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Study of isotope shifts, isotone shifts and nuclear compressibility from the analysis of muonic x-ray transitions

K V SUBBA RAO and A A KAMAL
Physics Department, Osmania University, Hyderabad 500 007, India

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Abstract. Muonic x-ray transitions in various spherical nuclei in the region $13 \leqslant Z \leqslant 83$ have been analysed and the isotope and isotone shifts in charge radius R are investigated. Assuming $R = r_0 A^{1/3}$, the isotopic and isotonic behaviour of the parameter r_0 $(= RA^{-1/3})$ is also studied. The variation of r_0 with mass number A reveals the variation of average nucleon density, which in turn sheds light on the compressibility of nuclear matter. The isotope and isotone shifts in R exhibit the shell effects in the vicinity of magic neutron and proton numbers: 20, 28, 50, 82 and 126. The results indicate that neutron-proton interaction is maximum at the beginning of a major neutron shell and decreases gradually as the shell gets filled up. The behaviour of parameter r_0 clearly suggests that low-Z nuclei are highly compressible while high-Z nuclei are more or less incompressible. The parameter r_0 too is observed to exhibit profound shell effects.

Keywords. Muonic atoms; charge radius; isotope shifts; isotone shifts; nuclear compressibility.

1. Introduction

In the uniform model, the nuclear charge is assumed to be uniformly distributed in the nucleus with radius $R = r_0 A^{1/3}$. The parameter r_0 $(= RA^{-1/3})$ would then be directly proportional to the cube root of volume per nucleon or inversely proportional to the cube root of average nucleon density. Although R is the charge radius, it is reasonable to assume that r_0 represents volume per nucleon or average nucleon density, because the charge and mass distributions are more or less identical. The $A^{1/3}$ dependence of the radius R is supported in general by a great variety of experiments. In many investigations like scattering experiments, study of isotope and isotone shifts in optical spectra and in the evaluation of nuclear binding energies, the value of r_0 is assumed to be constant—each experiment choosing a different value. Since it is proportional to average nucleon density, constancy of r_0 invariably means the acceptance of incompressibility of nuclear matter. That this is not strictly valid is well established and many experimental discrepancies are attributed to this assumption. Analysis of muonic x-ray transitions under the assumption of uniform charge density facilitates the investigation of the variation of nuclear parameter r_0 with mass number A, which in effect represents the behaviour of average nucleon density. This in turn sheds light on nuclear compressibility. Assumption of uniform nuclear charge density simplifies the analysis of muonic x-ray transitions to a great extent and it was also found to yield results which are in good agreement with the experimental results (Subba Rao and Kamal 1980).

In the present study, the muonic x-rays have been analysed in various spherical nuclei ranging from $Z=13$ to $Z=83$. Isotope and isotone shifts in charge radius R have been estimated and shell effects have been observed. Shera *et al* (1976) analysed muonic x-ray transitions in a model-independent way and carried out an extensive study of isotope and isotone shifts in Fe-Zn region. Most of our observations in this region agree with the results of Shera *et al.* Isotopic and isotonic variation of nuclear parameter r_0 would furnish information regarding the variation of average nucleon density. It is observed that nucleon density increases with mass number A, quite rapidly in the beginning upto $A \approx 70$ and rather slowly thereafter. It is also found to exhibit profound shell effects.

2. Isotope and isotone shifts

Uniform charge distribution generates harmonic oscillator potential of the form:

$$V(r) = -\frac{3\,Ze^2}{2R} + \frac{Ze^2}{2R^3}r^2 \qquad r \leqslant R. \qquad \text{(1a)}$$

$$= -Ze^2/r \qquad r \geqslant R. \qquad \text{(1b)}$$

The energy eigenvalues for the muon in the electrostatic field of the nucleus are obtained by solving the Dirac equation, which upon reducing to a purely radial equation for a two-component wavefunction with small component f and large component g, yields a pair of coupled equations:

$$\frac{df}{dr} = \frac{kf}{r} - \frac{1}{\hbar C}[W - V(r) - \mu C^2]\,g, \qquad \text{(2a)}$$

$$\frac{dg}{dr} = \frac{1}{\hbar C}[W - V(r) + \mu C^2]\,f - \frac{kg}{r}. \qquad \text{(2b)}$$

The above equations are solved for the harmonic oscillator potential (equation (1)), the energy levels thus obtained are then corrected for vacuum polarisation effect. The method of solving the above equations and correcting the energy levels for vacuum polarisation are detailed in Barrett (1977) and Subba Rao and Kamal (1980).

