Pramāṇa, Vol. 20, No. 6, June 1983, pp. 547–558. © Printed in India.

Dead time corrections to photocounts and clipped correlations of Gaussian light

S K SRINIVASAN*
Department of Mathematics, National University of Singapore,
Kent Ridge, Singapore 0511
*On leave of absence from Indian Institute of Technology, Madras 600 036, India

MS received 12 January 1983

Abstract. A general method of dealing with the photocount distribution is proposed when dead time effects are to be taken into account. The clipped correlations of photocounts are investigated in detail and pointwise bounds for the correlation functions are explicitly provided when the dead times are not necessarily small compared to the sample time.

Keywords.

1. Introduction

The theoretical analysis of the photocount distribution of Gaussian light has of late gained practical importance. The spectral properties of light are best brought out by the detailed characteristics of photo-electron counting statistics. In particular the correlational structure of the point process of counting brings to the fore the detailed properties of the spectra of the incident beam. A number of surveys are now available (Cummins and Pike 1974; Pike 1969; Mehta 1970; Perina 1971) that describe the state-of-art of the subject. The recent book of Saleh (1978) and the article by Barkat and Blake (1980) deal with all the aspects of the different techniques of calculation. The continuing impact of the theoretical investigations on instrumentation and practical photometry is brought out in the recent survey by Meade (1981). The general method of attack originally due to Mandel (1963) makes good use of the fact that the photocounts are governed by the Poisson distribution characterized by the parameter W given by

$$W = \alpha E(T), \tag{1}$$

where α is the photoefficiency of the detector and $E(T)$ is the time integrated intensity over an interval of length T:

$$E(T) = \int_t^{t+T} I(t')\, \mathrm{d}t' \tag{2}$$

where $I(\cdot)$ is the (time) intensity function. The photocount distribution is arrived at by making an ensemble average of the Poisson distribution over E. For instance the one-fold generating function of the photocounts in an interval $(t, t+T)$ defined by

$$G_1(s, t, T) = \langle \exp - sw \rangle \tag{3}$$

essentially brings out the main characteristics of the integrated intensity. On the other hand if we shift our emphasis on the correlational structure, we may have to resort to a more detailed description of the counting process. This is achieved by studying the m-fold generating function $G_m(s_1, t_1, T_1; s_2, t_2, T_2; \ldots; s_m, t_m, T_m)$ where

$$G_m(s_1, t_1, T_1; s_2, t_2, T_2; \ldots; s_m, t_m, T_m) = \left\langle \exp - \sum_{i=1}^{m} s_i \int_{t_i}^{t_i+T_i} I(t)\,dt \right\rangle \tag{4}$$

Of course the explicit evaluation of the m-fold generating function in a handy closed form expression is by itself exacting. For the Gaussian light beam having a Lorentzian profile, explicit expressions for $G_1(s,t,T)$ and $G_2(s_1,t_1,T; s_2,t_2,T)$ were provided by Bedard (1966); Jakeman and Pike (1968) and Jakeman (1970). Srinivasan and Sukavanam (1971, 1972, 1977) and Srinivasan (1974a) had outlined a general method of arriving at $G_1(s,t,T)$ for light beams of fairly general spectral profiles. There were other attempts to obtain the general statistical characteristics by higher order photocount distribution particularly by Dialetis (1969) and Cantrell (1971). Srinivasan *et al* (1973) had indicated a general method of evaluating G_n for arbitrary intervals $(t_i, t_i + T_i)$. While these developments can be considered to bring the state-of-art of photocounting statistics to a satisfactory culmination point, there is still an aii of incompleteness from theoretical as well as experimental point of view in that the dead time effects are not taken into account. Viewed from a general angle inclusion of dead time in the type of analysis presented in the literature is fraught with difficulties. The dead time corrections to electron counters (with Poisson input) is treated extensively in the literature. When the input is a general renewal process, we obtain what is known in the literature (in probability) as censors and extensive results relating to the censors are also available (see for example the survey by Smith 1958). In the context of photo-electron statistics, De Lotto *et al* (1964) obtained corrections to the photon counting statistics. This was improved by Bedard (1967) who provided an explicit expression for the probability distribution of the number of photons detected in an arbitrary interval when the dead time is small compared to the duration of the time interval. Cantor and Teich (1975) had also discussed the corrections for various cases when dead time is small. The usefulness of these results is discussed in a recent paper by Mandel (1980). However the distribution of the photocounts for arbitrary a is a fairly open problem. More recently Srinivas (1981) has discussed the problem; however his results are expressed in the form of expected values not capable of being evaluated without recourse to the approximation mentioned above.

