ the following set of equations is obtained.

$$(H - E_k)\, \chi_k^0 (\mathbf{r}, 0) = 0 \tag{11}$$

$$(H - E_k - \hbar\omega)\, \chi_k^1 (\mathbf{r}, -\, \omega) = (e,\, \mathbf{E}\cdot\mathbf{r}/2)\, \chi_k^0 (\mathbf{r}, 0) \tag{12}$$

$$(H - E_k + \hbar\, \omega)\, \chi_k^1 (\mathbf{r}, \omega) = (e\, \mathbf{E}\cdot\mathbf{r}/2)\, \chi_k^0 (\mathbf{r}, 0) \tag{13}$$

Clearly $\chi_k^0 (\mathbf{r}, 0)$, $\chi_k^1 (\mathbf{r}, -\omega)$, $\chi_k^1 (\mathbf{r}, +\, \omega)$ represents zero photon, one photon absorption and one photon emission processes respectively.
Similar to (9) the second order solution can be written as

$$\chi_k^2 (\mathbf{r}, t) = \chi_k^2 (\mathbf{r}, 2\, \omega)\, \exp\,[i\, (2\, \omega - \omega_k)]\, t + \chi_k^2 (\mathbf{r}, 0)$$

$$\exp\,(-\, i\, \omega_k\, t) + \chi_k^2 (\mathbf{r}, -\, 2\omega)\, \exp\,[-\, i\, (2\, \omega + \omega_k)]\, t \tag{14}$$

Substituting (14) and (9) into (7) and equating the coefficients of $\exp\,[i\, (2\omega - \omega k)\, t]$. $\exp\,(-\, i\omega_k\, t)$ and $\exp\,[-\, i\, (2\omega + \omega_k)]\, t$ the following set of equations is obtained,

$$(H - E_k + 2\, \hbar\, \omega)\, \chi_k^2 (\mathbf{r}, +\, 2\omega) = (e\, \mathbf{E}\cdot\mathbf{r}/2)\, \chi_k^1 (\mathbf{r}, +\, \omega) \tag{15}$$

$$(H - E_k)\, \chi_k^2 (\mathbf{r}, 0) = (e\, \mathbf{E}\cdot\mathbf{r}/2)\, (\chi_k^1 (\mathbf{r}, \omega) + \chi_k^1 (\mathbf{r}, -\, \omega)) \tag{16}$$

$$(H - E_k - 2\, \hbar\, \omega)\, \chi_k^2 (\mathbf{r}, -\, 2\omega) = (e\, \mathbf{E}\cdot\mathbf{r}/2)\, \chi_k^1 (\mathbf{r}, -\, \omega) \tag{17}$$

The solutions of equations (15) to (17), (i.e. $\chi_k^2 (\mathbf{r}, -\, 2\omega)$, $\chi_k^2 (\mathbf{r}, 0)$ $\chi_k^2 (\mathbf{r}, +\, 2\omega)$) represent two-photon absorption, zero photon and two photons emission processes respectively.
Similarly, the time-independent equations corresponding to higher order processes can be obtained.

3. First order processes

For a linearly polarized light and polarization along the polar axis we have

$$\mathbf{E}\cdot\mathbf{r} = (4\pi/3)^{1/2}\, |\, \mathbf{E}\, |\, r\, y_1^0 (\theta, \phi) \tag{18}$$

Substituting (18) into (12) we obtain

$$(H - E_k - \hbar\,\omega)\,\chi_k^1(\mathbf{r}, -\omega) = (4\pi/3)^{1/2}\, e\,|\,\mathbf{E}\,|\, r\, y_1^0(\theta,\phi)\,\chi_k^0(\mathbf{r},\omega) \quad (19)$$

Expanding $\chi_k^1(\mathbf{r},\omega)$ and $\chi_k^0(\mathbf{r},\omega)$ in terms of radial and angular part we have

$$\chi_k^1(\mathbf{r}, -\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \phi_{k,l}^1(\mathbf{r},\omega)\, y_l^m(\theta,\phi)$$

and

$$\chi_k^0(\mathbf{r}, 0) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \phi_{k\,l}^0(\mathbf{r},0)\, y_l^m(\theta,\phi) \quad (20)$$

