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Soliton-like solutions of some nonlinear theories and transformations among them

G P MALIK, J SUBBA RAO and GAUTAM JOHRI*
School of Environmental Sciences, Jawaharlal Nehru University, New Delhi 110 067, India
*Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India

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Abstract. The observation that the soliton-like solutions of a given second-order nonlinear differential equation define the separatrix of the equivalent autonomous system is used to obtain the one-soliton solutions for the ϕ^4 theories (the usual and the one with the wrong sign of the mass term), the ϕ^6, the ϕ^8, the sine-Gordon theories and the KdV equation. Transformations are given which transform the sine-Gordon equation into an equation belonging to the ϕ^{2n} class of theories. A procedure is evolved for obtaining the two-soliton solutions for the sine-Gordon theory without the use of Backlund transformations; it is suggested that this procedure may be useful for investigating the existence of similar solutions for theories of the polynomial type.

Keywords. Solitons; separatrix; transformations; multi-soliton solutions; nonlinear theories.

1. Introduction

Nonlinear theories which admit of soliton-like solutions have found increasing applications in physics in recent years. The phenomena for the description of which these theories are used or are likely to be relevant are as diverse as the behaviour of spin glasses (Bak and Jensen 1982), structural phase transition in ferroelectric crystals (Behera and Khare 1980), muscle-contraction (Davydof 1981), evolution of the early universe (Linde 1979), extended objects in field theory (Dashen *et al* 1974), chaos and turbulence (Bak and Jensen 1982), formation of bound states (Malik and Johri 1982), etc. The importance of the physics of solitons is also evident by the recent appearance of specialized texts on the subject (Eilenberger 1981; Rajaraman 1982).

This paper is motivated, in part, by the observation that in most of the above studies a soliton-like solution is simply written down after the relevant nonlinear equation is set up. It is then not clear to the uninitiated whether such a solution is found by trial and error or by criteria that fix the relevant constants of integration, etc., leading to the requisite solution in an unambiguous manner. This is the problem dealt with in § 2, where it is pointed out that the one-soliton solution of the original second order nonlinear differential equation may be found by converting it into an equivalent autonomous system of equations in two variables. For the latter system, the equation for the family of its paths in the phase plane is obtained containing one arbitrary constant. This constant is fixed so as to obtain the equation for the separatrix of the system. Then, a simple integration of the last equation yields the requisite one-soliton solution. Thus, the simple observation that a soliton-like solution defines the separatrix of the equivalent autonomous system may be used to obtain this solution unambiguously. We illustrate the method by solving the equations for the ϕ^4 theories (with the usual and the wrong sign of the mass terms), the ϕ^6, the ϕ^8, the sine-Gordon (SG) theories and the KdV equation.

Another motivation for the present paper is to point out links between the SG theory in $1+1$ dimensions and the class of ϕ^{2n} theories in $1+1$ dimensions: in § 3, we present transformations that transform the former to a member of the latter class. By means of these transformations, we obtain, from the known one-soliton solution of the SG equation, the one-soliton solutions for the two ϕ^4 theories, the ϕ^6 and the ϕ^8 theories. It then becomes natural to examine whether the knowledge of the multi-soliton solutions of the SG equation sheds any light on the existence of similar solutions for the theories of the polynomial class. With this in view, we take another look at the SG equation: from the point of view of § 2, rather than through the use of Backlund transformations, we reobtain the two-soliton solutions for this equation. Although we have been unable to construct any multi-soliton solutions for a theory of the type ϕ^{2n} it does seem that considerations of this section may prove to be useful in investigating their existence.

2. The separatrix and the one-soliton solutions

We adopt the following procedure to obtain the one-soliton solution of a given second order nonlinear differential equation: (a) convert the original equation into an equivalent autonomous system of equations, (b) find the critical points of this system; (c) obtain the equation for the family of paths in the phase plane of the system—this equation contains one arbitrary constant; (d) fix the constant so as to obtain the equation of the separatrix; then, (e) a simple integration of the last equation yields the requisite one-soliton solution unambiguously. Illustrative examples are given below.

