Pramāṇa, Vol. 20, No. 4, April 1983, pp. 293-303. © Printed in India.

The structure of the space of solutions of Einstein's equations coupled with scalar fields

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MS received 3 August 1981; revised 10 February 1982

Abstract. Following the work of Arms, Fischer, Marsden and Moncrief, it is proved that the space of solutions of Einstein's equations coupled with self-gravitating massless scalar fields has conical singularities at each spacetime possessing a compact Cauchy surface of constant mean curvature and a nontrivial set of simultaneous Killing fields, either all spacelike or including one (independent) timelike.

Keywords. Einstein's field equations; self-gravitating scalar fields; Hamiltonian formalism; linearization stability; Killing fields; Taub's conserved quantities.

1. Introduction

Fischer *et al* (1980) and Arms *et al* (1982) (hereafter referred to as FMM and AMM respectively) have settled the fundamental question regarding linearized stability of vacuum Einstein's equations of general relativity by proving that in the presence of Killing fields (one or many spacelike and a timelike), vanishing of Taub's conserved quantities (certain second order conditions) is both necessary and sufficient for solution of linearized Einstein's equations to be integrable, *i.e.* for the corresponding spacetime to be linearization stable. The necessary part was noted earlier (Fischer and Marsden 1979) and the sufficient part has been proved recently in the papers mentioned above. Arms (1977, 1979, 1981) extended some of this work for Einstein's equations coupled with Maxwell-fields, Yang-Mills fields and pure Yang-Mills fields. The sufficiency part of the above problem for Einstein-Yang-Mills fields has also been proved in AMM. Following the earlier work of Fischer and Marsden (1979), Saraykar and Joshi (1981, erratum 1982) proved some results regarding linearization stability of Einstein's equations coupled with self-gravitating massless scalar fields. For this particularly coupled case, we now complete the programme following the work of Arms, Fischer, Marsden and Moncrief cited above.

The aim of this paper is two-fold. Following the work of Saraykar and Joshi (1981, erratum 1982) we wish to show that the results of AMM hold true for coupled gravitational and scalar fields. In doing so, we discuss the ideas of proofs of the results in AMM. The work in AMM is based on FMM and Arms *et al* (1981) which in turn is based on the earlier work of Fischer and Marsden extended over nearly a decade followed by important contributions of Moncrief (1975 a, b and 1976). A very good survey of this work until 1979 is given in Fischer and Marsden (1979).

FMM discusses the structure of the set of solutions of vacuum Einstein's equations in the presence of one Killing field (spacelike as well as timelike) whereas AMM discusses the same in the presence of many spacelike Killing fields including one timelike. The techniques used in the two papers are different. FMM used the Morse lemma as the major tool whereas AMM used the Kuranishi map from deformation theory for spacelike symmetries and the inverse of this Kuranishi map and parametrized Morse lemma when a timelike symmetry is also included. Use of Ebin-Palais slice theorem (Ebin 1970) is common to both the papers. (Actually, use of the inverse Kuranishi map is implicitly made in FMM and the technique discovered after the paper was written). In both the cases, the problem is reduced to finite-dimensional case by using Liapunov-Schmidt procedure from bifurcation theory. Since results of AMM obviously imply those of FMM, we only show that the results of AMM hold true for our coupled case.

Moncrief (1975a) has shown that the space of Killing fields that a spacetime admits is isomorphic to the kernel of $D\Phi\,(g, \pi)^*$ where $\Phi(g, \pi) = 0$ are the constraint equations (see below). Moreover, spacetimes which are not linearization stable are precisely those that admit one or more Killing fields. Thus points of linearization instability in the set of solutions of Einstein's equations are precisely the points with symmetry. In other words, symmetric solutions are singularities in the space of solutions. FMM and AMM discuss the structure of these singularities. They prove that the singularities are conical in the sense that the neighbourhood of a singular solution has the structure of the product of a cone with a manifold. The manifold points represent nearby solutions with the same symmetry as the given one whereas the cones represent the branching of the solution set to solutions of lower symmetry. These cones are determined by the second order conditions, which require the Taub's conserved quantities to vanish. This is proved in the process of proving that the second order conditions are sufficient for linearization stability. The problem is discussed by reducing it to the study of solutions of the constraint equations (cf. Fischer and Marsden 1979).