Substituting (20) and (21) in (19) and using the property of spherical harmonics

$$y_1^0(\theta,\phi)\, y_l^m(\theta,\phi) = \left[\left\{\frac{(l+1-m)}{(2l+1)} \times \frac{(l+1-m)}{(2l-1)}\right\}^{1/2} y_{l+1}^m(\theta,\phi)\right.$$

$$\left. + \left\{\frac{(l+m)}{(2l+1)}\,\frac{(l-m)}{(2l-1)}\right\}^{1/2} y_{l-1}^m(\theta,\phi)\right] \quad (21)$$

we obtain

$$(H - E_k - \hbar\,\omega)\,\phi_{k,l}^1(\mathbf{r}, -\omega)\, y_l^m(\theta,\phi) = (4\pi/3)^{1/2}\, e\,|\,\mathbf{E}\,|\,\mathbf{r}\,\phi_{k',l}^0(\mathbf{r},0)$$

$$\times\left[\left\{\frac{(l+1+m)}{(2l+1)}\,\frac{(l+1-m)}{(2l-3)}\right\}^{1/2} y_{l+1}^m(\theta,\phi)\right.$$

$$\left. + \left\{\frac{(l+m)}{(2l+1)}\,\frac{(l-m)}{(2l-1)}\right\}^{1/2} y_{l-1}^m(\theta,\phi)\right] \quad (22)$$

Multiplying (23) by $y_{l'}^{m'}(\phi,\theta)$ and integrating over $d\Omega$ the following radial equation is obtained

$$(h(r) - E_k - \hbar\,\omega)\,\phi_{k,l'}^1(\mathbf{r}, -\omega) = (4\pi/3)^{1/2}\, e\,|\,E\,|\, r$$

$$\times\left[\left\{\frac{(l+1-m)}{(2l+1)}\,\frac{(l+1-m)}{(2l+3)}\right\}^{1/2} \delta_{l',\,l+1}\,\phi_{k',l}(\mathbf{r},0)\right.$$

$$\left. + \left\{\frac{(l+m)}{(2l+1)}\,\frac{(l-m)}{(2l-1)}\right\}^{1/2} \delta_{l',\,l-1}\,\phi_{k,l}^0(\mathbf{r},0)\right] \quad (23)$$

where $h(r)$ is the radial part of H and is given by

$$h(r) = -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right) + \frac{\hbar^2}{2m}\,\frac{l'(l'+1)}{r^2} + v(\mathbf{r}) \quad (24)$$

Using the property of Kronecker δ, (24) reduces to (changing the notation l' to l later on)

$$(h(r) - E_k - \hbar\omega)\,\phi^1_{k,l}(\mathbf{r}, -\omega) = A_l(k, \mathbf{r}) \tag{25}$$

where $$A_l(k, \mathbf{r}) = (4\pi/3)^{1/2}\, e\,|E|\, r \times \left[\left\{ \frac{(l+m)}{(2l-2)} \frac{(l-m)}{(2l+1)} \right\}^{1/2} \phi^0_{k,l-1}(\mathbf{r}, 0) + \left\{ \frac{(l+m+1)}{(2l+1)} \frac{(l-m+1)}{(2l-3)} \right\}^{1/2} \phi^0_{k,l+1}(\mathbf{r}, 0) \right]$$

The above equation is an inhomogeneous second order differential equation and the solution of this can be found out by finding the solution of the homogeneous part i.e.

$$(h(r) - E_k - \hbar\omega)\,\phi^1_{k,l}(\mathbf{r}, -\omega) = 0 \tag{26}$$

where $E_k = \hbar^2 k^2/2m$ is the energy of the free particle which is defined after one photon absorption as

$$E_{k_1} = \hbar^2 k_1^2/2m = E_k + \hbar\omega = \hbar^2 k^2/2m + \hbar\omega \tag{27}$$