2.1 *The usual ϕ^4-theory*

For the Lagrangian density

$$L = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{4}\lambda \phi^4, \tag{1}$$

the Euler-Lagrangian equation is

$$\phi_{tt} - \phi_{xx} = -m^2 \phi + \lambda \phi^3. \tag{2}$$

Let

$$\phi \to \frac{m}{\sqrt{\lambda}}\phi,\ x_\mu \to \frac{\sqrt{2}}{m} x_\mu, \tag{3}$$

then the scaled equation is

$$\phi_{tt} - \phi_{xx} = -2\phi + 2\phi^3. \tag{4}$$

For the static case, we have

$$\phi_{xx} = 2\phi - 2\phi^3. \tag{5}$$

The equivalent autonomous system is:

$$\phi_x = \eta + a,$$

$$\eta_x = 2\phi(1 - \phi^2), \tag{6}$$

and there is no loss of generality in setting $a=0$.
The critical points, (η, ϕ), are:

(0, 0): saddle point,

(0, $\pm$ 1): centres. (7)

The equation for the family of paths of (6) is

$$\tfrac{1}{2}\eta^2 = \phi^2 - \tfrac{1}{2}\phi^4 + k, \tag{8}$$

where k is a constant of integration; since the separatrix passes through the saddle point, $k=0$. (9)

Integration of (8) now yields

$$\phi(x) = \sqrt{2}\,\text{sech}\,[\pm\sqrt{2}(x + k_1)], \ (\phi^2 < 2), \tag{10}$$

where the constant k_1 signifies the translational invariance of the theory.

2.2 *ϕ^4-theory with wrong sign of the mass term*

We have

$$L = \tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}m\phi^2 - \tfrac{1}{4}\lambda\phi^4. \tag{11}$$

The scaled Euler-Lagrangian equation for the static case is

$$\phi_{xx} = -2\phi + 2\phi^3. \tag{12}$$

The equation of the separatrix of the equivalent system is

$$\tfrac{1}{2}\eta^2 = -\phi^2 + \tfrac{1}{2}\phi^4 + \tfrac{1}{2}, \tag{13}$$

whence $\phi(x) = \tan\text{h}\,(\pm x + k_1)$, (14) or, in terms of the original variables,

$$\phi(x) = \frac{m}{\sqrt{\lambda}} \tan h \left(\frac{m x}{\sqrt{2}} + k_1 \right). \tag{15}$$

This soliton-like solution lies on the separatrix for all values of λ and m.

We note that for the above theory another class of solutions has been given; see, for example, Aubry (1976):

$$\phi(x) = sn \left[\left(m^2 - \frac{\lambda}{2} \right)^{1/2} x \,\middle|\, \frac{\lambda}{2m^2 - \lambda} \right]. \tag{16}$$

This solution may be obtained as follows: the original equation,

$$\phi_{xx} = - m^2 \phi + \lambda \phi^3,$$

may be equivalently written as

$$\eta^2 = (\phi_x)^2 = C - m^2 \phi^2 + \tfrac{1}{2} \lambda \phi^4, \tag{17}$$

whence for the choice

$$C = m^2 - (\lambda/2). \tag{18}$$

Equation (16) emerges as the solution. While the solution in (16) is well-behaved and wave-like in the limit $\lambda \to 0$ unlike the solution in (15), it is not soliton-like for arbitrary λ, m. However, in the limit $\lambda = m^2$ the solution in (16) becomes $\tan h \ (mx/\sqrt{2})$ which is soliton-like. We note that in this limit the trajectory defined by (17) and (18) is, indeed, the separatrix of the equivalent system.

2.3 *The ϕ^6-theory*

We have (Behera and Khare 1980)

$$L = \tfrac{1}{2} (\partial_\mu \phi)^2 - B \phi^2 + A \phi^4 - C \phi^6. \tag{19}$$

The static Euler-Lagrangian equation is

$$\phi_{xx} = 2 B \phi - 4 A \phi^3 + 6 C \phi^5. \tag{20}$$

The equation for the separatrix of the system is

$$\tfrac{1}{2} \eta^2 = B \phi^2 - A \phi^4 + C \phi^6, \tag{21}$$

whence $$\phi(x) = \pm (A/4C)^{1/2} \left[1 + \tan h \sqrt{2B} (x + k_1) \right]. \tag{22}$$

In obtaining the solution (22) we have used the following condition for the vacuua to be degenerate

$$A^2 = 4\,BC \tag{23}$$

2.4 *The ϕ^8-theory*

We have (Lohe and O'Brien 1981)

$$L = \tfrac{1}{2}\,(\partial_\mu \phi)^2 - \lambda^3\,(\phi^2 - v^2)^2\,(\phi^2 - av^2)^2. \tag{24}$$

The static Euler-Lagrangian equation is

$$\phi_{xx} = 4\lambda^3\,\phi\,(\phi^2 - v^2)\,(\phi^2 - av^2)\,[2\phi^2 - (1 + a)\,v^2]. \tag{25}$$