In earlier contributions Srinivasan (1975, 1978) had discussed the corrections that arise when dead times are of arbitrary duration. While an exact result was derived for the stationary rate of dead time corrected photocounts, the variance of the corrected counts could not be estimated to the same degree of accuracy; a lower bound for the variance was provided. In view of the recent advances in instrumentation techniques, it is necessary to see how the dead time corrections modify other characteristics of the photocounts. For instance the clipped correlation functions are extensively used in instrumentation. Jakeman *et al* (1971) had discussed this problem and provided corrections to clipped correlations of photocounts when the dead time is small compared to the lengths of the time intervals during which the channels are open. In view of the difficult and intractable nature of the problem, dead time corrections were investigated only under this approximation (see for example Cantor and Teich 1975; Mandel 1980). In a more recent communication (Srinivasan and Singh 1981), the corrections to the double clipped correlations were provided. The object of this contribution is to present further results in the same direction.

The layout of the paper is as follows. In § 2 we formulate the problem in terms of the counting process and summarize the results. In § 3, we deal with the corrections to the clipped correlations that arise when one of the channels is clipped at level 0. In § 4, we deal with the situation when one of the channels is clipped at higher levels.

2. Distribution of photocounts and dead time corrections

At the outset we note that the photocounts form a stationary point process on the time axis by virtue of the stationary Gaussian nature of the optical field. The point process is described by the family of probabilities $p_m(n_1, t_1, T_1; n_2, t_2, T_2; \dots; n_m, t_m, T_m)$ corresponding n_i photocounts in the interval $(t_i, t_i + T_i)$ $(i = 1, 2, \dots, m)$. If $V(t)$ represents the analytic signal, then corresponding to a specified sample function $V_s(t)$ of $V(t)$, p_m is given by (see Mandel 1963)

$$p_m(n_1, t_1, T_1; n_2, t_2, T_2; \dots; n_m, t_m, T_m)$$

$$= \sum_{i=1}^{m} \left(\alpha \int_{t_i}^{t_i+T_i} I_s(t)\,\mathrm{d}t\right)^{n_i} \left(\exp - \alpha \int_{t_i}^{t_i+T_i} I_s(t)\,\mathrm{d}t\right) \Big/ n_i! \tag{5}$$

where $\quad I_s(t) = V_s^*(t)\, V_s(t). \tag{6}$

Since $V_s(t)$ is the sample function of the optical field, an ensemble average over the right side of (5) leads to the final distribution of the number of photocounts. If $g_m(u_1, t_1, T_1; u_2, t_2, T_2; \dots; u_m, t_m, T_m)$ is the generating function of the probabilities p_m defined by

$$g_m(u_1, t_1, T_1; u_2, t_2, T_2; \dots; u_m, t_m, T_m)$$

$$= \sum P_m(n_1, t_1, T_1; n_2, t_2, T_2; \dots; n_m, t_m, T_m)$$

$$u_1^{n_1} u_2^{n_2} \dots u_m^{n_m} \, (|u_i| < 1, i = 1, 2, \dots, m) \tag{7}$$

then it follows that g_m is related to G_m (defined by 4), by

$$g_m(u_1, t_1, T_1; u_2, t_2, T_2; \dots; u_m, t_m, T_m)$$

$$= G_m[\alpha(u_1 - 1), t_1, T_1; \alpha(u_2 - 1), t_2, T_2; \dots; \alpha(u_m - 1), t_m, T_m]. \tag{8}$$

Thus the many fold (counting) characteristics of the photocounts are best described by the sequence of generating function G_m.