2.1 *Low-Z region* ($13 \leqslant Z \leqslant 20$)

The shift in the $1s$ level of the muon due to the finite extension of the nuclear charge is about 50% in the region $Z \sim 82$ whereas it is only about 6% in the region $Z \sim 20$. In fact, the shift in the low-Z region is comparable with the experimental error and hence the analysis in this region is not very accurate. However, the accuracy is sufficient to study the trend in the behaviour of the nuclear parameter r_0.

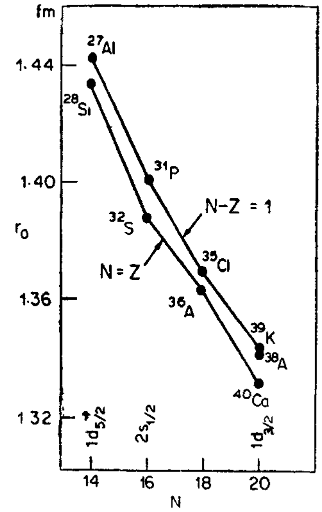

Figure 1. Variation of parameter r_0 with neutron number N in the low-Z region. The lines are drawn to guide the eye.

The values of muonic energy levels given by Barrett (1977) have been used in the present analysis. For the isotopes of $_{18}$A, experimental data from Engfer *et al* (1974) have been used. The Dirac equation is solved for a few trial values of charge radius R and in each case $1s_{1/2}$ level is calculated. The value of R is adjusted till the calculated value coincides with that given by Barrett (1977). In figure 1, the parameter r_0 calculated in each case is plotted against neutron number N. The value of r_0 is always less for even-Z nucleus as compared with its isotonic neighbour with odd-Z. This indicates that even-Z nuclei have smaller volume per nucleon or larger average nucleon density than their isotonic odd-Z neighbours. Due to lack of muonic x-ray data in this region for more isotopes, the isotope and isotone shifts in charge radius R could not be studied.

2.2 *Fe-Zn region* ($26 \leqslant Z \leqslant 30$)

In this region one can observe the effect of addition of nucleons on the nuclear radius and density in the vicinity of proton magic number 28. Isotope and isotone shifts have been accurately estimated and the shell effects are clearly observed. In evaluating the charge radius R, the experimental values of $2p_{3/2}-1s_{1/2}$ muonic transition energy (Shera *et al* 1976) have been used. The R value is determined according to the procedure outlined in § 2.1. In this region, the finite extension of nuclear charge density has the most pronounced effect on the binding energy of the $1s$ state and hence only $2p - 1s$ transitions show a finite size effect which is larger than the achievable experimental accuracy (Shera *et al* 1976). The experimental error leads to corresponding error in the estimation of R. The R value is evaluated in most of the nuclei to an accuracy of about 0·75 mfm. The estimated values of parameter r_0 are plotted against N, the neutron number in figure 2. In all the isotone sequences shown in figure 2, Ni has the least value of r_0, *i.e.*, the volume per nucleon is minimum in Ni isotopes. Thus a shell effect is clearly suggested.

