

Bounds on the effective thermal conductivity of two-phase systems

R N PANDE, V KUMAR and D R CHAUDHARY
Department of Physics, University of Rajasthan, Jaipur 302 004, India.

MS received 27 September 1982; revised 9 February 1983

Abstract. The weighted geometric mean of resistors considered for determining the effective thermal conductivity K_E of two-phase systems has been optimised. Solutions of the equations lead to a useful set of bounds. When compared with other bounds the present bounds give the better results in estimating the upper and lower values of the effective thermal conductivity of a two-phase system.

Keywords. Bounds; effective thermal conductivity; variational principles; thermal conductivity.

1. Introduction

The lack of a general theoretical expression to determine the effective thermal conductivity K_E , of all types of two-phase systems has led to attempts to find bounds on K_E . The bound-technique is important in estimating the optimum values of K_E of a two-phase system. Better the set of bounds, lesser the difference between two optimum values of K_E . In the literature different bounds (Hashin and Shtrikman 1962; Kumar and Chaudhary 1980; Prager 1969; Schulgasser 1976; Beran 1965; Miller 1969; Beran *et al* 1971 and Hori 1973) have been proposed. But no serious attempt has been made to obtain bounds considering a system as made of resistors, whereas the resistor models (Sugawara and Yoshizawa 1961; Tye 1969) are more suitable in predicting the K_E of two-phase systems. The present work which is a continuation of our previous work (Kumar *et al* 1980) considers the bounds on K_E of two-phase systems.

2. Theory

Taking the weighted geometric mean of resistors, the K_E of two-phase system is given (Chaudhary and Bhandari 1967) by

$$K_E = K_{\parallel}^n K_{\perp}^{1-n}, \quad (1)$$

where $K_{\parallel} = (1 - \phi) K_s + \phi K_f,$ (2a)

and $K_{\perp} = \left\{ \frac{(1 - \phi)}{K_s} + \frac{\phi}{K_f} \right\}^{-1}$ (2b)

Here K_f and K_s are the thermal conductivities of fluid and solid phases respectively and ϕ is the volume fraction of fluid phase in the two-phase system, n and $(1-n)$ are the corresponding probabilities of orientation of the sample in parallel and perpendicular directions of heat flow.

On optimising K_E with respect to n , we have,

$$\frac{1}{K} (\partial K / \partial \phi) = \frac{K_f - K_s}{\phi (K_f - K_s) + K_s}, \text{ whence } n \rightarrow 1, \quad (3)$$

and
$$\frac{1}{K} (\partial K / \partial \phi) = \frac{K_f - K_s}{K_f - \phi (K_f - K_s)}, \text{ whence } n \rightarrow 0. \quad (4)$$

Applying the boundary conditions of ϕ to (3) and (4), we have,

$$K_s = K_f \left\{ 1 - \frac{1}{K} \left(\frac{\partial K}{\partial \phi} \right) \right\}, \text{ for } n \rightarrow 1, \phi \rightarrow 1, \text{ and } n \rightarrow 0, \phi \rightarrow 0. \quad (5)$$

and
$$K_f = K_s \left\{ 1 + \frac{1}{K} \left(\frac{\partial K}{\partial \phi} \right) \right\}, \text{ for } n \rightarrow 1, \phi \rightarrow 0, \text{ and } n \rightarrow 0, \phi \rightarrow 1. \quad (6)$$

The solutions of the differential equations (5) and (6) (Kumar and Chaudhary 1980) yield bounds on K_E of that system, whose phase particles are oriented either parallel or perpendicular to the direction of heat flow. Also the volume fraction of dispersed phase, considered in the solution, is either zero or unity, but a natural sample does not agree with (5) or (6).