In extending the results of AMM to our coupled case, we note that many of the arguements in the introduction and § 1 of AMM follow word to word with Φ for the coupled system and with (g, π) replaced by (g, ϕ, π, σ) (see below for notations). Moncrief's decomposition, its interpretation, the orthogonal slice, construction of the Kuranishi map F and its properties all hold for the coupled system with no essential change in the proof. These topics are discussed in §§ 2.3, 2.4 and 3.1. Notations are given in § 2.1 and relationship between linearization stability and Killing fields is discussed in § 2.2. In §§ 3.2 and 3.3 we deal with momentum (spacelike) and Hamiltonian (timelike) constraints respectively corresponding to particular symmetries. This requires a characterization of these symmetries on a constant mean curvature hypersurface. This is given in § 2.2. § 4 contains some remarks.

2. Preparatory material

2.1 *Notations and preliminaries*

Let V denote a four dimensional Lorentz manifold with Lorentz metric ${}^4g$. Let ${}^4\phi$ be a scalar field on V. Einstein's equations coupled with self-gravitating massless scalar fields are given by $G_{\mu\nu} = \chi\, T_{\mu\nu}$ where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, {}^4g_{\mu\nu}$ and $T_{\mu\nu} = \beta$

$(2\,{}^4\phi_{,\mu}\,{}^4\phi_{,\nu} - {}^4g_{\mu\nu}\,{}^4\phi_{,\rho}\,{}^4\phi^{,\rho})$. Here $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and β is a positive constant related to the choice of units. The sign conventions are determined by the Ricci commutation formula

$$X^{\alpha}_{;\mu;\nu} - X^{\alpha}_{;\nu;\mu} = -R^{\alpha}_{\delta\mu\nu}\,X^{\delta}$$

where $\mu, \nu, \ldots = 0, 1, 2, 3$ and ; denotes covariant differentiation with respect to ${}^4g$. Then $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$. Using appropriate function spaces (for example the Sobolev spaces as in Fischer and Marsden 1979), the functions Ric $({}^4g) = R_{\mu\nu}$, $R = R^{\alpha}_{\alpha}$ and G are C^{∞} in ${}^4g$ and $T_{\mu\nu}$ is C^{∞} in ${}^4\phi$. We write the above field equations in the form $E\,({}^4g, {}^4\phi) = E\,({}^4x) = G_{\mu\nu} - T_{\mu\nu} = 0$ (letting $\chi = 1$). The solution of this equation is denoted by a spacetime $(V, {}^4g, {}^4\phi) = (V, {}^4x)$. Given this spacetime, a compact three-dimensional manifold M and a spacelike embedding $i : M \to V$ define

(i) $g = i^*\,({}^4g)$, $\phi = i^*({}^4\phi)$: a Riemannian metric and a scalar field respectively on M;

(ii) k = the second fundamental form of the embedding, a symmetric 2-tensor on M, with the sign convention $k_{ij} = -{}^4Z_{i;j}$ where ${}^4Z$ is the forward pointing unit normal to $\Sigma = i\,(M) \subset V$;

(iii) $\bar{k}$ = trace k, the mean curvature of the embedding;

(iv) $\pi' = (\bar{k}g - k)^{\#}$ where $\#$ denotes the contravariant form of the tensor with respect to g.

(v) $\pi = \pi' \otimes \mu(g)$ where $\mu\,(g)$ is the Riemannian volume element of g. π serves as the quantity conjugate to g in the Hamiltonian formulation of geometrodynamics;

(vi) γ denotes a quantity conjugate to ϕ and let $\sigma = 4\beta\gamma\mu(g)$. π, σ are to be obtained by using the so-called Christodoulou-DeWitt metric as described in Francaviglia (1977). The reader should refer to this article for more details.

(vii) Let $G^{\perp\perp} = Z^{\alpha}\,Z^{\beta}\,G_{\alpha\beta}$, the perpendicular-perpendicular projection of G, a scalar function on M; $(G^{\perp}_{\parallel})_i = -Z_{\alpha}\,G^{\alpha}_{i}$, the perpendicular-parallel projection of G, a one-form on M.

(viii) Geometric part of the Hamiltonian $\mathcal{H}_G$ is then $\mathcal{H}_G = -2G^{\perp\perp}\,\mu(g) = \{\pi' \cdot \pi' - \frac{1}{2}(\mathrm{tr}\,\pi')^2 - R(g)\}\,\mu(g)$, a scalar density on M; geometric part of the momentum constraint $\mathcal{J}_G = -2G^{\perp}_{\parallel}\,\mu(g) = -2\,(\delta\pi)^{\oplus}$, a one-form density on M. ($\oplus$ denotes lowering of the index with respect to g). In coordinates, $(\mathcal{J}_G)_i = 2\pi_i{}^{j}{}_{|j}$, a vertical bar denoting covariant differentiation on M with respect to g. (There is a sign change as compared to FMM due to the sign change in the shift).