If at this stage, we introduce a dead time or resolution time of the detector, the resulting counting process (censored process) assumes great importance from both theoretical and experimental point of view. We shall assume that the dead time is of duration $\underset{\sim}{a}$ where $\underset{\sim}{a}$ is fully determinate. We denote the resulting process* of photocounts by $N^{\mathrm{I}}(\tau)$. Two of the main characteristics of the censored process $N^{\mathrm{I}}(\tau)$ are $f_a(\cdot)$ the stationary probability density function of the time interval between two successive counts and $h_1^c(\cdot)$ the stationary conditional intensity function (or conditional product density of degree one) defined by (see Srinivasan 1974b)

$$f_a(x) = \lim_{\Delta,\Delta'\to 0} \Pr\{N^{\mathrm{I}}(t_0+x, t_0+x+\Delta) = 1$$

$$> N^{\mathrm{I}}(t_0, t_0+x) \mid N^{\mathrm{I}}(t_0-\Delta', t_0) = 1\}/\Delta \tag{9}$$

$$h_1^c(x) = \lim_{\Delta,\Delta'\to 0} \Pr\{N^{\mathrm{I}}(t_0+x, t_0+x+\Delta) = 1 \mid N^{\mathrm{I}}(t_0-\Delta', t_0) = 1\}/\Delta \tag{10}$$

where for convenience we have used the notation $N^{\mathrm{I}}(a, b)$ to represent the number of dead time corrected photocounts in the interval (a, b). At this juncture it must be specially noted that while censors are treated quite extensively for stationary renewal processes, few results are available for more general processes of the type we encounter here. For Gaussian beams, Srinivasan (1978) has obtained the following results

$$\bar{F}_a(x) = \frac{1}{\bar{I}} \frac{\partial^2 G_2(s_1, t_1, T_1; s_2, t_2, T_2)}{\partial s_1\, \partial t_1} \tag{11}$$

where the derivative on the right side is evaluated at the point $s_1=0$, $s_2=1$, $t_1=t_0$, $T_1=a$, $t_2=t_0+a$, $T_2=x-a$ and $\bar{F}_a(*)$ is related to $f_a(\cdot)$ by

$$\bar{F}_a(x) = \int_x^\infty f_a(y)\,\mathrm{d}y. \tag{12}$$

From the general theory of stationary point processes it follows that the stationary rate of dead time corrected photocounts is given by $\beta_I\,\bar{I}$ where

$$\beta_I = \int_0^\infty \bar{F}_a(x)\,\mathrm{d}x \tag{13}$$

There appears to be no way of obtaining $h_1^c(\cdot)$; Srinivasan (1978) had obtained a (pointwise) lower bound for the same by replacing the event $N^{\mathrm{I}}(t_0+x, t_0+x+\Delta)$

*We use the superscript I in conformity with the terminology in vogue in the literature on type I counters.

=1 an appropriate event. Based on the lower bound for $h_1^c(\cdot)$ and equation (11) through (13) explicit formulae for the expected value of the corrected counts as well as a lower bound for the variance of the counts over an arbitrary interval were obtained. These results were extended to cover the case when dead times corresponding to different counts are independent and identically distributed. We now proceed to the corrections that arise when we deal with clipped correlation counts.

3. Single clipped correlation functions

We consider the experimental set-up when two independent counters are arranged to record the number of photons in the time intervals $(0,T)$ and $(t, t+T)$ where $t>T$ so that the two time intervals do not overlap. We shall also use the superscript I to denote the dead time corrected counts. In the photodetectors, the dead time a has the significance that it stands for the resolution time so that arrivals of photons if any during the dead time will not be counted. However there can be a fictitious counter whose dead time gets prolonged by a period equal to a (measured from the epoch of its arrival). Such counters are known as type II counters and since we shall make frequent use of such a process in obtaining bounds, we reserve the superscript II to denote the counts corresponding to such a censored process. Two types of clipped correlation functions are studied in the literature (see Jakeman *et al* 1971) corresponding to the situation when the photocounts in one or both of the channels $(0, T)$ $(t, t+T)$ are clipped at certain levels. These can be described by the functions $C_1(\alpha, k, t)$ $(\alpha = 1, 2)$ and $C_2(k, k', t)$ defined by

$$C_1(1, k, t) = E\,[N_k(0, T)\,N(t, t+T)], \tag{14}$$

$$C_1(2, k, t) = E\,[N(0, T)\,N_k(t, t+T)], \tag{15}$$

$$C_2(k, k', t) = E\,[N_k(0, T)\,N_{k'}(t, t+T)], \tag{16}$$

where $$N_k(t, t+T) = 1 \text{ if } N(t, t+T) > k = 0 \text{ otherwise.} \tag{17}$$

The dead time corrected clipped correlation functions will be distinguished from the corresponding uncorrected ones by a superscript D. In this section we deal with the functions that arise when clipping is done at the lowest level ($k = 0$).