Equation (26) is a second order radial equation with Coulomb potential $v(r)$ and can be solved easily (Mott and Massey 1965; Bethe and Salpeter 1957). It has two solutions one regular and another irregular defined by $L_l(kr)$ and $K_l(kr)$ respectively. The functional form and their behaviours near the origin and in the asymptotic limit is given in the appendix A. The solution of the inhomogeneous equation (Mott and Massey 1965) can thus be written as

$$\phi^1_{k,l}(\mathbf{r}, -\omega) = -L_l(k_1, r) \int_r^\infty K_l(k_1, r')\, A_l(k_1, r')\, r'^2\, \mathrm{d}r' - K_l(k_1, r) \int_0^r L_l(k_1, r')\, A_l(k_1, r')\, r'^2\, \mathrm{d}r' \tag{28}$$

From (28) the corresponding asymptotic solution is given by

$$\phi^1_{k,l}(\mathbf{r}, -\omega) \underset{r\to\infty}{\simeq} -K_l(k_1, r\to\infty) \int_0^\infty L_l(k_1, r')\, A_l(k_1, r')\, r'^2\, \mathrm{d}r' \tag{29}$$

For the determination of the scattering amplitude we are only interested in the outgoing solution. Therefore substituting $K_l(k_1, r\to\infty)$ from (24) $A_l(k, r')$ from (25) and taking the coefficient of the term

$$\exp[i(k_1 r - \alpha \log 2k_1 rr)]/r$$

the amplitude for one photon absorption is obtained as $f_l'(k_1, r)\, y_l^m(\theta, \phi)$ where

$$f_l'(k_1, r) = -\frac{1}{2ik}\left(\frac{4\pi}{3}\right)^{1/2} e\,|E|$$

$$\times \left\{\left[\frac{(l+m)}{(2l-1)}\frac{(l-m)}{(2l+1)}\right]^{1/2} \int_0^\infty \mathrm{d}r'\, r'^3 L_l(k_1, r')\, \phi^0_{k,l-1}(r')\right.$$

$$\left.+\left[\frac{(l+m+1)}{(2l+1)}\frac{(l-m+1)}{(2l+3)}\right]^{1/2} \int_0^\infty \mathrm{d}r'\, r'^3 L_l(k_1, r')\, \phi^0_{k,l+1}(r')\right\} \quad (30)$$

Taking the regular solution (Bethe and Salpeter 1957) for $\phi^0_{k,l+1}(r)$, substituting (A1) into (30) and performing the integral (Landau and Lifshitz 1959) we obtain

$$f_l'(k_1, r, -\omega) = -\left(\frac{\pi}{3}\right)^{1/2} \frac{e\,|E|}{(ik)} \exp\left[-\pi/2(\alpha+\alpha_1)\right] |\Gamma(l+1+i\alpha_1)|\, (4kk_1)^l$$

$$\times \left\{\left[\frac{(l+m)}{(2l-1)}\frac{(l-m)}{(2l+1)}\right]^{1/2} |\Gamma(2l+1+i\alpha)|\, (2k_1)\, J_\nu^{3,-2}(\beta, \beta_1)\right.$$

$$\left.+\left[\frac{(l+m+1)}{(2l+1)}\frac{(l-m+1)}{(2l+3)}\right]^{1/2} |\Gamma(l+i\alpha)|\, (2k_1)^{-1} J_\nu^{1,-2}(\beta, \beta_2)\right\} \quad (31)$$

where $\nu = (2l+2)$, $\beta = i\alpha_1 + l + 1$, $\beta_1 = i\alpha + 2l + 1$ and $\beta_2 = i\alpha + l$.

The term $J_\nu^{s,p}$ occurring in the above equation can be determined through the recurrence relation

$$J_\nu^{s,p}(\beta, \beta') = \frac{\nu - 1}{(-2ik_1)} \times \left\{J_\nu^{s,p-1}(\beta, \beta') - J_{\nu-1}^{s,p-1}(\beta - 1, \beta')\right\} \quad (32)$$

where $$J_\nu^{s+1,0}(\beta, \beta') = 1/(k^2 - k_1^2) \times \left\{[\nu(k - k_1) + 2i(k_1\beta - k\beta' + ks)]\, J_\nu^s(\beta, \beta')\right.$$