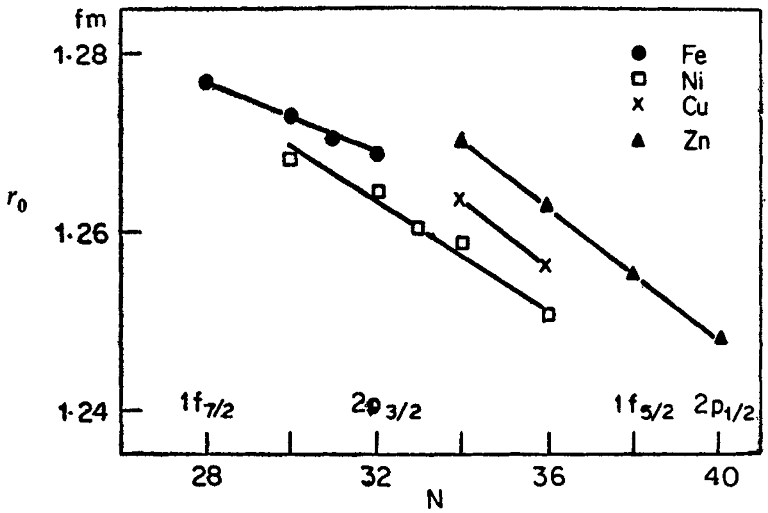

Figure 2. Variation of parameter r_0 with neutron number N in the Fe–Zn region. The lines are drawn to guide the eye.

Table 1. Differences in the charge radii for $\Delta Z = 2$.

Isotones	ΔR (mfm)
^{58}Ni – ^{56}Fe	38·61
^{60}Ni – ^{58}Fe	38·51
^{64}Zn – ^{62}Ni	96·76
^{66}Zn – ^{64}Ni	99·42
^{116}Sn – ^{114}Cd	16·3
^{126}Te – ^{124}Sn	59·6
^{140}Ce – ^{138}Ba	45·6

Another striking feature of the isotopic effect on r_0 is the uniform decrease with increasing N.

2.2a *Shifts between even nuclei.* Isotone shifts in R between even nuclei clearly indicate a strong shell effect (table 1). The addition of the two protons which complete the $1f_{7/2}$ proton shell (^{58}Ni-^{56}Fe or ^{60}Ni-^{58}Fe) causes an increase in R of about 38 mfm. But the addition of two more protons *i.e.*, the first two protons in the $2p_{3/2}$ shell (^{64}Zn-^{62}Ni or ^{66}Zn-^{64}Ni) causes an increase of about 98 mfm—a value which is more than twice as large as that for $1f_{7/2}$ shell. This feature is consistent with the shell model prediction that when an orbital is filled, additional protons must enter a higher shell and hence the isotone shift should increase. In other words, the $2p_{3/2}$ shell has a larger radius than the $1f_{7/2}$ shell.

The isotope shift in R for $\Delta N = 2$ is plotted in figure 3. The data cover the region between the beginning of the $2p_{3/2}$ neutron shell at $N = 28$ and the closure of the $2p_{1/2}$ neutron shell at $N = 40$. A prominent feature observed is the uniform decrease of isotope shift with increasing N. This implies that the neutron-proton interaction decreases in effect as one progresses through a major neutron

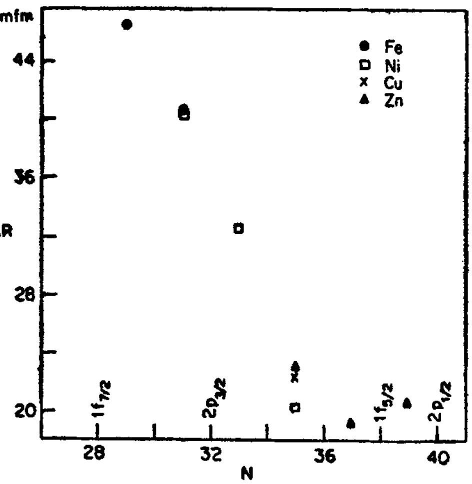

Figure 3. Isotope shifts in charge radius R for $\Delta N = 2$ in the Fe — Zn region.

Table 2. Differences in charge radii for $\Delta N = 2$.