We first proceed to obtain a lower bound for $C_1^D(1, 0, t)$. From the definition it follows

$$C_1^D(1, 0, t) = \sum_{m=0}^{\infty} m \Pr\{N^{\mathrm{I}}(0, T] \geqslant 1, N^{\mathrm{I}}(t, t+T] = m\}. \tag{18}$$

The right side of (18) cannot be easily estimated unless we spell out the epochs at which the various photons are detected in the channel $(t, t+T]$, a procedure which leads to the expectation value of an unwieldy expression. Hence it is prudent to replace the event $N^{\mathrm{I}}(t, t+T) = m$ by $N^{\mathrm{II}}(t, t+T) = m$ which in turn provides a lower bound. Thus we have

$$C_1^D(1, 0, t) \geqslant \sum m \Pr\{N^{\mathrm{I}}(0, T)] \geqslant 1, N^{\mathrm{II}}(t, t+T] = m\}. \tag{19}$$

The probability occurring on the right side of (19) can be written down for a specified sample path of the amplitude. We let $I(\cdot)$ denote the corresponding intensity fun- tion; then since the detector that detects the photons in the time interval is open at the beginning of the interval [0, T], we have

$$\Pr\{N^{\mathrm{I}}(0, T) \geqslant 1\}_{V(\cdot)}{}^{*} = \int_0^T \left[\exp - \int_0^u I(\tau)\,\mathrm{d}\tau\right] I(u)\,\mathrm{d}u \tag{20}$$

Next we note that the 'event' $N^{\mathrm{II}}(t, t+T) = m$ is independent of the 'event' $N^{\mathrm{I}}(0, T) \geqslant 1$ for a specified sample path of $V(\cdot)$ in as much as t is greater than T which in turn is usually of the order of several times a, the duration of the dead time. Thus we have

$$\sum_m m \Pr\{N^{\mathrm{I}}(0, T] \geqslant 1, N^{\mathrm{II}}(t, t+T] = m\}_{V(\cdot)}$$

$$= \Pr\{N^{\mathrm{I}}(0, T) \geqslant 1\}_{V(\cdot)}\, E\{N^{\mathrm{II}}(t, t+T]\}_{V(\cdot)}. \tag{21}$$

The conditional expected value of the censored photocounts in the interval $[0, T]$ can be expressed in terms of the unconditioned intensity function $h_1(\cdot)$ where

$$h_1(v) = \lim_{\Delta \to 0} \Pr\{N^{\mathrm{II}}(v, v+\Delta) = 1\}_{V(\cdot)}\}/\Delta; \tag{22}$$

$$E\{N^{\mathrm{II}}(t, t+T]\}_{V(\cdot)} = \int_t^{t+T} h_1(v)\,\mathrm{d}v. \tag{23}$$

An expression for $h_1(v)$ is obtained by arguing that in order that a photocount may be detected at the epoch v, it is necessary that no photon arrives in the interval $(v-a, v)$. Moreover since the counter is open only in the interval $(t, t+T)$, it is sufficient that the left hand end of the interval is t whenever $v - a$ is less than t. Thus we have for $T < t \leqslant v \leqslant t + T$,

$$h_1(v) = \exp\left[-\int_{\max(v-a,\, t)}^{v} I(\xi)\,\mathrm{d}\xi\right] I(v) \tag{24}$$

*We use the subscript $V(\cdot)$ to the braces under probability to signify that the probability in question is a conditional one.