(ix) The corresponding field quantities are obtained from $T_{\mu\nu}$ in a similar way:

$$\mathcal{H}_F = 2T^{\perp\perp}\,\mu(g) = 2\beta\,(\gamma^2 + \phi_{,i}\,\phi^{,i})\,\mu(g)$$

$$\mathcal{J}_F = 2\,(T^{\perp}_{\parallel})_i\,\mu\,(g) = -2\,T_{0i}\,\mu\,(g) = -4\,\beta\,\gamma\,\mu\,(g)\,\phi_{,i} = -\sigma\,\phi_{,i}.$$

Thus the total Hamiltonian $\mathcal{H} = \mathcal{H}_G + \mathcal{H}_F = -2E({}^4x)^{\perp\perp}\mu(g)$

$= \{\pi' \cdot \pi' - \frac{1}{2}(\text{tr}\,\pi')^2 - R(g) + 2\beta(\gamma^2 + \phi_{,i}\phi^{,i})\}\,\mu(g).$

The total momentum $\mathcal{J} = \mathcal{J}_G + \mathcal{J}_F = -2(E({}^4x)^{\perp}_{\|})_i\,\mu(g) = 2\pi^j_{i\,|\,j} - \sigma\phi_{,i}$.

(x) Set $\Phi(x) = \Phi(g, \phi, \pi, \sigma) = (\mathcal{H}(x), \mathcal{J}(x))$. Given a slicing of $(V, {}^4x)$, *i.e.* a curve i_t in the space Emb $(M, V, {}^4x)$ of spacelike embeddings of M into V which foliate a neighbourhood of $\Sigma_0 = i_0(M)$ in V, the coupled Einstein's equations $E({}^4x) = 0$ are equivalent to the following system

$$\Phi(x) = 0 \quad \text{(constraint equations)}$$

$$\frac{\partial x}{\partial t} = J \circ D\,\Phi(x)^*.\,(N, X) \quad \text{(evolution equations).}$$

Here N and X are perpendicular and negative of parallel components of ${}^4X_t = \mathrm{d}/\mathrm{d}t\,(i_t)$, the tangent to the curve of embeddings, so ${}^4X : M \to TV$ and covers i_t.

Explanation of evolution equations is given in full details in Fischer and Marsden (1979) (see also FMM and Saraykar and Joshi (1981, erratum 1982)). We are following sign conventions of Fischer and Marsden (1979), so our J (giving complex structure on the configuration space) coincides with that in Fischer and Marsden (1979) and differs in sign from that in FMM. In the coupled case we have to work with the configuration space $\mathcal{M} \times \mathcal{F}$ where $\mathcal{M}$ is the space of $W^{s,p}$ (Sobolev class)—Riemannian metrics and $\mathcal{F}$ is the space of $W^{s,p}$ — real valued functions on M, $3/p + 1 < s \leqslant \infty$. We also keep in mind that $T(\mathcal{M} \times \mathcal{F}) = \mathcal{M} \times \mathcal{F} \times S_2 \times F$, S_2 denoting 2-covariant symmetric tensors on M. Similarly $T^*(\mathcal{M} \times \mathcal{F}) = \mathcal{M} \times \mathcal{F} \times S^2_d \times \mathcal{F}_d$, S^2_d denoting 2—contravariant symmetric tensor densities π and $\mathcal{F}_d$ denoting scalar densities σ on M. The weak Riemannian metric and weak symplectic structure on $T^*(\mathcal{M} \times \mathcal{F})$ is defined in an obvious manner following FMM. For relation among these and the complex structure J, see FMM. $D\,\Phi(x)^*$ is the adjoint operator, adjoint to $D\,\Phi(x)$, taken relative to the above metric on $T^*(\mathcal{M} \times \mathcal{F})$. For any unexplained notation that follows the reader should consult FMM and Saraykar and Joshi (1981).

In what follows, by a Killing field we always mean a simultaneous Killing field *i.e.* $L_{{}^4X}{}^4g = 0$ and $L_{{}^4X}{}^4\phi = 0$. In the context of constraint equations, by a Killing field on M, we mean $L_X g = 0$ and $L_X \phi = 0$ whenever of course $\phi \neq 0$.