Isotopes	ΔR (mfm)	Isotopes	ΔR (mfm)
^{56}Fe — ^{54}Fe	46·67	^{98}Mo — ^{96}Mo	38·8
^{58}Fe — ^{56}Fe	40·74	^{109}Ag — ^{107}Ag	26·1
^{60}Ni — ^{58}Ni	40·64	^{118}Sn — ^{116}Sn	13·5
^{62}Ni — ^{60}Ni	32·70	^{120}Sn — ^{118}Sn	19·0
^{64}Ni — ^{62}Ni	20·40	^{138}Ba — ^{136}Ba	9·8
^{65}Cu — ^{63}Cu	22·46	^{142}Ce — ^{140}Ce	35·7
^{66}Zn — ^{64}Zn	23·06	^{205}Tl — ^{203}Tl	13·2
^{68}Zn — ^{66}Zn	19·20	^{206}Pb — ^{204}Pb	14·5
^{70}Zn — ^{68}Zn	20·82	^{208}Pb — ^{206}Pb	13·5

shell. The values of ΔR also seem to indicate that isotope shift values are independent of Z. In fact, basing their argument on a similar observation Shera *et al* (1976) arrived at such a conclusion, which they pointed out as rather surprising. Independence of isotope shift from Z would mean that the added neutrons interact with the entire proton core wheıeas dependence on Z would imply that the added neutrons interact with the valence protons alone. It is reasonable to expect that the effect on the nuclear radius by the addition of neutrons to the closed proton shell in Ni isotopes would be considerably less than the corresponding effect in Cu or Zn; a closer inspection of the isotope shift values (figure 3) in the pairs (^{64}Ni, ^{62}Ni), (^{65}Cu, ^{63}Cu) and (^{66}Zn, ^{64}Zn) reveals that this is so. In the above pairs Ni has the least value of ΔR whereas in the pairs (^{58}Fe, ^{56}Fe) and (^{60}Ni, ^{58}Ni) ΔR is almost the same (table 2). The difference between the value of ΔR for Ni and for Cu and Zn isotopes ($\sim$ 2·3 mfm) is larger than the error ($\sim$ 1 mfm) on ΔR and hence cannot be ignored. This is clearly due to the effect of proton shell closure at $Z = 28$ on the isotope shift values. A possible explanation for the above observation is as follows. In Fe, the proton configuration is two short of the magic number 28, while the proton shell is completely filled in Ni. It is reasonable then to expect that the two neutrons 29th and 30th, added to the nuclei of Fe and Ni cannot distinguish the difference in the proton configuration in the nearly filled (Fe) and completely filled (Ni) proton shells. So, the interaction between the added neutrons and proton core in Fe and Ni is almost the same. Hence, the value of ΔR is same for the two pairs (^{58}Fe, ^{56}Fe) and (^{60}Ni, ^{58}Ni). The situation in Cu and Zn is altogether different. The 29th proton in Cu and 29th and 30th protons in Zn can be considered as valence protons and the added neutrons in these nuclei can be expected to interact predominantly with these valence protons. Consequently, the added neutrons clearly distinguish the difference in proton configuration in the nuclei Ni, Cu and Zn. Therefore, in the pairs (^{64}Ni, ^{62}Ni), (^{65}Cu, ^{63}Cu) and (^{66}Zn, ^{64}Zn) the value of ΔR is significantly different and has the least value in Ni isotopes.

2.2b *Odd-even shifts.* When comparing nuclei which differ by $\Delta A = 1$, the effect of an unpaired nucleon becomes evident. The radius R_A of an odd-mass isotope is closer to the neighbouring lighter even-mass isotope (R_{A-1}) than the neighbouring heavier even-mass isotope (R_{A+1}). This phenomenon is referred to as odd-even staggering. Odd-mass isotones are also found to exhibit such a staggering effect. To depict this effect, an odd-even staggering parameter γ is defined (Shera *et al* 1976) as

$$\gamma(A+1) = \frac{R(A+1) - R(A)}{1/2\{R(A+2) - R(A)\}}, \tag{3}$$

where A is even.