Combining (21), (23) and (24) and taking expectation over all the sample paths of $V(\cdot)$, we have

$$\sum_m m \Pr\{N^{\mathrm{I}}(0, T) \geqslant 1, N^{\mathrm{II}}(t, t+T] = m\}$$

$$= \left\langle \int_0^T \left[\exp - \int_0^u I(\tau)\, d\tau\right] I(u)\, du \int_t^{t+T} I(v) \right.$$

$$\left. \left[\exp - \int_{\max(v-a,\, t)}^{v} I(\xi)\, d\xi\right] dv \right\rangle. \tag{25}$$

Splitting the range of integration over v, we find

$$\sum_{m=0}^{\infty} m \Pr\{N^{\mathrm{I}}(0, T) \geqslant 1, N^{\mathrm{II}}(t, t+T) = m\} = J_1 + J_2, \tag{26}$$

where
$$J_1 = \int_0^T du \int_{t+a}^{t+T} \left\langle \left[\exp - \int_0^u I(\tau)\, d\tau\right] I(u) \right.$$

$$\left. \left[\exp - \int_{v-a}^{v} I(\xi)\, d\xi\right] I(v) \right\rangle dv \tag{27}$$

$$J_2 = \int_0^T du \int_t^{t+a} \left\langle \left[\exp - \int_0^u I(\tau)\, d\tau\right] I(u) \right.$$

$$\left. \left[\exp - \int_t^{v} I(\xi)\, d\xi\right] I(v) \right\rangle dv. \tag{28}$$

The integrands of J_1 and J_2 can be identified as the deriative of G_2 at appropriate points. Thus we have

$$J_1 + J_2 = \int_0^T du \int_t^{t+T} \rho_2(u, v, a, t)\, dv, \tag{29}$$

where
$$\rho_2(u, v, a, t) = \frac{\partial^4 G_2(s_1, t_1, T_1; s_2, t_2, T_2)}{\partial s_1\, \partial s_2\, \partial T_1\, \partial T_2}, \tag{30}$$

where the derivative is evaluated at the point $s_1 = 1$, $s_2 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = v - a$, $T_2 = a$ if $t + a < u < t + T$; if u lies in the interval $(t, t + a)$, the derivative is evaluated at the point $s_1 = 1$, $s_2 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = t$, $T_2 = v - t$. An explicit expression for G_2 is available in the literature (see Srinivasan *et al* 1973) for beams with a Lorentzian spectral profile:

$$G_2(s_1, t_1, T_1; s_2, t_2, T_2) = g_1(T_1)\, g_2(T_2)\, [\exp - \Gamma(T_1 + T_2)].$$

$$- \frac{1}{4}\left(\frac{\Gamma}{p_1} - \frac{p_1}{\Gamma}\right)\left(\frac{\Gamma}{p_2} - \frac{p_2}{\Gamma}\right) \sinh p_1 T_1 \sinh p_2 T_2$$

$$\exp\{\Gamma[(T_1 - T_2) + 2(t_1 - t_2)]\}, \tag{31}$$

$$g_i(T) = \cosh p_i T + \frac{1}{2}\left(\frac{\Gamma}{p_i} + \frac{p_i}{\Gamma}\right) \sinh p_i T. \tag{32}$$

$$p_i^2 = \Gamma^2 + 2\Gamma \langle I \rangle s_i. \tag{33}$$

Thus (19) taken along with (26) through (33) provides a lower bound for $C_1^D(1, t, 0)$.

We next proceed to provide an upper bound. One bound is obviously $C_1(1, 0, t)$; however we would like to provide a better one. We observe from direct arguments

$$C_1^D(1, 0, t) = \sum_{m=0}^{\infty} m \Pr\{N^{\mathrm{I}}(t, t+T) = m\}$$

$$- \sum_{m=0}^{\infty} m \Pr\{N^{\mathrm{I}}(t, t+T) = m, N^{\mathrm{I}}(0, T) < 1\}. \tag{34}$$

The first term on the right side represents the expected value of the dead time corrected photocounts in the interval $(t, t+T)$ and hence equals $\beta_I, \bar{I}T$ where β_I is given by (13). Hence we can obtain an upper bound of $C_1^D(1, 0, t)$ by simply replacing $N^{\mathrm{I}}(t, t+T)$ by $N^{\mathrm{II}}(t, t+T)$* in (34). Thus, using the same type of arguments as before, we have

$$\sum_{m=0}^{\infty} m \Pr\{N^{\mathrm{I}}(t, t+T) = m, N^{\mathrm{I}}(0, T) < 1\}$$

$$> \sum_{m=0}^{\infty} m \Pr\{N^{\mathrm{II}}(t, t+T), N(0, T) = 0\}$$

$$= \left\langle \left[\exp - \int_0^T I(\xi)\, d\xi\right] \int_t^{t+T} \left[\exp - \int_{\max(t, v-a)}^{v} I(\tau)\, d\tau\right] I(v)\, dv \right\rangle. \tag{35}$$

*It is to be noted that since the detector is open only in the interval $(0, T)$, $N^{\mathrm{I}}(0, T) = 0$ and $N(0, T) = 0$ represent the same event.