2.2 *Linearization stability and Killing fields*

As in FMM and AMM, it is enough to study the constraint equations. If $E({}^4x) = 0$ and $D\,E({}^4x) \cdot {}^4y = 0$, ${}^4y$ is called an *infinitesimal deformation.* An actual *deformation* is a smooth curve ${}^4x(t) = ({}^4g(t), {}^4\phi(t))$ of solutions through $({}^4g_0, {}^4\phi_0) = {}^4x_0$; i.e. $E({}^4x(t)) = 0$. ${}^4y$ is said to be *integrable* if for every compact set $C \subset V$ there is an actual deformation ${}^4x(t)$ defined on C such that ${}^4x(0) = {}^4x_0$ on C and

$d/dt\,({}^4x\,(t))|_{t=0} = {}^4y$ on C. Then every integrable ${}^4y$ is an infinitesimal deformation. A spacetime is called *linearization stable* if every infinitesimal deformation is integrable.

In the presence of Killing fields, any infinitesimal deformation must satisfy a necessary second order condition in order to be integrable as follows:

Let $E\,({}^4x) = 0$, ${}^4X$ be a Killing field of ${}^4x_0$ and let ${}^4y$ be integrable. Then the following Taub's conserved quantities vanish identically when integrated over any compact spacelike hypersurface Σ:

$$\int_{\Sigma} {}^4X \cdot [D^2\,E\,({}^4x_0) \cdot ({}^4y, {}^4y) \cdot {}^4Z_{\Sigma}\,\mathrm{d}^3\,\Sigma] = 0$$

where ${}^4Z_{\Sigma}$ is the forward pointing unit normal to Σ.

Saraykar and Joshi (1981) proved that if $(V, {}^4x)$ has no Killing fields, then it is linearization stable. The converse of this is also true following Arms and Marsden (1979):

Theorem 1: If $(\mathrm{V}, {}^4x)$ is linearization stable, then it has no Killing fields.

Discussion of proof: Linearization stability implies the second order condition. Hence it is enough to prove that if V admits a nontrivial Killing field, then there exists an element $y = (h, \psi, \omega, \tau) \in \ker\, D\Phi(x)$ violating the second order condition. Further, Killing fields are in one-to-one correspondence with $\ker\, D\Phi(x)^*$ (see Saraykar and Joshi 1981). Thus we start with assuming $(N, X) \not\equiv 0$ in $\ker\, D\Phi(x)^*$. Then the argument goes exactly as in Arms and Marsden (1979) since the addition of scalar fields adds purely algebraic terms to the differential operators involved.

The treatment in § 3 dealing with spacelike and timelike constraints corresponding to particular symmetries requires a characterization of those symmetries on a constant mean curvature hypersurface Σ. The arguments of Saraykar and Joshi (1981) imply the following:

Theorem 2: Let Σ be a smooth compact hypersurface of constant mean curvature with induced quantities $x_0 = (g_0, \phi_0, \pi_0, \sigma_0)$. Then

(i) If one of ϕ_0, π_0, σ_0 is non-zero or g_0 is not flat, $\ker\, D\Phi(x_0)^* = \{(0, X)/$

$$L_X g_0 = L_X \phi_0 = L_X \pi_0 = L_X \sigma_0 = 0\}$$

and (ii) if $\phi_0 = \pi_0 = \sigma_0 = 0$ and g_0 is flat, $\ker\, D\Phi(x_0)^* = \{(0, X)/L_X g_0 = 0\} \cup \{(N, 0)/N \text{ is constant}\}$.

In particular, if $\Sigma = i(M)$ has constant mean curvature and one of π_0, ϕ_0, σ_0 is not zero or g_0 is not flat, then any Killing field on V must be tangent to Σ.

In case (i), the spacelike case, there is a basis of $\ker\, D\Phi^*$ of the form $\{(0, X_i)$: $i = 1, \ldots, n$ and $L_{X_i}\, x_0 = 0\}$; in case (ii), the timelike case, there is a basis with $(1, 0)$ as one element and the rest of the basis like that in case (i).