$$\left.+ s(\nu - 1 + s - 2\beta')\, J_\nu^{s-1,0}(\beta, \beta') + 2\beta' s\, J_\nu^{s-1,0}(\beta', \beta' + 1)\right\} \quad (33)$$

$$J_\nu^{0,0}(\beta, \beta_1) = (2l+2)\,\Gamma(2l+2)\,(k + k_1)^{\beta+\beta_1-\nu}$$

$$\times (k - k_1)^{-\beta}\,(k_1 - k)^{-\beta_1}\, F(\beta, \beta_1\, \nu, -4kk_1/(k - k_1)^2 \quad (34)$$

and $$J_\nu^{0,0}(\beta, \beta_2) = (2l+2)\,\Gamma(2l+2)\,(k + k_1)^{\beta+\beta_2-\nu}$$

$$\times (k - k_1)^{-\beta}\,(k_1 - k)^{-\beta_2}\, F(\beta, \beta_2\, \nu, -4kk_1/(k - k_1)^2) \quad (35)$$

Similarly the amplitude for one photon emission can be obtained from (31) by replacing k_1 by k_1' where k_1' is determined as

$$k_1' = (k^2 - 2m_1\omega/\hbar)^{1/2} \tag{36}$$

where m_1 is the electron mass.

Further replacement of k_1 by k_1' in (28) gives the first order solution for one photon emission. Thus substitution of $X_k'(r, \omega)$ and $X_k'(r, -\omega)$ as described in the above paragraph in (9), gives the first-order term occurring in (4) which describes the one photon absorption and emission processes. In § 4 the second order term and emission processes are discussed.

4. Second-order processes

Proceeding as in §2 and substituting (18) in (17) we get

$$(H-E_k-2\hbar\omega)\, X_k^2(\mathbf{r},-2\omega) = (4\pi/3)^{1/2}\, e\,|E|\, r\, y_1^0(\theta,\phi)\, \chi_k^1(\mathbf{r},-\omega) \tag{37}$$

for the plane-polarized light with polarization along the polar axis.

Expanding $\chi_k^2(\mathbf{r}, 2\omega)$ in terms of radial and angular part we have

$$\chi_k^2(\mathbf{r}, -2\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \phi_{k,l}^2(\mathbf{r}, -2\omega)\, y_l^m(\theta,\phi) \tag{38}$$

Substituting (38) and (20) into (37), using (A1), and providing as in §3 the following radial equation for the two photon absorption process is obtained

$$(h(r) - E_k - 2\,\hbar\omega)\, \phi_{k,l}^2(\mathbf{r}, -2\omega) = C_l(k_1, \mathbf{r}) \tag{39}$$

where
$$C_l(k_1, \mathbf{r}) = (4\pi/3)^{1/2}\, e\,|E|\, r \left\{ \left[\frac{(l+m)}{(2l-1)}\frac{(l-m)}{(2l+1)}\right]^{1/2} \phi_{k,l-1}^1(\mathbf{r}, -\omega) + \left[\frac{(l+m+1)}{(2l+1)}\frac{(l-m+1)}{(2l+3)}\right]^{1/2} \phi_{k,l-1}^1(\mathbf{r}, -\omega)\right\} \tag{39a}$$

The above equation is again a second-order inhomogeneous differential equation and the solution is given by

$$\phi_{k,l}^2(\mathbf{r}, -2\omega) = -L_l(k_2, r)\int_r^{\infty} K_l(k_2, r')\, C_l(k_1, r')\, r'^2\, dr' - K_l(k_2, r)\int_0^{r} L_l(k_2, r')\, C_l(k_1, r')\, r'^2\, dr' \tag{40}$$

where $L_l(k_2, r')$, $K_l(k_2, r')$ are the regular and irregular solutions defined in (A1), (A3) respectively with k_1 replaced by k_2 defined by

$$k_2 = (k^2 + 4\, m_1\, \omega/\hbar)^{1/2}$$

Clearly from (40) the asymptotic solution can be easily evaluated and is given by

$$\phi^2_{k,l}(\mathbf{r}, -2\omega) \underset{r\to\infty}{\simeq} K_l(k_2, r\to\infty) \int_0^\infty L_l(k_2, r')\, C_l(k_1, r')\, r'^2\, \mathrm{d}r' \tag{41}$$

Putting (39a) for $C_l(k_1, r)$, (A1) for $L_l(k_2, r)$ in the above equation, and performing the resulting integral as in § 2, the amplitude $f_l(k_2, r, 2\omega)$ for two photon absorption is obtained by taking the coefficient of the term $\exp[i(kr - \alpha \ln 2kr)]/r$ in the expansion of $K_l(k_2, r\to\infty)$.