The equation for the separatrix of the system is

$$\tfrac{1}{2}\,\eta^2 = \lambda^3\,(\phi^2 - v^2)^2\,(\phi^2 - av^2)^2, \tag{26}$$

whence $$\ln\left[\left(\frac{v + \phi}{v - \phi}\right)\left(\frac{v\sqrt{a} - \phi}{v\sqrt{a} + \phi}\right)^{1/\sqrt{a}}\right] = \pm\,\mu\,x + k_1, \tag{27}$$

where $\mu^2 = 8\,\lambda^3\,v^6\,(a - 1)^2.$

2.5 *The SG theory*

We have (Rajaraman 1975)

$$L = \frac{1}{2}(\partial_\mu \psi)^2 + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\,\psi\right) - 1\right], \tag{28}$$

with $\psi \to \dfrac{m}{\sqrt{\lambda}}\,\psi, \quad x_\mu \to \dfrac{1}{m}\,x_\mu,$

the static equation is

$$\psi_{xx} = \sin\psi. \tag{29}$$

The equation for the separatrix of the system is

$$\tfrac{1}{2}\,\eta^2 = -\cos\psi + 1, \tag{30}$$

whence $$\psi = 4\tan^{-1}[\exp(\pm\,x + k_1)]. \tag{31}$$

2.6 The KdV equation

We have (Eilenberger 1981)

$$\phi_t = 6\,\phi\,\phi_x - \phi_{xxx}. \tag{32}$$

For the case

$$\phi = \phi\,(x - vt) \equiv \phi(z), \tag{33}$$

we have

$$\frac{\partial}{\partial z}(3\phi^2 + v\phi - \phi_{zz}) = 0, \tag{34}$$

whence $\phi_{zz} = 3\phi^2 + v\phi + c.$

If we set $C = 0$, the equation for the separatrix is

$$\tfrac{1}{2}\,\eta^2 = \phi^3 + \tfrac{1}{2}\,v\phi^2, \tag{35}$$

and we obtain the following solitary wave solution

$$\phi = -\tfrac{1}{2}\,q^2 \operatorname{sec\,h}^2 [\tfrac{1}{2}\,q\,(x - vt)], \tag{36}$$

where $q = v^2$.

Without going into details, let us note that the one-soliton solutions in respect of other theories discussed in the literature, e.g., theories with interactions of the type

$$A \cos^4\left(\frac{\sqrt{\lambda}}{m}\,\psi\right) \text{ and } A\,\phi^2 \cos^2 [B \ln \lambda\,\phi^2]$$

can also be obtained by the above procedure. We conclude this section by pointing out that for theories with degenerate vacuua, the uniqueness of the separatrix implies uniqueness of the corresponding one-soliton solutions, upto a constant signifying translational invaiiance.

3. Links between the SG and the ϕ^{2n} theories in $1+1$ dimensions

It is interesting to note that in the static case the SG equations can be transformed to a polynomial theory of type ϕ^{2n}. For the soliton-like solutions that we are interested in, it is imperative that the transformations which convert the former to a member of the latter class satisfy the following criteria: the critical points of the SG equation and its separatrix be mapped, respectively, into the critical points and the separatrix of the requisite ϕ^{2n} theory. Some examples of the transformations which bring about such conversions are given below (see Appendix A).

Transformation	*Transformed theory*
$\sin \frac{\psi}{2} = \phi$	The usual ϕ^4-theory
$\cos \frac{\psi}{2} = \phi$	ϕ^4-theory with wrong sign of the mass term
$\cos \frac{\psi}{2} = 1 - \phi^2$	ϕ^6-theory
$\cos \frac{\psi}{2} = \phi^3 - (27/4)^{1/3}\phi$	ϕ^8-theory (with $\alpha = 4$, see (24))
$\sin \frac{\psi}{2} = \phi^3 - (27/4)^{1/3}\phi$	ϕ^8-theory with changed sign of the potential

We note that the above transformations ensure that the minima of the transformed theories are degenerate. Further, it is easy to check that in each of the above cases, the given transformation may be used to obtain the one-soliton solution of the transformed theory directly from the known one-soliton solution of the SG theory.

It is now natural to ask: does the above link between the SG theory and the theories of the polynomial class extend beyond the one-soliton solutions? The fact that multi-soliton solutions exist in the case of the SG theory and that these may be obtained through the Backlund transformations are of no avail, since Backlund transformations are not known to exist in the case of the theories of the polynomial class. It thus seems worthwhile to evolve a procedure to obtain the multisoliton solutions of the SG equation without the use of the Backlund transformations. In what follows, we confine ourselves to the consideration of the two-soliton solution.