The values of staggering parameter now calculated are given in table 3 along with the values obtained by Shera *et al* (1976). The agreement between both the results is quite good. In isotopes, the odd-even staggering is clearly indicated. Among the isotone sequences (table 3), ^{59}Co, which is just below the major shell $Z = 28$, exhibits a large staggering effect ($\gamma = 0{\cdot}65$). In contrast, both ^{63}Cu and ^{65}Cu, which are immediately above the closed shell, exhibit small staggering ($\gamma = 0{\cdot}95$). All these observations provide independent evidence for the existence of strong shell effects in the vicinity of magic proton number 28.

Table 3. Odd-even staggering parameter values.

Isotopes	Present value	Value of Shera *et al* (1976)
^{56}Fe – ^{57}Fe – ^{58}Fe	0·89	0·88
^{60}Ni – ^{61}Ni – ^{62}Ni	0·76	0·72
^{206}Pb – ^{207}Pb – ^{208}Pb	0·74	...
Isotones		
^{58}Fe – ^{59}Co – ^{60}Ni	0.65	0.66
^{62}Ni – ^{63}Cu – ^{64}Zn	0.94	0.94
^{64}Ni – ^{65}Cu – ^{66}Zn	0·95	0.94
^{114}Cd – ^{115}In – ^{116}Sn	0.43	...
^{138}Ba – ^{139}La – ^{140}Ce	0.80	...

2.3 *Medium-Z region* ($39 \leqslant Z \leqslant 58$)

In this region one can observe the effect of addition of nucleons on the nuclear radius and density in the vicinity of magic neutron numbers 50 and 82, and the magic proton number 50. In most of the nuclei analysed, experimental values of both $2p$-$1s$ and $3d$-$2p$ transition energies are available (Engfer *et al* 1974). The $3d$-$2p$ transition energy in these nuclei is sensitive, though to a smaller extent, to the finite extension of nuclear charge density. Hence, both $2p$-$1s$ and $3d$-$2p$ transition energies have been used in the evaluation of charge radius R. But, only one transition energy from each of the two sets $3d$-$2p$ and $2p$-$1s$ can be used in evaluating R. This is because the energies of fine structure doublets of p and d levels are not so much different in their sensitivity to nuclear charge density as to elicit any further significant information (Devons and Duerdoth 1969). The Dirac equation is solved for a few trial values of R and in each case $2p_{1/2}$-$1s_{1/2}$ and $3d_{3/2}$-$2p_{1/2}$ transition energies are evaluated. Comparing these transition energies with the experimental values, χ^2 for each trial value of R is calculated. The χ^2 value calculated is the sum of chi-squares evaluated separately for the two transitions $2p_{1/2} - 1s_{1/2}$ and $3d_{3/2} - 2p_{1/2}$. The value of R, for which χ^2 is minimum, leads to the best value of charge radius. The error that is associated with R in medium and high-Z regions is about 1-5 mfm.

2.3a *Shifts between even nuclei.* The isotope shift ΔR for $\Delta N = 2$ is plotted in figure 4. The magic numbers 50 and 82 are covered by the data. The most remarkable feature of the isotope shift is that it is maximum at the beginning of two major neutron shells at 50 and 82 ($^{98-96}$Mo and $^{142-140}$Ce) and decreases gradually as the shell gets filled up. In the vicinity of the magic number 82, the shell effect is very prominent. The general tendency exhibited by ΔR in this region is quite similar to that in Fe-Zn region (figure 3). As pointed out earlier, this tendency clearly indicates that neutron-proton interaction is maximum at the beginning of a major neutron shell and decreases gradually as one progresses through the shell. The data available are not sufficient to investigate the Z-dependence of isotope shift.