Also we can obtain $\phi^2_{k,l}(r\to\infty, 2\omega)$ and corresponding amplitude for two photon emission from (41) by replacing k_2 by k_2' where k_2' is defined as

$$k_2' = (k^2 - 4\, m_1\, \omega/\hbar)^{1/2}$$

Similarly the second order term for zero-photon process can be obtained i.e. $\phi^2_{k,l}(r, 0)$. The corresponding asymptotic solution i.e. $\phi^2_{k,l}(r\to\infty, 0)$ obtained is given by

$$\phi^2_{k,l}(r\to\infty, 0) = -K_l(k, r\to\infty) \int_0^\infty L_l(k, r')\, B_l(k, r')\, r'^2\, \mathrm{d}r' \tag{42}$$

where $$B_l(k_1, r) = (4\pi/3)^{1/2}\, e\,|E|\, r \left[\frac{(l+m)}{(2l-1)}\frac{(l-m)}{(2l+1)}\right]^{1/2} \{\phi^1_{k,l-1}(\mathbf{r}, \omega) + \phi^1_{k,l-1}(\mathbf{r}, -\omega)\}$$

$$+ \left[\frac{(l+m+1)}{(2l+1)}\frac{(l-m+2)}{(2l+3)}\right]^{1/2} \{\phi^1_{k,l+1}(\mathbf{r}, \omega) + \phi^1_{k,l+1}(\mathbf{r}, -\omega)\}$$

Substituting $\phi^1_{k,l+2}(\mathbf{r}, \omega)$ from (28); $L_l(k, r')$, $K_l(k, r\to\infty)$ from Appendix A, and performing the integral in (42) as in § 2, we can easily obtain the second order elastic scattering amplitude.

Also substitution of the solutions thus obtained, i.e. $\phi^2_{k,l}(\mathbf{r}, \pm 2\omega)$ and $\phi^2_{k,l}(\mathbf{r}, 0)$ second order term $\psi^2_k(\mathbf{r}, t)$ can be obtained from (14).

In the next § the elastic scattering in the presence of laser plus coulomb field is discussed.

5. Elastic scattering

The wave-function (time independent) of elastic scattering can now be easily written from (4) by collecting the terms representing zero photon processes (*e.g.* $\chi^{2n}_k(\mathbf{r}, 0)$

where $n = 0, 1, 2, \ldots$etc.) from it. Expanding these terms in terms of radial and angular parts as in (21), the elastic scattering wavefunction is finally obtained as

$$\psi_{k,\,\text{elastic}}(\mathbf{r}) = \sum_{l=0}^{\infty} \xi_{k,\,l}(r)\, P_l(\cos\theta) \tag{43}$$

where $\xi_{k,\,l}(r) = (4\pi/2l+1)^{\frac{1}{2}} \times \{\phi^0_{k,\,l}(r, 0) + \phi^2_{k,\,l}(r, 0) + \phi^4_{k,\,l}(r, 0) + \ldots\}$ for $m = 0$ (or ϕ-independent). Also as the radial parts etc. are solutions of the second-order differential equation these can be written as the sum of the regular $L_l(k, r)$ and irregular solution $K_l(k, r)$, i.e.

$$\phi^0_{k,l}(r, 0) = A_1 L_l(k, r) + B_1 K_l(k, r) \tag{44}$$

$$\phi^2_{k,l}(r, 0) = A_2 L_l(k, r) + B_2 K_l(kr) \tag{45}$$

and so on. Substituting (44) and (45), etc. in (43) we obtain

$$\psi_{k,\,\text{elastic}}(r) = \sum_{l=0}^{\infty} G_l(kr)\, P_l(\cos\theta) \tag{46}$$

where
$$G_l(k, r) = (4\pi/2l+1)^{1/2} \times \{(A_1 + A_2 + \ldots)\, L_l(kr) + (B_1 + B_2 + \ldots)\, K_l(K r)\} \tag{46a}$$