In the present work attempts have been made to obtain governing equations under the general conditions of n and ϕ . The values of the slope $1/K(\partial K/\partial \phi)$ obtained from (5) and (6), correspond to a case when a two-phase system approaches a single phase system, which does not agree with the basic assumption of a system being a two-phase system. Therefore, solutions of $1/K(\partial K/\partial \phi)$ are to be evaluated under the condition $0 < \phi < 1$. For this purpose, we have obtained two values of $1/K(\partial K/\partial \phi)$. One is obtained by taking products of (5) and (6) and the other by the division of relations (5) and (6).

The multiplication of (5) and (6), gives,

$$\frac{1}{K} (\partial K / \partial \phi) = \frac{K_f - K_s}{(K_s K_f)^{1/2}} = a \text{ (say)}. \quad (7)$$

Their division, leads to

$$\frac{1}{K} (\partial K / \partial \phi) = \left(\frac{K_f^2 - K_s^2}{K_f^2 + K_s^2} \right) = b \text{ (say)}. \quad (8)$$

Solutions of (7) and (8) are of the form

$$K = a e^{\beta \phi} + \delta,$$

where α , β and δ are constants, β is a of (7) and b of (8); α and δ are evaluated through the phase boundary conditions,

$$K \rightarrow K_s \text{ when } \phi \rightarrow 0,$$

$$\text{and } K \rightarrow K_f \text{ when } \phi \rightarrow 1. \quad (9)$$

On substituting values of α , β and δ in the respective solutions of (7) and (8); we find

$$K = K_1 = \frac{1}{(e^a - 1)} \{(K_f - K_s) e^{a\phi} + K_s e^a - K_f\}, \quad (10)$$

$$\text{and } K = K_2 = \frac{1}{(e^b - 1)} \{(K_f - K_s) e^{b\phi} + K_s e^b - K_f\}. \quad (11)$$

2.1 Limits of K_1 and K_2

The relations for K_1 and K_2 have been derived, keeping in view the basic physical situation, that under varying dispersion the system always remains a two-phase system. Here we examine the upper and lower values of K_1 and K_2 .

When $K_s < K_f$

From (7) and (8), when $K_s \ll K_f$:

$$a \rightarrow (K_f / K_s)^{1/2}, \text{ where } (K_f / K_s)^{1/2} > 1,$$

$$\text{and } b \rightarrow 1.$$

Now since, $K_1 = \frac{1}{(e^a - 1)} \{(K_f - K_s) e^{a\phi} + K_s e^a - K_f\}$, and $a > 1$.

$$\text{Hence } K_1 \rightarrow \{(K_f - K_s) e^{-a(1-\phi)} - K_f e^{-a}\}. \quad (12)$$

$$\text{and } K_2 = \frac{1}{(e^b - 1)} \{(K_f - K_s) e^{b\phi} + K_s e^b - K_f\},$$

$$\text{for } b \rightarrow 1, K_2 > \{(K_f - K_s) e^{-(1-\phi)} - K_f e^{-1}\},$$

$$\text{and also } K_2 > \{(K_f - K_s) e^{-a(1-\phi)} - K_f e^{-a}\}, \text{ for any } a > 1. \quad (13)$$

On comparing (12) and (13), we have

$$K_2 > K_1. \quad (14)$$

When $K_f \ll K_s$

From (7) and (8), when $K_f \ll K_s$;

$$a \rightarrow - (K_s/K_f)^{1/2}, \text{ where } - (K_s/K_f)^{1/2} < -1,$$

and $b \rightarrow -1$.

$$\text{Hence } K_1 \rightarrow \{(K_s - K_f) e^{a\phi} - K_s e^a\}, \text{ for } K_f \ll K_s. \quad (15)$$

$$\text{For } b \rightarrow -1 \quad K_2 > \{(K_s - K_f) e^{-\phi} - K_s e^{-1}\},$$

and also

$$K_2 > \{(K_s - K_f) e^{a\phi} - K_s e^a\}, \text{ for any } a < -1. \quad (16)$$