2.3 *Moncrief's decomposition*

Since, as in the vacuum case, $D\Phi(x)^*$ and $J \circ D\Phi(x)^*$ are elliptic in our case also (see Saraykar and Joshi 1981), we get the following decomposition as in FMM or Arms *et al* (1981): If $\Phi(x)=0$, then the tangent space $T_x(T^*(\mathcal{M}\times\mathcal{F})) \approx S_2\times\mathcal{F}\times S_d^2\times\mathcal{F}_d$ splits L_2-orthogonally as follows:

$$T_x(T^*(\mathcal{M}\times\mathcal{F})) = \text{range}\,(J \circ D\Phi(x)^*) \oplus \text{range}\,(D\Phi(x)^*) \oplus$$

$$[\ker\,(D\Phi(x) \circ J) \cap \ker\, D\Phi(x)].$$

As in FMM, the range $(J \circ D\Phi(x)^*)$ represents the infinitesimal gauge transformations, range $(D\Phi(x)^*)$ is the orthogonal complement to the linearized constraints ker $D\Phi(x)$ and ker $D\Phi(x) \cap$ ker $(D\Phi(x) \circ J)$ is the space of linearized 'true' degrees of freedom: a generalization of the usual 'TT' component. The latter is a symplectic subspace of $T_x(T^*(\mathcal{M}\times\mathcal{F}))$ (*cf.* Arms *et al* 1981 for the vacuum case). The orthogonal complement of the gauges plays the role of the slice for the action of diffeomorphism group of spacetime. We discuss this in the next sub-section.

2.4 *Slice in $T^*(\mathcal{M}\times\mathcal{F})$ for the action of $\mathscr{D}^3$*

The central idea of the slice-construction is as follows: the three-dimensional diffeomorphism group $\mathscr{D}^3$ acts as an honest group on $T^*(\mathcal{M}\times\mathcal{F})$ by pull-back. The moment for this action is $P \circ \Phi$, where $P(N, X) = X$. Since $D\Phi(x)^*$ is elliptic, it follows that $(P \circ D\Phi)^*$ is also elliptic, and so is $J \circ (P \circ D\Phi)^*$. The tangent space to the orbit through x_0 is given by range $J \circ (P \circ D\Phi(x_0))^*$ and its orthogonal complement is (by ellipticity) ker $(P \circ D\Phi(x_0) \circ J)$. The slice S_{x_0} at x_0 is then given by $\{x_0\}$ + a neighbourhood of zero in ker $(P \circ D\Phi(x_0) \circ J)$.

To justify negative signs of Lie derivatives in our expression for

$$J \circ D\Phi(x_0)^* \cdot (O, X) = J \circ [P \circ D\Phi(x_0)^* \cdot (N, X)],$$

take the action by inverse pull-back. This does not affect the smoothness of maps involved since $\eta \to \eta^{-1}$ is a smooth map, $\eta \in \mathscr{D}^3$ (*cf.* Ebin 1970).

The above slice construction follows from the standard theory about momentum maps (see Arms *et al* 1981). A separate proof is given in FMM, § 5. For more details on slice-theorem and its role in general relativity, see Isenberg and Marsden (1982).

3. Main results and discussion of proofs

3.1 *Kuranishi map and its properties*

Along with the results in §§ 2 and 2.2–2.4, Kuranishi map is the major tool which enables us to prove our main results. It enables us to deal with all the momentum constraints and a certain projection of the Hamiltonian constraint. When combined with a Morse-lemma type-argument this then enables us to deal with the Hamiltonian constraint. Its definition uses the fact that $D\Phi(x)^*$ is an elliptic operator. We recall its definition from AMM in our context:—Let $x_0 \in \Phi^{-1}(0)$ be fixed and write $\Delta = D\Phi(x_0) \circ D\Phi(x_0)^*$ which, by ellipticity of $D\Phi(x_0)^*$, is an isomorphism of range $D\Phi(x_0)$ to itself. Let P denote the orthogonal projection to range $D\Phi(x_0)$ and set $G = \Delta^{-1} \circ P$, the Green's function for Δ. Write $y = x - x_0$ and let the remainder be given by