In the asymptotic limit $G_l(kr)$ from (46a) caa be written as

$$G_l(kr) \simeq C_l \sin(kr - l\pi/2 - \alpha \log 2kr + \eta_l + \sigma_l) \tag{47}$$

where σ_l is the partial wave-phase shift due to EM interaction and is given by

$$\tan\sigma_l = (B_1 + B_2 + \ldots)/(A_1 + A_2 + \ldots)$$

while normalization constant C_l must be chosen so that we still have the Coulomb modified incoming plane wave plus an outgoing spherical wave namely

$$\sum_l G_l(kr)\, P_l(\cos\theta) \simeq \exp[-ikz + i\alpha \log(r - z) + [f_c(\theta) + f_m(\theta)]\frac{1}{r}\exp(ikr - i\alpha \log 2kr) \tag{48}$$

where f_c is the Coulomb scattering amplitude.

Using the asymptotic relation for Coulomb function, (48) can be written as

$$\sum_l G_l(kr)\, P_l(\cos\theta) \simeq \sum (2l+1)\, i^l \exp(i\eta_l)\, L_l(k, r \to \infty)\, P_l(\cos\theta) + f_m(\theta) \exp[i(kr - \alpha_1 \log 2\, kr)]\,/\,r \tag{49}$$

Also the amplitude due to non-Coulomb potential for interaction potential can be expanded as

$$f_m(\theta) = \sum_l a_l P_l(\cos\theta) \tag{50}$$

Substituting (47) and (50) on the left and right sides respectively of (49), we obtain

$$\begin{aligned} C_l \sin(kr - l\pi/2 - (\alpha \log 2kr + \eta_l + \sigma_l)) \\ = (2l+1)\, i^l \exp(i\eta_l) \sin(kr - l\pi/2 - \alpha \log 2kr + \eta_l) \\ + ka_l \exp i(kr - \alpha \log 2\, kr) \end{aligned} \tag{51}$$

where $L_l\,(k, r \to \infty)$ is given by (B2).

Equating the coefficient of $\exp\, i(kr - \alpha \log 2kr)$ and $\exp - i(kr - \alpha \log 2kr)$ on the left and right sides of (51) we get

$$C_l = (2l+1)\, i^l \exp i(\eta_l + \sigma_l) \tag{52}$$

and
$$a_l = (1/2ik)\,(2l+1) \exp(2i\eta_l) \exp(2i\sigma_l) - 1 \tag{53}$$

Substituting (53) into (52) the scattering amplitude in the presence of EM field and Coulomb field is obtained as

$$f_m(\theta) = 1/2ik \times \sum_{l=0}^{\infty} (2l+1) \exp(2i\eta_l)\, [\exp(2i\sigma_l) - 1]\, P_l(\cos\theta) \tag{54}$$

Substituting (54) into (48) the differential cross-section for elastic scattering in the presence of laser plus Coulomb field (Mott and Massey 1965) is obtained as

$$\frac{d\sigma}{d\Omega} = |f_c(\theta) + f_m(\theta)|^2 = R \frac{d\sigma}{d\Omega_c} \tag{55}$$

where
$$R = \left| 1 + \frac{f_m(\theta)}{f_c(\theta)} \right|^2;$$

$$f_c(\theta) = \alpha/2k \sin^2\theta/2 \exp[-i\alpha \ln(\sin^2\theta/2) + i\pi + 2i\eta_0]$$

$$f_m(\theta)/f_c(\theta) = -\,2\alpha \sin^2\theta/2 \exp(i\alpha) \ln(\sin^2\theta/2)$$

$$\times \sum_{l=0}^{\infty} (2l+1) \sin\sigma_l \exp(2i(\eta_l - \eta_0)) \times P_l(\cos\theta)$$

and
$$\eta_0 = \arg\Gamma(1 + i\alpha)$$

Equation (55) thus provides a formal solution of the elastic scattering problem in the presence of a laser beam. In the next section the results thus obtained are discussed.