Comparing (15) and (16), we again establish the relation (14) *i. e.*

$$K_2 > K_1. \quad (17)$$

As such we find that for all values of K_s and K_f , K_2 is the upper bound and K_1 the lower bound on the thermal conductivity of the two-phase system. The effective thermal conductivity K_E of the two-phase system should lie between these two values *i.e.*

$$K_2 > K_E > K_1 \quad (18)$$

3. Comparison with experimental results

The K_1 and K_2 value for two-phase systems of practical importance have been calculated and presented in table 1 along with the experimental values of K_E of these systems. It is evident that the experimental values of K_E lie around K_2 for the systems numbered 2, 3, 6, 7, 8, 9, 10, 11, 13 and 14 and that the upper bound K_2 is very sharp, because the largest difference between K_2 and experimental is 6%. The experimental values of K_E for the systems numbered 1, 4 and 5 lie around K_1 . However, in the case of the system numbered 12, the experimental value of K_E lies between K_1 and K_2 .

4. Discussion and results

In literature we find that most of the derived bounds use variational principles (Beran 1965). These are specific and do not predict K_E of all types of two-phase systems. In addition, Prager and Schulgasser bounds (Prager 1969; Schulgasser 1976) require an understanding of the experimental value of K_E of that system, whose K_f/K_s value is equivalent to the K_f/K_s value of the system under consideration. A more suitable bound is the one due to Hashin-Shtrikman which uses the variational principle. The present bound has, therefore, been compared with that of Hashin-Shtrikman bound.

Tables 1 and 2 show that the experimental values of K_E lie well within the bounda-

Table 1. Calculated and measured values of K_E of various two-phase systems.

Systems	K_f/K_s	ϕ	K_a $WM^{-1}K^{-1}$	K_1 $WM^{-1}K^{-1}$	$K_{\text{expt.}}$ $WM^{-1}K^{-1}$
Uranium oxide and molybdenum (Gilchrist <i>et al</i> 1975)	14.375	0.217	22.60	11.70	15.70
		0.1715	19.10	10.70	12.60
		0.282	28.00	13.50	18.30
Uranium oxide and sodium (slurry) (Huetz 1972)	12	0.25	18.16	11.07	17.00
		0.50	34.31	20.00	33.60
		0.75	55.00	39.70	54.00
Foresterite and magnesia (Kingrey 1959)	6	0.868	2.52	2.22	2.48
Silicon rubber and glass beads (Hayashi <i>et al</i> 1976)	5.195	0.089	0.241	0.226	0.231
		0.198	0.310	0.261	0.283
		0.306	0.373	0.311	0.325
		0.505	0.530	0.428	0.462
Silicon rubber and glass particles (Hayashi <i>et al</i> 1976)	5.195	0.088	0.242	0.226	0.241
		0.308	0.379	0.312	0.336
		0.505	0.530	0.428	0.504
Uranium oxide and sodium potassium (slurry) (Huetz 1972)	4.5	0.25	9.60	7.95	9.10
		0.50	13.70	11.22	13.50
		0.75	19.20	16.73	18.80
Plasticizer (Nahas and Couper 1966)	3.03×10^{-1}	0.90	0.21	0.21	0.21
Cellosize and polypropylene (Nahas 1966)	2.674×10^{-1}	0.90	1.52	0.92	1.55
Water and petrol (Knudson and Wand 1958)	2.61×10^{-1}	0.80	0.21	0.20	0.21
Copper and Solder (Lee and Taylor 1976)	1.96×10^{-1}	0.9876	79.50	78.50	78.80
		0.9714	82.70	80.40	82.00
		0.9493	87.00	83.00	83.70
		0.9004	97.10	89.10	85.40
		0.7623	128.00	110.00	119.00
Zirconium oxide powder (dry) (Waterman and Goldsmith 1961)	1.71×10^{-2}	0.09	1.52	0.92	1.55
Rajasthan desert sand(dry) (Kumar and Chaudhary 1980)	7.875×10^{-3}	0.385	1.675	0.067	0.387
Uranium oxide powder (dry) (Waterman and Goldsmith 1961)	3×10^{-3}	0.267	5.66	0.089	5.67
Aluminium oxide powder (dry) (Waterman and Goldsmith 1961)	7.5×10^{-4}	0.234	22.78	0.033	22.26