$$R(y) = \Phi(x) - D\Phi(x_0) \cdot y.$$

Define the Kuranishi map F by

$$F(x) = x + D\,\Phi\,(x_0)^* \circ G \circ R\,(y).$$

It then turns out that F is a diffeomorphism of a neighbourhood of x_0 onto a neighbourhood of x_0. Moreover F maps the slice S_{x_0} at x_0 to itself. Not only this, F turns out to be a local chart for $\mathscr{C}_P$ and when restricted to $\mathscr{C}_P \cap S_{x_0}$, F is a local symplectic diffeomorphism of $\mathscr{C}_P \cap S_{x_0}$ to $x_0 + [\ker\ D\ \Phi\,(x_0) \cap \ker\ (D\ \Phi\,(x_0) \circ J)]$. Recall that $\mathscr{C}_P = \{x/P\ \ \Phi\,(x) = 0\}$, the projected constraint set which is a smooth manifold in a neighbourhood of x_0 with tangent space at x_0 given by $\ker D\ \Phi\ (x_0)$ (see FMM, proposition 3.2). For these properties of F in the vacuum case and their proofs, see AMM, propositions 1.1 to 1.4. The proofs follow word-to-word in our case with obvious changes in notations, hence we omit them.

Next we need the inverse of F restricted to $\mathscr{C}_P$. Its determination uses the isomorphism of Δ. It will be used to deal with the Hamiltonian constraint. Since $\mathscr{C}_P$ is tangent to $\ker D\,\Phi\,(x_0)$ at x_0, by inverse function theorem, there is a unique smooth map Ψ: $\ker\ D\ \Phi\ (x_0) \rightarrow$ range $D\ \Phi\ (x_0)^*$ defined on a neighbourhood of zero such that $\Psi\ (0) = 0$, $D\ \Psi\ (0) = 0$ and such that $\mathscr{C}_P$ is the graph of Ψ; *i.e.* locally

$$\mathscr{C}_P = \{x = x_0 + y + \Psi\,(y)/y\ \in\ \ker\ D\ \Phi\ (x_0)\}.$$

If we write $\Psi\ (y) = D\ \Phi\ (x_0)^*\ [C\ (y),\ Y\ (y)]$, then C and Y are determined by the nonlinear elliptic system

$$P\ \Phi\ (x_0 + y + D\ \Phi\ (x_0)^*\ .\ (C,\ Y)) = 0.$$

The derivative of the left hand side with respect to $(C,\ Y)$ at $y = 0$ and $(C,\ Y) = 0$ in the direction of $(C',\ Y')$ is

$$\Delta\,(C',\ Y') = D\ \Phi\ (x_0) \circ D\ \Phi\ (x_0)^*\ (C'\ \ Y').$$

Since Δ is an isomorphism of range $D\ \Phi\ (x_0)$ to itself, if $(C,\ Y) \in$ range $D\ \Phi\ (x_0)$, we can uniquely solve the above system for $(C,\ Y)$ as functions of y and thereby determine Ψ.

Result 3.1*a:* The map of $\{x_0\}+\ker D\ \Phi\ (x_0)$ to $\mathscr{C}_P$ given by $x_0+y \longmapsto x_0+y+\Psi(y)$ is the inverse of the Kuranishi map restricted to $\mathscr{C}_P$.

The proof is as in AMM, proposition 1.5.

This solves the part of the constraint equations that can be dealt with by the inverse function theorem, namely $P\,\Phi\,(x)=0$. We now split the remaining part $(I-P)\,\Phi\,(x) = 0$ into timelike and spacelike parts. As in 2.2, let $(0, X_1), \ldots (0, X_n)$ be an L^2-orthogonal basis of elements of ker $D\,\Phi\,(x_0)^*$. Then we have by theorem 2,

$$L_{X_i} x_0 = 0, \quad i = 1, \ldots, n.$$

If (i) there is no timelike Killing field, these span all of ker $D\,\Phi(x_0)^*$ and if (ii) there is a timelike Killing field, then g_0 is flat, $\pi_0 = \phi_0 = \sigma_0 = 0$, and in this case (1,0) is the other basis element of this kernel.

Let, then, $P_{\mathcal{J}}$ be the L^2-orthogonal projection onto the span of $\{(0, X_i)\}$ and $P_{\mathcal{H}}$ the projection onto (1, 0). Thus $I\text{-}P = P_{\mathcal{J}}$ if there is no timelike Killing field and $I\text{-}P = P_{\mathcal{J}} \oplus P_{\mathcal{H}}$ if there is a timelike killing field.