6. Discussions

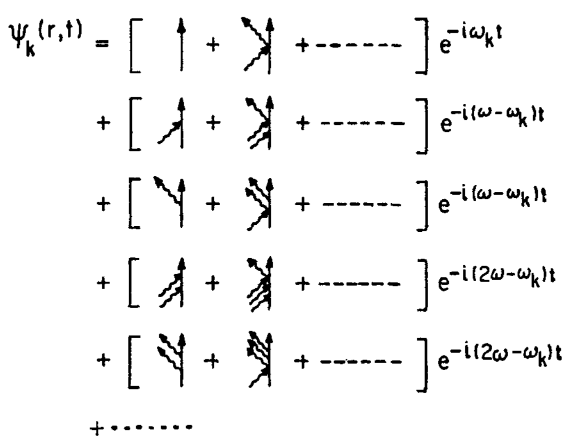
As described in earlier sections the positive energy state of a charge particle *i.e.* $\psi_k(\mathbf{r}, t)$ can be obtained using the Fourier analysis technique, in the presence of laser plus strong Coulomb field.

Substituting (3), (9) and (14) into (4) $\psi_k(\mathbf{r}, t)$ can be expanded in terms of χ's as

$$\begin{aligned}\psi_k(\mathbf{r}, t) = {} & [\chi_k^0(\mathbf{r}, 0) + \chi_k^2(\mathbf{r}, 0) + \ldots] \exp(-i\omega_k t) \\ & + [\chi_k^1(\mathbf{r}, -\omega) + \chi_k^3(\mathbf{r}, -\omega) + \ldots] \exp[-i(\omega + \omega_k)t] \\ & + [\chi_k^1(\mathbf{r}, \omega) + \chi_k^3(\mathbf{r}, \omega) + \ldots] \exp[i(\omega - \omega_k)t] \\ & + [\chi_k^2(\mathbf{r}, -2\omega) + \chi_k^4(\mathbf{r}, -2\omega) + \ldots] \exp[-i(2\omega + \omega_k)t] \\ & + [\chi_k^2(\mathbf{r}, +2\omega) + \chi_k^4(\mathbf{r}, 2\omega) + \ldots] \exp[-i(2\omega - \omega_k)t]\end{aligned} \tag{58}$$

where the first term represents zero photon process, the second and third terms represent one photon absorption and emission respectively, while the fourth and fifth terms represent two photon absorption or emission respectively, etc.

Equation (58) can be represented in terms of simple diagrams as shown below showing clearly the zero, one, two and higher order processes, *i.e.*



(59)

wherein (59) the symbol (↑) represents the potential line (either ionic or atomic depending upon the scattering system); wavy line towards potential line represents photon absorption; while wavy lir e away from potential line represents photon emission. Also (59) shows that $\psi_k(r, t)$ is a dressed state.

It is quite evident from (1) that for low values of electric field strength E(a.u.) $= 5 \times 10^{-6}$ (or $I \lesssim 10^6$ W/cm^2) the Coulomb field is quite dominant over the EM interaction term (*i.e.* $e|E|r/\sqrt{2}$) from small to larger values of radial distances (*e.g.* for r near the origin to r (a.u.) = 100). Thus for intensities $I \lesssim 10^6$ W/cm^2, the higher order terms in (58) will be smaller than the preceding lower order terms, resulting the series in (58) to be convergent. Therefore the positive energy state $\psi_k(r, t)$ obtained here for intensities $I \lesssim 10^6$ W/cm^2 will give fairly accurate results.