ries of the present bounds and the span ($K_2 - K_1$) between upper and lower bounds of the present set of bounds is more confined as compared to the Hashin-Shtrikman bound. These bounds can be further compared by evaluating the area covered by the various bounds.

Table 2. Comparison of $(K_s - K_1)$ with Hashin-Shtrikman bounds.*

Systems	K_f/K_s	ϕ	Present Bounds			Hashin-Shtrikman Bounds		
			K_s WM ⁻¹ K ⁻¹	K_1 WM ⁻¹ K ⁻¹	$K_s - K_1$ WM ⁻¹ K ⁻¹	K_s WM ⁻¹ K ⁻¹	K_1 WM ⁻¹ K ⁻¹	$K_s - K_1$ WM ⁻¹ K ⁻¹
Uranium oxide and molybdenum (Gilchrist <i>et al</i> 1975)	14.375	0.10	14.50	9.50	5.00	15.62	10.14	5.48
		0.20	22.00	11.50	10.50	23.74	12.69	11.05
		0.30	30.00	14.50	15.50	32.42	15.79	16.63
		0.50	48.50	23.60	24.90	51.68	24.57	27.11
		0.75	77.66	50.40	27.26	80.14	45.95	34.19
Uranium oxide and sodium potassium (slurry) (Huertz 1972)	4.5	0.25	9.60	7.95	1.65	9.97	8.72	1.25
		0.50	13.59	11.22	2.37	14.53	12.43	2.10
		0.75	19.20	16.75	2.45	19.81	17.73	2.08
Copper and solder (Lee and Taylor 1976)	1.96×10^{-1}	0.9876	79.50	78.50	1.00	79.97	78.76	1.21
		0.9004	97.10	89.10	8.00	100.50	91.06	8.44
		0.7623	128.00	110.00	18.00	135.76	113.89	21.87
Zirconium oxide dry powder (Waterman and Goldsmith 1961)	1.71×10^{-2}	0.09	1.52	0.924	0.596	1.56	0.62	0.94

*Hashin-Shtrikman bounds are

$$K_s + \frac{\phi}{\left(\frac{1}{K_f - K_s}\right) + \left(\frac{1-\phi}{3K_s}\right)} \geq K_E \geq K_f + \frac{1-\phi}{\left(\frac{1}{K_s - K_f}\right) + \left(\frac{\phi}{3K_f}\right)}$$

4.1 Area of bounds

The area covered by bounds is A where,

$$A = \int_0^1 (K_2 - K_1) d\phi. \quad (19)$$

The area of the present bounds is,

$$A_{PB} = (K_f - K_s) \left(\frac{1}{b} - \frac{1}{a} \right) + K_s \left(\frac{e^b}{e^b - 1} - \frac{e^a}{e^a - 1} \right) - K_f \left(\frac{1}{e^b - 1} - \frac{1}{e^a - 1} \right). \quad (20)$$

The area covered by the Hashin-Shtrikman bound when $K_f \geq K_s$, is given by,

$$A_{HS} = \frac{1}{2} (K_f - K_s) - \frac{4}{9} K_s \left(1 - \frac{K_s}{K_f} \right) - \frac{1}{18} K_f \left(1 - \frac{K_s}{K_f} \right) - \frac{4}{9} K_s \left(1 - \frac{K_s}{K_f} \right)^2, \quad (21)$$

and when $K_f \leq K_s$, the area covered by the same bound is,

$$A'_{HS} = \frac{1}{2} (K_s - K_f) - \frac{1}{2} \frac{K_f}{K_s} (K_s - K_f) - \frac{1}{18} \frac{1}{K_s} (K_s - K_f)^2 - \frac{4}{9} \frac{K_f}{K_s^2} (K_s - K_f)^2. \quad (22)$$