Thus, identifying the span of $\{(0,X_i)\}$ with R^n,

$$P_{\mathcal{J}}\,\Phi\,(x) = \Big(\int_M X_1 \cdot \mathcal{J}\,(x), \ldots, \int_M X_n \cdot \mathcal{J}\,(x)\Big)$$

$$= \Big(\int_M -(L_{X_1} g)\,\pi - (L_{X_1}\phi)\,\sigma, \ldots \int_M -(L_{X_n} g)\,\pi - (L_{X_n}\phi)\,\sigma\Big),$$

and $$P_{\mathcal{H}}\,\Phi\,(x) = \int_M \mathcal{H}(x).$$

It follows that if $\mathscr{C}_{\mathcal{J}} = \{x / P_{\mathcal{J}}\,\Phi\,(x) = 0\}$ and

$\mathscr{C}_{\mathcal{H}} = \{x / P_{\mathcal{H}}\,\Phi\,(x) = 0\}$, then

$$\mathscr{C} = \begin{cases} \mathscr{C}_P \cap \mathscr{C}_{\mathcal{J}} & \text{in case (i)} \\ \mathscr{C}_P \cap \mathscr{C}_{\mathcal{J}} \cap \mathscr{C}_{\mathcal{H}} & \text{in case (ii).} \end{cases}$$

3.2 *The momentum constraints*

Let the cone $\mathscr{C}_{\mathcal{J}}$ be defined by

$$\mathscr{C}_{\mathcal{J}} = \{x_0\} + \{y \in \text{Ker } D\,\Phi\,(x_0) \cap \text{Ker } (D\,\Phi\,(x_0) \circ J) / \int (L_{X_i} h)\,\omega + \int (L_{X_i} \psi)\,\tau = 0, i = 1, \ldots, n\}$$

Here $y = (h, \psi, \omega, \tau)$. Then we have

Theorem 3: The map F takes a neighbourhood of x_0 in $\mathscr{C}_P \cap \mathscr{C}_J \cap S_{x_0}$ one-one and onto a neighbourhood of x_0 in $\mathscr{C}_{\mathcal{J}}$. Compare theorem 3 of Arms *et al* (1981). If there are no timelike Killing fields, this gives the structure of $\mathscr{C} \cap S_{X_0}$. After removing the gauges as explained in FMM and Isenberg and Marsden (1982), we get the desired structure of $\mathscr{C}$.

The proof is analogous to that of theorem 2.1 of AMM. It uses crucially the gauge invariance of $D^2\Phi(x_0)$ which follows in our case as in FMM, proposition 1.12. Then all that we have to do is to define $Q\,(y) \in R^n$ by

$$Q\,(y) = \left(\int (L_{X_i} h)\,\omega + \int (L_{X_i} \psi)\,\tau\right)_{i\,=\,1,\,\ldots,\,n}$$

and use the fact that the scalar part of

$$P_J\, D^2\, \Phi\, (x) \cdot (y, y_1) \text{ is given by } -\,(\tau_1 \nabla \psi + \tau \nabla \psi_1).$$

Finally use $\int (L_X \psi)\,\tau = -\int (L_X \tau)\,\psi$, etc. The rest follows as in AMM.

3.3 *The Hamiltonian constraint*

This is the case (ii) mentioned above and here g_0 is flat and $\phi_0 = \pi_0 = \sigma_0 = 0$. Denote $(g_0, 0, 0, 0)$ by x_0^0. The Kuranishi map F takes $\mathscr{C}_P \cap \mathscr{C}_{\mathscr{J}} \cap S_{x_0^0}$ to the cone $C_{\mathscr{J}}$. To study the Hamiltonian constraint *i.e.* the intersection

$$\mathscr{C} \cap S_{x_0^0} = \mathscr{C}_{\mathscr{H}} \cap \mathscr{C}_{\mathscr{J}} \cap \mathscr{C}_P \cap S_{x_0^0}$$

we need lemma 2·4 of AMM which is valid in our case since g_0 is flat and $\phi_0 = \pi_0 = \sigma_0 = 0$. Then the result 3.1a is used to get the following decomposition for points $(g, \phi, \pi, \sigma) \in \mathscr{C}_P \cap S_{x_0^0}$ near x_0^0:

$$g = g_0 + h^{TT} + \tfrac{1}{3}\,\alpha_1\, g_0 - \text{Hess}\; C - g_0\, \Delta\, C,$$

$$\phi = 0 + \phi,$$

$$\pi = \omega^{TT} + \tfrac{1}{3}\,\alpha_2\, g_0^{\#}\, \mu_0 - (L_Y\, g_0)^{\#}\, \mu_0,$$

$$\sigma = 0 + \sigma,$$

$\mu_0 = \mu(g_0)$ being the volume element of g_0.