Thus we can define $\rho_2(v, a, t)$ by

$$\rho_2(v, a, t) = -\frac{\partial^2 G_2(s_1, t_1, T_1; s_2, t_2, T_2)}{\partial s_2 T_2}, \tag{36}$$

where the derivative is evaluated at the point $s_1 = 1$, $t_1 = 0$, $T_1 = T$, $s_2 = 1$, $t_2 = v - a$, $T_2 = a$ for $t + a < v \leqslant t + T$ and at the point $s_1 = 1$, $t_1 = 0$, $T_1 = T$, $s_2 = 1$, $t_2 = t$, $T_2 = v - t$ for $t \leqslant v \leqslant t + a$; then we have the final result

$$C_1^D(1, 0, t) \leqslant \bar{I}\beta T - \int_t^{t+T} \rho_2(v, a, t)\,dv \tag{37}$$

Next we look at the dual problem when channel 2 is clipped at level 0. The clipped correlation function

$$C_1^D(2, 0, t) = \sum_m \Pr\{N^{\mathrm{I}}(0, T) = m,\ N^{\mathrm{I}}(t, t+T) > 0\}\, m. \tag{38}$$

Replacing the event $N^{\mathrm{I}}(0, T) = m$ by the event $N^{\mathrm{II}}(0, T) = m$ and proceeding as before, we finally obtain

$$C_1^D(2, 0, t) \geqslant \sum_m \Pr\{N^{\mathrm{II}}(0, T) = m,\ N^{\mathrm{I}}(t, t+T) > 0\}\, m$$

$$= \int_0^T du \int_t^{t+T} \rho_3(u, v, a, t)\,dv, \tag{39}$$

where $\rho_3(u, v, a, t)$ is given by

$$\rho_3(u, v, a, t) = \frac{\partial^4 G_2(s_1, t_1, T_1; s_2, t_2, T_2)}{\partial s_1\, \partial s_2\, \partial T_1\, \partial T_2}, \tag{40}$$

where the derivative is evaluated at the point $s_1 = s_2 = 1$, $t_1 = u - a$, $T_1 = a$, $t_2 = t$, $T_2 = v - t$ if u lies in the interval (a, T); on the other hand if u lies in the interval $(0, a)$, the derivative is evaluated at the point $s_1 = s_2 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = t$, $T_2 = v - t$.

To obtain an upper bound, we rewrite the right side of (37) as

$$C_1^D(2, 0, t) = \sum_m \Pr\{N^{\mathrm{I}}(0, T) = m\}\, m$$

$$- \sum \Pr\{N^{\mathrm{I}}(0, T) = m,\ N^{\mathrm{I}}(t, t+T) = 0\}\, m. \tag{41}$$

Using the independent behaviour of the detectors corresponding to the two channels, we replace the event $N^{\mathrm{I}}(0, T) = m$ by $N^{\mathrm{II}}(0, T) = m$ in the second term thus obtaining an upper bound:

$$C_1^D(2, 0, t) \leqslant \beta_I \bar{I} T - \int_0^T \left\langle \left[\exp - \int_t^{t+T} I(\xi)\, \mathrm{d}\xi\right] \right.$$

$$\left. \left[\exp - \int_{\max(0,\, u-a)}^{u} I(\tau)\, \mathrm{d}\tau\right] I(u) \right\rangle \mathrm{d}u$$

$$= \beta_I \bar{I} T - \int_0^T \rho_4(u, a, t)\, \mathrm{d}u, \tag{42}$$

where $\rho_4(u, a, t)$ is given by

$$\rho_4(u, a, t) = -\frac{\partial^2 G_2(s_1, t_1, T_1; s_2, t_2, T_2)}{\partial s_1\, \partial T_1}, \tag{43}$$

where the derivative of G_2 is evaluated at the point $s_1 = 1, s_2 = 1, t_1 = u - a, T_1 = a, t_2 = t, T_2 = T$ if $a < u \leqslant T$; if u lies in the interval $[0, a]$, the derivative is evaluated at the point $s_1 = 1, s_2 = 1, t_1 = 0, T_1 = u, t_2 = t, T_2 = T$.