The area covered by series and parallel resistor bounds, when $K_s \geq K_f$, is,

$$A_{SPR} = \frac{1}{2} (K_s - K_f) \frac{K_f}{2 K_s} (K_s - K_f) \quad (23)$$

and, when $K_s \leq K_f$, the area is,

$$A_{SPR} = \frac{1}{2} (K_f - K_s) - \frac{K_s}{2 K_f} (K_f - K_s), \quad (24)$$

The areas covered by different bounds, using relations (20), (21), (22), (23) and (24) for systems, whose K_f/K_s values varies from 16 to 10^{-2} are calculated. These are given in table 3. It is found that the area covered by the present bounds is considerably less as against that covered by Hashin-Shtrikman bounds. For example, when (K_f/K_s) is 2×10^{-2} , the area covered by the present bounds is only 20% of the area covered by the Hashin-Shtrikman bounds.

Table 3. Comparison of area between the present bounds and Hashin-Shtrikman bounds.

K_f/K_s	Area of series and parallel bounds (A_{SPB})	Area of Hashin-Shtrikman bounds (A_{HS})	Area of present bounds (A_{PB})
16	0.44 K_f	0.33 K_f	0.1645 K_f
2×10^{-1}	0.32 K_s	0.217 K_s	0.0527 K_s
6.2×10^{-1}	0.44 K_s	0.393 K_s	0.1645 K_s
1.0×10^{-2}	0.49 K_s	0.432 K_s	0.315 K_s

Acknowledgements

The authors are thankful to Prof. R C Bhandari for giving valuable suggestions. One of the authors (RNP) is grateful to UGC for the award of a teacher fellowship.

References

- Beran M 1965 *Nuovo Cimento* **38** 771
 Beran M J and Silnuitzer R 1971 *J. Compos. Mater.* **5** 246
 Chaudhary D R and Bhandari R C 1967 *Indian J. Pure Appl. Phys.* **5** 413
 Gilchrist K E, Brocklehurst J E and Ware J O 1975 *Rev. Int. Htres. Temp. et Refract.* **12** 146-152
 Huetz J 1972 *Progress heat mass transfer* (Pergamon Press) Vol. 5, p. 285
 Hayashi K, Saijo Y and Uei I 1976 *Yogyo-Kyokai-Shi* **82** 318
 Hori M 1973 *J. Math. Phys.* **14** 514
 Hori M 1973 *J. Math. Phys.* **14** 1942
 Hashin Z and Shtrikman S 1962 *J. Appl. Phys.* **33** 3125
 Kingrey W D 1959 *J. Am. Ceram. Soc.* **42** 617
 Knudson J G and Wand R H 1958 *Ind. Eng. Chem.* **50** 1667
 Kumar V and Chaudhary D R 1980 *Indian J. Pure Appl. Phys.* **18** 984
 Lee H J and Taylor R E 1976 *J. Appl. Phys.* **47** 148
 Miller M N 1969 *J. Math. Phys.* **10** 1988, 2005
 Nahas N C and Couper J R 1966 *Res. Rep. Ser. No. 7*, University of Arkansas, Arkansas
 Prager S 1969 *J. Chem. Phys.* **50** 4305
 Schulgasser K 1976 *J. Math. Phys.* **17** 382-387
 Sugawara A and Yoshizawa 1961 *Aust. J. Phys.* **14** 469
 Tye R P 1969 *Thermal conductivity* (London: Academic Press) Vol. 1
 Waterman and Goldsmith 1961 *Handbook thermophysical properties of materials* (Macmillan Company) Vol. 3, 17, 37 and 319