Here $(C, Y) \in$ domain of $D\Phi^*$, is a function of h^{TT}, α_1, ω^{TT}, α_2 given by the Ψ map of result 3·1a. Since g, π in the above decomposition are unchanged from § 2 of AMM, lemma 2·5 of AMM remains valid in the coupled case. Lemma 3·1 of AMM is unchanged except for the addition of a *positive* quadratic term

$$2\beta \int_M (\gamma^2 + \nabla\phi \cdot \nabla\phi^{\#})\, \mu_0.$$

Then the rest of the proof of the conical structure of $\mathscr{C}_P \cap \mathscr{C}_{\mathscr{J}} \cap \mathscr{C}_{\mathscr{H}}$ follows exactly as in § 3, AMM. We briefly describe the procedure:
Substitute the above decomposition of (g, ϕ, π, σ) in $\int_M \mathscr{H}(x)$ and consider Taylor expansion of $\int_M \mathscr{H}(x)$ in the variables h^{TT}, α_1, ω^{TT}, α_2 of which it is a smooth function. Then $\mathscr{A} = \{(g_0 + h^{cc}, 0, 0, 0) \in T^* (\mathscr{M} \times \mathscr{F}) / h^{cc}$ is covariant constant with respect to $g_0\}$, an affine submanifold of $T^*(\mathscr{M} \times \mathscr{F})$, turns out to be a nondegenerate critical manifold for $\int_M \mathscr{H}(x)$ in the sense of FMM, § 6. To eliminate the higher order terms in the Taylor expansion make a change of coordinates by using the parametrized Morse lemma (see FMM, § 6 and Golubitsky and Marsden 1982). Then, from this emerges a cone with two branches (defined by two values of α_2) corresponding to the Hamiltonian constraint, noting that the cone $C_{\mathscr{J}}$ does not mix with varible α_2, which is a key point. Denoting this cone by $C_{\mathscr{H}}$ we finally obtain the following main result.

Theorem 4: There is a one-one correspondence between a neighbourhood of x_0^0 in the cone $C_{\mathscr{J}} \cap C_{\mathscr{H}} \cap S_{x_0^0}$ and a neighbourhood of x_0^0 in the nonlinear constraint set $\mathscr{C} \cap S_{x_0^0}$ which maps straight lines in the cone through x_0^0 (a solution of the linearized equations satisfying the second order conditions) to a smooth curve in $\mathscr{C} \cap S_{x_0^0}$ with the same tangent at x_0^0.

The theorem says that the second order conditions on linearized perturbations are sufficient for the existence of an exact perturbation curve (*i.e.* for linearized stability). This was our main goal.

Gauge conditions should be removed by eliminating $S_{x_0^0}$ as described in FMM and Isenberg and Marsden (1982).

This completes the discussion of the case when a spacetime admits a timelike Killing field along with spacelike ones.

4. Remarks

(1) Following Choquet-Bruhat *et al* (1979), the present author (Saraykar 1982) has proved linearized stability of the present coupled system in an asymptotically flat spacetime. The space of solutions itself is a manifold and so, no structure theory is needed in the asymptotically flat case; the symmetries are irrelevant. However, as remarked in Isenberg and Marsden (1982), one does not know whether or not one should expect a slice for the action of ' asymptotically identity diffeomorphisms' on asymptotically flat spacetimes. It is expected that the space of gravitational degrees of freedom has dynamics remaining.

(2) Using the slice theorem in § 2 and the results of FMM and Arms *et al* (1981) (especially lemma 18) one can show that the space of solutions admitting a compact spacelike hypersurface of constant mean curvature modulo 4-diffeomorphisms is a stratified symplectic manifold; i.e. a stratified manifold, each stratum of which is symplectic. As in Isenberg and Marsden (1982) it also follows that the generic stratum consisting of equivalence classes of solutions of coupled Einstein

equations with no isometries (and hence nonsingular) is an open dense set Thus the generic symplectic stratum in the reduced space is also open and dense.

(3) In an earlier version of this paper we included extension of results of FMM to our coupled case which are deleted in this version since those results are obviously implied by the results here. Also details of the proofs are omitted in this version. Detailed proofs are available with the author and requests from readers for any of the proofs are most welcome.

Acknowledgements

The author is very grateful to the referees for pointing out a sign error in the energy expression in an earlier version and foı suggesting many valuable changes regarding the form and contents of the paper. The author is also grateful to Professors J E Marsden and V Moncrief for useful correspondence and for making available their paper as soon as it was ready in preprint form.

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