Since the two-soliton solutions involve both x and t, the equation we have to consider is

$$\psi_{tt} - \psi_{xx} = - \sin \psi; \tag{37}$$

let us now define

$$z = - \ln g(t) + \ln f(x), \tag{38}$$

where the logarithms have been used for convenience. Substituting (38) into (37), we obtain

$$\psi_{zz} \left[(g_t/g)^2 - (f_x/f)^2\right] + \psi_z \left[-\frac{g_{tt}}{g} - \frac{f_{xx}}{f} + \left(\frac{g_t}{g}\right)^2 + \left(\frac{f_x}{f}\right)^2 \right]$$

$$= - \sin \psi. \tag{39}$$

Now, motivated by our experience with the one-soliton situation in § 2, we demand that (39) reduces to

$$\psi_{zz} = \sin \psi, \tag{40}$$

where $\psi = 4 \tan^{-1} \exp (\pm z)$, (41)

which is tantamount to requiring that $\psi(z)$ lies on the separatrix of the system (40). This obviously imposes restrictions on z, since we are demanding that, as $x \to \pm \infty$, the solution $\psi(z)$ must go over to the vacuua of the system (37). From (39) and (40) we get

$$\psi_{zz} \, [(g_t/g)^2 - (f_x/f)^2 + 1] + \psi_z \left[-\frac{g_{tt}}{g} - \frac{f_{xx}}{f} + \left(\frac{g_t}{g}\right)^2 + \left(\frac{f_x}{f}\right)^2 \right] = 0. \quad (42)$$

It is interesting to note that (42) is satisfied if

$$g = \exp (\gamma v t), \; f = \exp (\gamma x) \left(\gamma = \frac{1}{\sqrt{1 - v^2}} \right), \quad (43)$$

which, upon substitution into (41), lead to the one-soliton solution.

To obtain the two-soliton solutions, we note that (41) gives

$$\psi_{zz}/\psi_z = - \tan \mathrm{h} \, z, \quad (44)$$

whence we obtain from (42)

$$- \tan \mathrm{h} \, z \left[\left(\frac{g_t}{g}\right)^2 - \left(\frac{f_x}{f}\right)^2 + 1 \right] + \left[-\frac{g_{tt}}{g} - \frac{f_{xx}}{f} + \left(\frac{g_t}{g}\right)^2 + \left(\frac{f_x}{f}\right)^2 \right] = 0. \quad (45)$$

The solutions now are

$$g = \sin \mathrm{h} \, \gamma v t, \quad f = v \cos \mathrm{h} \, \gamma x; \quad (46)$$

$$g = \cos \mathrm{h} \, \gamma v t, \quad f = v \sin \mathrm{h} \, \gamma x; \quad (47)$$

which correspond to the soliton-antisoliton and soliton-soliton solutions, respectively.

The above procedure may be applied to other theories. However, we have not been able to obtain solutions other than the equivalent of (43). Whether this implies that the procedure is inadequate or that such solutions may not exist is an open question.

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Appendix A

In this Appendix we show how the equation for the ϕ^4-theory with the wrong sign of the mass term may be transformed into the equation for the SG theory.

The equation for the SG theory is equivalent to

$$\psi_x = \eta,$$
$$\eta_x = \sin \psi, \tag{A1}$$

the separatrix is defined by

$$\tfrac{1}{2}\eta^2 = -\cos\psi + 1$$
$$\equiv -2\cos^2\frac{\psi}{2} + 2, \tag{A2}$$

and enters the critical point $(0, \pm 2n\pi)$. The equation for the ϕ^4 theory is equivalent to

$$\phi_x = \chi,$$
$$\chi_x = -2\phi + 2\phi^3, \tag{A3}$$

the separatrix is defined by

$$\tfrac{1}{2}\chi^2 = -\phi^2 + \tfrac{1}{2}\phi^4 + \tfrac{1}{2}, \tag{A4}$$

and enters the critical points $(0, \pm 1)$. The transformation

$$\phi = f(\psi). \tag{A5}$$

maps (A4) into

$$\tfrac{1}{2}f_\psi^2\,\eta^2 = -f^2 + \tfrac{1}{2}f^4 + \tfrac{1}{2}. \tag{A6}$$

By substituting (A2) into (A6) we get

$$\sin(\psi/2)\,f_\psi = \tfrac{1}{2}(f^2 - 1). \tag{A7}$$

Integrating, we obtain

$$\frac{1}{2}\log\frac{1+f}{1-f} = \frac{1}{2}\log\frac{1+\cos\psi/2}{1-\cos\psi/2}, \tag{A8}$$

whence $\quad f = \cos\psi/2$

i.e., $\quad \phi = \cos\psi/2, \tag{A9}$

which maps the critical points of the two systems appropriately onto one another.

The other transformations listed in § 3 can be similarly obtained.

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