4. Clipping at higher levels

So far we have dealt with the clipped correlation functions when one of the channels is clipped at zero level. The same technique can be used when clipping is effected at a higher level. For instance when clipping of channel 1 is effected at level 1, we have

$$C_1^D(1, 1, t) = \sum_m \Pr\{N^{\mathrm{I}}(0, T) \geqslant 2,\ N^{\mathrm{I}}(t, t+T) = m\}\, m,$$

or

$$C_1^D(1, 1, t) \geqslant \sum_m \Pr\{N^{\mathrm{I}}(0, T) \geqslant 2,\ N^{\mathrm{II}}(t, t+T) = m\}\, m. \tag{44}$$

The right side of inequality (44) can be evaluated explicitly. In view of the similarity of arguments, we skip the derivation and give the final result:

$$C_1^D(1, 1, t) \geqslant \int_0^{T-a} \mathrm{d}u \int_{u+a}^{T} \mathrm{d}v \int_t^{t+T} \rho_1^2(u, v, w, a, t)\, \mathrm{d}w, \tag{45}$$

where $\rho_1^2(u, v, w, a, t)$ is given by

$$\rho_1^2(u, v, w, a, t) = -\frac{\partial^6 G_3(s_1, t_1, T_1; s_2, t_2, T_2; s_3, t_3, T_3)}{\partial s_1\, \partial s_2\, \partial s_3\, \partial T_1\, \partial T_2\, \partial T_3}, \tag{46}$$

where the derivative is evaluated at the point $s_1 = s_2 = s_3 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = u + a$, $T_2 = v - u - a$, $t_3 = w - a$, $T_3 = a$ if w lies in the interval $(t + a, t + T)$; if w lies in the interval $(t, t + a)$, the derivative is evaluated at the point $s_1 = s_2 = s_3 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = u + a$, $T_2 = v - u - a$, $t_3 = t$, $T_3 = w - t$. Thus the bound can be explicitly calculated if we have an explicit expression for G_3. An expression for G_3 for Lorentzian profile has been derived by Srinivasan and Gururajan (1981) in the context of certain two-valued processes associated with a doubly-stochastic Poisson process:

$$1/G_3 (s_1, t_1, T_1; s_2, t_2, T_2; s_3, t_3, T_3)$$

$$= g_1 (T_1)\, g_2 (T_2)\, g_3 (T_3) - L_{12} (p_1, p_2) - L_{23} (p_2, p_3)$$

$$- L_{31} (p_3, p_1) - M (p_1)\, M (p_2)\, M (p_3), \tag{47}$$

where g_i and p_i are respectively defined by (32) and (33) and M and L_{ij} are given by

$$L_{ij} (p_i, p_j) = 2\, M (p_i)\, M (p_j) \exp [\Gamma (T_i - T_j) + 2\,\Gamma (t_i - t_j)] \tag{48}$$

$$M (p_i) = \frac{1}{2} \left(\frac{\Gamma}{p_i} - \frac{p_i}{\Gamma}\right) \sinh p_i\, T_i. \tag{49}$$

Thus (45) taken with (46) through (49) provides an explicit lower bound for $C_1^D (1, 1, t)$.

We finally state without proof the inequality expressing an upper bound for $C_1^D (1, 1, t)$:

$$C_1^D (1, 1, t) \leqslant \beta^I\, \bar{I} T - \int_t^{t+T} \rho_2 (v, a, t)\, \mathrm{d}v - \int_t^{t+T} \mathrm{d}v \int_0^{T-a} \rho_5 (u, v, a, t)\, \mathrm{d}u$$

$$- \int_t^{t+T} \mathrm{d}v \int_{T-a}^{T} \rho_6 (u, v, a, t)\, \mathrm{d}u, \tag{50}$$

where ρ_5 is given by

$$\rho_5 (u, v, a, t) = \frac{\partial^4 G_3 (s_1, t_1, T_1; s_2, t_2, T_2; s_3, t_3, T_3)}{\partial s_1\, \partial s_3\, \partial T_1\, \partial T_3}, \tag{51}$$