One can expect a large jump in the isotone shift in the vicinity of magic proton number 50, similar to that observed for the magic proton number 28. In fact, such a jump is very prominently observed (table 1). The addition of two protons which complete the $1g_{9/2}$ proton shell (^{114}Cd—^{116}Sn) causes an increase in R of about

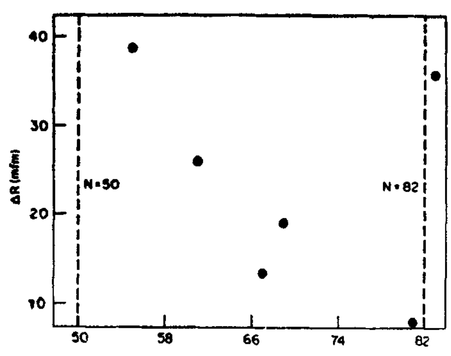

Figure 4. Isotope shifts in charge radius R for $\Delta N = 2$ in the medium-Z region.

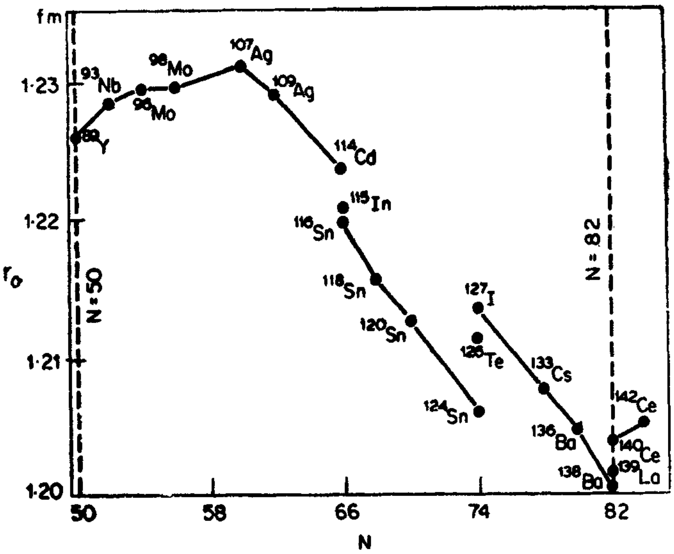

Figure 5. Variation of parameter r_0 with neutron number N in the medium-Z region. The lines are drawn to guide the eye.

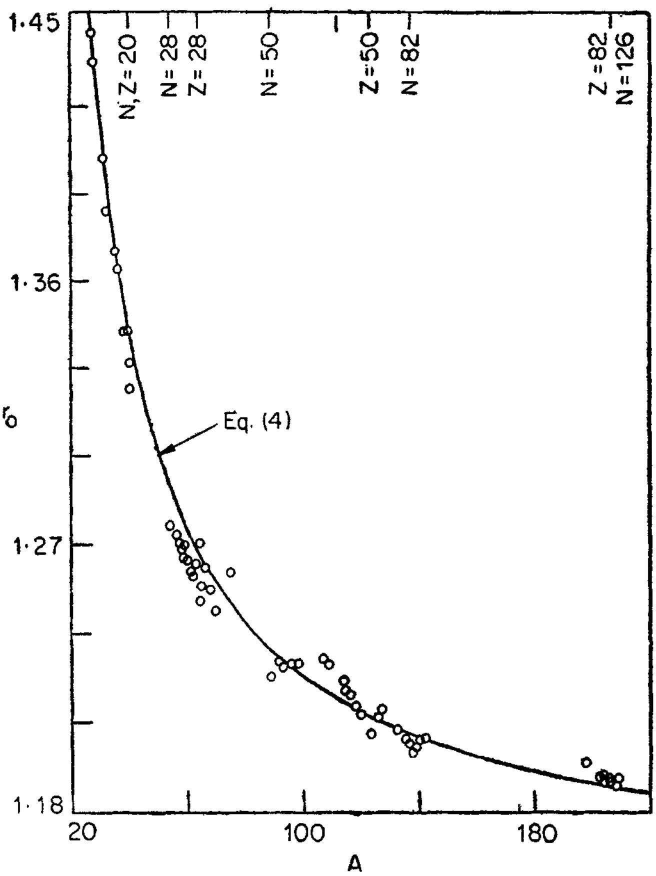

Figure 6. Parameter r_0 as a function of mass number A.