However, for higher value of intensities *e.g.* $E = 6\times10^{-2}$ (a.u.) (or $I = 10^{14}$ W/cm^2) the EM interaction term becomes equal to the Coulomb term at r(a.u.) = 5 beyond which the EM interaction dominates over Coulomb term, so that in the asymptotic limit, *i.e.* for higher values of radial distances and for $I \gtrsim 10^{14}$ W/cm^2, (1) reduces to the free-particle equation in the presence of the laser beam with the solution given by

$$\psi_k(\mathbf{r}, t) = \exp\{i\mathbf{k}\cdot\mathbf{r} - (i/\hbar)\sum_{-\infty}^{t} \tfrac{1}{2}m\,(\hbar k - (e\mathbf{E}/\omega)\sin\omega t')^2\,\mathrm{d}t' \tag{60}$$

as also derived by Kroll and Watson (1973).

Lastly, the analysis as described from §§ 2 to 4 is extended for both elastic and inelastic processes during charge-particle scattering in the presence of laser plus real potentials (*e.g.* Coulombic type in case of ions, etc.). As described in §4, the total elastic cross-section is found out by evaluating σ, the phase shift arising due to EM interaction.

It is hoped that the formulation of free-free transition with resonances in the presence of laser plus strong Coulomb field can be done exactly. Further work in this direction is in progress.

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Appendix

(A) The radial equation (26) is a second order differential equation and has the regular ($L_l(k_1, r)$) and irregular $K_l(k_1, r)$ positive energy states solutions which are defined as follows:

(a) *Regular solution*

$$L_l(k_1, r) = \exp(-\pi\alpha/2)\,|\Gamma(l+1+i\alpha_1)|\,(2k_1 r)^l \exp(-ik_1 r)\,[W_1 + W_2]$$

where $$W_1 + W_2 = F(i\alpha_1 + l + 1, 2l + 2, -2ik_1 r) \tag{A1}$$

with $$L_l(k_1 r) \underset{r\to\infty}{\simeq} 1/kr \sin(kr - l\pi/2 + \eta_l - \alpha_1 \log 2k_1 r) \tag{A2}$$

(b) *Irregular solution*

$$K_l(k_1 r) = i \exp(-\pi a/2) \,|\, \Gamma(l+1+i\alpha_1)/(2l+1)! \,|\, (2k_1 r)^l \exp(-ik, r)$$
$$\times \; [W_1(i\alpha_1 + l + 1, 2l+2, -2ik_1 r)$$
$$+ W_2(i\alpha_1 + l + 1, 2l+2, -2ik_2 r)] \qquad \text{(A3)}$$

with $$K_l(k_1 r) \underset{r\to\infty}{\simeq} 1/kr \cos(k_1, r - l\pi/2 + \eta_l - \alpha_1 \log 2k_1 r) \qquad \text{(A4)}$$

where $$W_1(a, b, z) = (\Gamma(b)/\Gamma(b-a))(-z)^{-a} g(a, a-b+1; -z) \qquad \text{(A5)}$$

$$W_2(a, b, z) = (\Gamma(b)/\Gamma(a)) e^z z^{a-b} g(1-a, b-a, z); \qquad \text{(A6)}$$

$$g(\alpha, \beta, z) \underset{z\to\infty}{\simeq} 1 + \alpha\beta/z + \alpha(\alpha+1)/z^2 \; \beta(\beta+1)/2! + \ldots; \qquad \text{(A7)}$$

$$\eta_l = \arg \Gamma(l + 1 + i\alpha_1) \qquad \text{(A8)}$$

and $$\alpha_1 = -ze^2/hv = -(ze^2/h^2)\, m_1/k_1 \qquad \text{(A9)}$$

Solutions near the origin $r \to 0$ takes the following form

$$L_l(k_1, r) \underset{r\to 0}{\simeq} C_l r^{l+1} \{1 + [(-2ik)/(l+1)]\, r + \ldots\} \qquad \text{(A10)}$$

and $$K_l(k_1, r) \underset{r\to 0}{\simeq} 1/(2l+1)\, C_l r^{-l} \left[1 + \begin{Bmatrix} 0\,(-2ikr \ln r) \text{ if } l = 0 \\ 0\,(-2ikr/l) \text{ if } l \neq 0 \end{Bmatrix}\right. \qquad \text{(A11)}$$

where $$C_l = 2^l \exp(-i\pi kr) \,|\, \Gamma(l + 1 - 2ikr)/(2l+1)! \,| \qquad \text{(A12)}$$

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