where the derivative is evaluated at the point $s_1 = s_2 = s_3 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = u + a$, $T_2 = T - u - a$, $t_3 = v - a$, $T_3 = a$ if u lies in the interval $(t + a, t+T)$; if u lies in the interval $(t, t + a)$ the derivative is evaluated at the point $s_1 = s_2 = s_3 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = u + a$, $T_2 = T - u - a$, $t_3 = t$, $T_3 = v - t$. The function ρ_6 is given by

$$\rho_6 (u, v, a, t) = \frac{\partial^4 G_2 (s_1, t_1, T_1; s_2, t_2, T_2)}{\partial s_1\, \partial s_2\, \delta T_1\, \partial T_2} \tag{52}$$

where the derivative is evaluated at the point $s_1 = s_2 = 1$, $t_1 = 0$, $T_1 = u$, $t_2 = v - a$, $T_2 = a$.

5. Conclusions

In an exactly similar way, we can obtain bounds for the function $C_1^D(2, 1, t)$. Finally we note that clipped correlation functions $C_1^D(a, k, t)$ for higher values of k can be expressed in terms of higher order generating functions of the photocount distribution. The evaluation of these generating functions is by no means easy and the method of obtaining bounds for the correlation functions becomes clumsy. It may be worthwhile to look for alternative methods of handling the problem. The author believes that this should involve direct handling of the physical process of detection. In the limited context of Lorentzian profiles, it is interesting to note that the Markov nature of the process has not been exploited. Until a new method of dealing with the detection process becomes available, one has to be content with the results reported in the present paper.

References

Barakat R and Blake J 1980 *Phys. Rep.* **60** 225
Bedard G 1966 *Phys. Rev.* **151** 1038
Bedard G 1967 *Proc. Phys. Soc. (London)* **A90** 131
Cantor B I and Teich M C 1975 *J. Opt. Soc. Am.* **65** 786
Cantrell C D 1971 *J. Math. Phys.* **12** 1005
Cummins H A and Pike E R 1974 *Light beating* and *Photon correlation spectroscopy* (New York: Plenum Press)
De Lotto L, Manfredi P F and Principo P 1964 *Energia Nucleare* **11** 557
Dialetis D 1969 *J. Phys. A* **2** 229
Jakeman E 1970 *J. Phys. A* **3** 201
Jakeman E, Oliver C J and Pike E R 1971 *J. Phys.* **4** 827
Jakeman E and Pike E R 1968 *J. Phys. A* **1** 128
Mandel L 1963 in *Progress in Optics* (ed) E Wolf (Amsterdam: North Holland) Vol. **2** Chap. 5
Mandel L 1980 *J. Opt. Soc. Am.* **70** 873
Meade M L 1981 *J. Phys. E.* **14** 909
Mehta C L 1970 in *Progress in Optics* (ed) E Wolf (Amsterdam: North Holland) Vol. **8** Chap. 8
Perina J 1971 *Coherence of Light* (London: van Nostrand Reinhold)
Pike E R 1969 *Piv Nuovo Cimento* **1** 271
Saleh B E A 1978 *Photoelectron statistics* (Berlin: Springer)
Smith W L 1958 *J. R. Stat. Soc.* **B20** 243
Srinivas M D 1981 *Pramana* **17** 203
Srinivasan S K 1974a *Phys. Lett.* **A47** 151
Srinivasan S K 1974b *Stochastic point processes and their applications* (London: Griffin)
Srinivasan S K 1975 *Phys. Lett.* **A50** 277
Srinivasan S K 1978 *J. Phys. A* **11** 2333
Srinivasan S K and Gururajan M 1981 *J. Math. Phys. Sci.* **15** 297
Srinivasan S K and Singh M M 1981 *Opt. Acta.* **28** 1619
Srinivasan S K and Sukavanam S 1971 *Phys. Lett.* **A35** 81
Srinivasan S K and Sukavanam S 1972 *J. Phys. A* **5** 682
Srinivasan S K and Sukavanam S 1977 *Phys. Lett.* **A60** 287
Srinivasan S K, Sukavanam S and Sudarshan E C G 1973 *J. Phys. A* **6** 1910