16 mfm. But the addition of two more protons *i.e.*, the first two protons in the $1g_{7/2}$ shell (^{124}Sn—^{126}Te) causes an increase of about 60 mfm—a value which is nearly four times as large as in the former case.

2.3b *Odd-even shifts.* In the two isotone sequences studied, ^{115}In, which is just below the major shell $Z = 50$, exhibits quite a large staggering effect ($\gamma = 0{\cdot}43$). A smaller but significant staggering effect ($\gamma = 0{\cdot}80$) is observed in ^{139}La, which is much above the $Z = 50$ shell. In figure 5, which shows the variation of r_0 with N, the tendency of r_0 to decrease gradually as N increases can be prominently observed. But the effects of neutron shell closure at $N = 50$ and 82, and proton shell closure at $Z = 50$ are superimposed over this general behaviour. There is a perceptible increase, in the value of r_0 at the beginning of major neutron shells $N = 50$ and 82 (^{89}Y, ^{91}Nb, ^{138}Ba and ^{142}Ce). In the two isotone sequences ^{114}Cd, ^{115}In, ^{116}Sn and ^{124}Sn, ^{126}Te, ^{127}I, the isotopes of Sn have the least value of r_0—as in the case of Ni (figure 2). Thus, the effects of both neutron shell closure at $N = 50$ and 82 and proton shell closure at $Z = 50$ are clearly evident from the foregoing results.

2.4 *High-Z region* ($Z \simeq 82$)

Experimental data (Engfer *et al* 1974) available for a few spherical nuclei in the high-Z region have been analysed and charge radius R is evaluated (figure 7) according to the procedure outlined in § 2.3. The data available are not sufficient to warrant any comments on the isotopic and isotonic behaviour of ΔR and r_0 in this region. However, a few shell effects are observed. The value of r_0 decreases very slowly (figure 6) with A and is minimum for the doubly magic ^{208}Pb. A prominent staggering

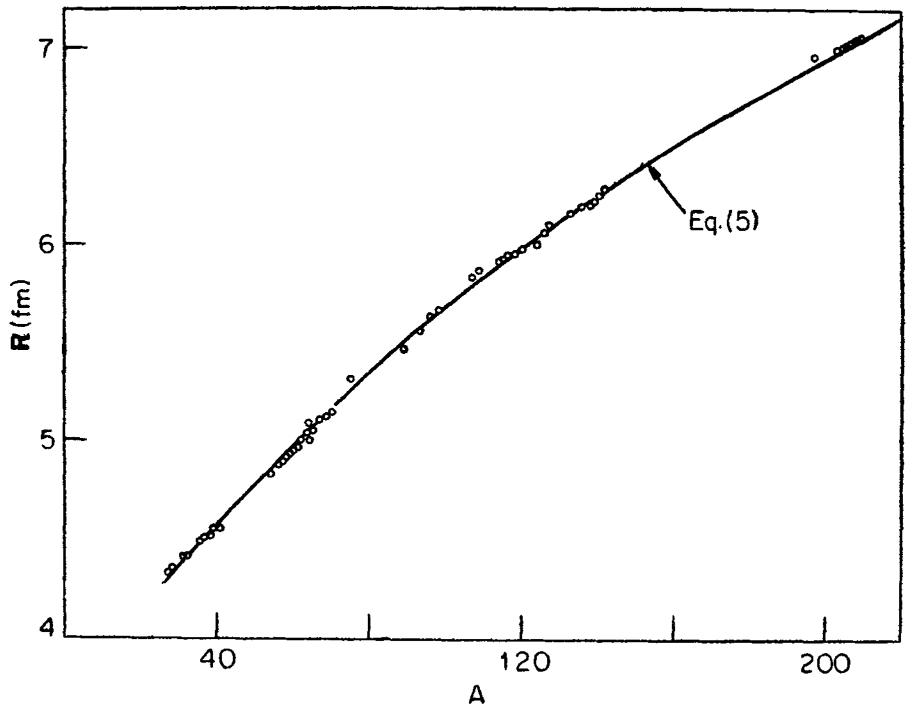

Figure 7. Charge radius R as a function of mass number A.

effect is exhibited by ^{207}Pb, which lies just below the closed neutron shell $N = 126$ (table 3).

3. Average nucleon density and nuclear compressibility

Isotopic behaviour of the parameter r_0 (figures 1, 2 and 5) clearly reveals the tendency of average nucleon density to increase uniformly with neutron number N. This apart, the effects of neutron shell closure at $N = 50$ and 82, and proton shell closure at $Z = 28$, 50 and 82 are clearly observed. In figure 6 the variation of parameter r_0 with mass number A is depicted. It is observed that r_0 decreases from a value of about 1.44 fm at $A = 27$ to a value of about 1.19 fm at $A = 209$. But the decrease is very rapid in the region $A \leqslant 70$. Thereafter it decreases slowly and in the high-Z region its variation is very small. In other words, this behaviour indicates that the average nucleon density increases very rapidly upto $A \approx 70$, gradually thereafter and becomes almost constant in the region of heavy nuclei. This phenomenon indicates that heavy nuclei are compressible to a very small extent while the light nuclei are highly compressible. Medium heavy nuclei exhibit a behaviour which is intermediate between these two extreme cases. The general behaviour of average nucleon density is punctuated with profound shell effects (figure 6). Shell effects exhibited by average nucleon density curve provide an independent evidence for the nuclear shell structure.

As already pointed out, the behaviour of parameter r_0 depicts the compressible nature of nuclei. It is observed that the behaviour of parameter r_0 can be approximately represented by an equation of the form

$$r_0 = a \exp (b/A), \tag{4}$$

where a and b are constants. The best fit to the values of r_0 yields the values: $a = 1.1553$ fm and $b = 5{\cdot}837$. As (4) represents the gross behaviour of r_0 the shell effects are not expected to be reproduced (figure 6).

In view of (4), the general relation for charge radius, $R = r_0 A^{1/3}$ gets modified as

$$R = a \exp (b/A) A^{1/3}. \tag{5}$$

The values of R estimated from (5) are plotted in figure 7 along with the values of R determined in the present work, the agreement between the both being quite close. In view of the exponential term in it, (4) is not valid in very light nuclei ($A \leqslant 25$).

4. Conclusions

It is clearly observed that isotope and isotone shifts in charge radius R exhibit strong shell effects (figures 3 and 4; tables 1 and 2) in the vicinity of neutron and proton magic numbers 20, 28, 50, 82 and 126. The odd-even staggering is prominently observed and the staggering parameter exhibits shell effects very clearly (table 3). The results also indicate that neutron-proton interaction is maximum at the beginning of a major neutron shell and decreases gradually as the shell gets filled up. The effect of proton shell closure on isotope shift in R is clearly observed at $Z = 28$. It is quite reasonable to conclude from this observation that atleast at the beginning of a major proton shell, the added neutrons interact predominantly with valence protons rather than with the whole proton core. It is possible that as the proton shell gets filled up, the added neutrons interact with the whole proton core and consequently the isotope shifts become Z-independent. However, the data available in the vicinity of other major proton shells ($Z = 20$, 50, and 82) are not sufficient to generalise such a conclusion.

The present study demonstrates that analysis of muonic atoms provides an elegant method to investigate nuclear compressibility. As evidenced from the behaviour of parameter r_0, low-Z nuclei are highly compressible whereas high-Z nuclei are compressible to a very small extent; apart from this profound shell effcts are also observed (figure 6).

The behaviour of r_0 can be incorporated into the semi-empirical mass formula through the Coulomb energy term. This procedure permits investigation of the effect of nuclear compressibility on nuclear binding energies. Such an investigation is in progress.

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