Pramāṇa, Vol. 20, No. 3, March 1983, pp. 245–249. © Printed in India.

Viscous fluid motion around the Kerr-Newman black hole

S K SAHA*
Department of Physics, Presidency College, Calcutta 700 073, India
*Permanent address: Department of Physics, Stewart College, Cuttack 753 002, India

MS received 16 November 1982; revised 17 January 1983

Abstract. In some recent papers, it has been shown that electrovac spacetimes may sometimes be interpreted as space-time filled with a viscous fluid—one such spacetime being the Kerr-Newman metric. Here the possible fluid motion in that case is studied in some detail and some interesting features of the motion in the black hole region are pointed out.

Keywords. General relativity; Kerr-Newman black hole; viscous fluid motion.

1. Introduction

It has recently been shown that electrovac solutions of Reissner and Nordstrom as well as the Kerr-Newman metric may be alternatively interpreted as due to viscous fluid distribution (Tupper 1981). With this interpretation we would have a steady flowing fluid in the black hole regions also. It does not seem unreasonable to hope that the study of this fluid motion will throw some interesting light on black hole physics. However, in the description of the fluid motion, the shear viscosity η appears as a free parameter, the only restriction demanded by the second law of thermodynamics being that η must be positive (*i.e.* viscosity must tend to convert mechanical energy to heat). With this in mind, the present paper reports the studies on the viscous fluid motion in the Kerr-Newman black hole.

2. Kerr-Newman black hole

The Kerr-Newman metric is given by

$$\begin{aligned} ds^2 = & \left[1 + \frac{e^2 - 2mr}{\epsilon}\right] dt^2 - \frac{\epsilon}{\lambda} dr^2 - \epsilon d\theta^2 \\ & - \left[(r^2 + a^2)\sin^2\theta - \frac{a^2 \sin^4\theta}{\epsilon}(e^2 - 2mr)\right] d\Phi^2 \\ & - \frac{2}{\epsilon}(e^2 - 2mr)\, a \sin^2\theta \, d\Phi \, dt, \end{aligned} \tag{1}$$

where $\epsilon = r^2 + a^2 \cos^2\theta$, and $\lambda = r^2 + a^2 + e^2 - 2mr.$

The symbols m, e and a signify the mass, charge and the angular momentum per unit mass respectively of the source. The electromagnetic field form associated with this solution is

$$F_{\mu\nu} = e\, \epsilon^{-2} (r^2 - a^2 \cos^2 \theta)\, \mathrm{d}r \wedge [\mathrm{d}t - a \sin^2 \theta\, \mathrm{d}\Phi]$$
$$- 2e\epsilon^{-2}\, ar \cos \theta \sin \theta\, d\theta \wedge [a\, \mathrm{d}t - (r^2 + a^2)\, \mathrm{d}\Phi]. \tag{2}$$

The energy momentum tensor components of the electromagnetic field for the above metric are given by

$$E^1_1 = - E^2_2 = e^2/8\pi\epsilon^2,$$
$$E^3_3 = - E^0_0 = - \frac{e^2}{8\pi\epsilon^3} [r^2 + a^2 (1 + \sin^2 \theta)], \tag{3}$$
$$E^0_3 = ae^2/4\pi\epsilon^3,$$
$$E^0_3 = - \frac{ae^2 \sin^2 \theta}{4\pi\epsilon^3} (r^2 + a^2).$$

To recapitulate the viscous fluid interpretation we recall that we have to consider a viscous fluid whose energy stress tensor agrees exactly with E^μ_ν *i.e.*

$$E^\mu_\nu = (\rho + p)\, v^\mu v_\nu - p\delta^\mu_\nu + 2\eta\sigma^\mu_\nu. \tag{4}$$

Utilising (3) and (4) we find

$$\rho = 3p = e^2/8\pi\epsilon^2, \tag{5}$$

and the non-vanishing velocity components to be v^1, v^3 and v^0 where we have numbered the co-ordinates r, θ, Φ and t on 1, 2, 3 and 0 respectively.

Using the identity (*cf.* Raychaudhuri and Saha 1981)

$$\rho_{,\mu}\, v^\mu = \frac{4\rho^2}{3\eta} - \frac{4\rho\theta}{3}, \tag{6}$$

for the above case we get

$$\frac{\mathrm{d}v^1}{\mathrm{d}r} - \frac{rv^1}{\epsilon} = \frac{1}{8\pi\eta} \frac{e^2}{\epsilon^2}. \tag{7}$$

This may be formally integrated to give

$$v^1\, \epsilon^{-1/2} = \frac{e^2}{8\pi} \int \frac{\epsilon^{-5/2}}{\eta} \mathrm{d}r + \text{const.} \tag{8}$$

The other non-vanishing velocity components can be written in terms of v^1 using the normalisation condition $v^\mu v_\mu = 1$ and (3) and (4) and are given by

$$v^3 = \frac{a}{\epsilon^{1/2}\lambda}[\lambda + \epsilon(v^1)^2]^{1/2}, \tag{9}$$

$$v^0 = \frac{(r^2 + a^2)}{\epsilon^{1/2}\lambda}[\lambda + \epsilon(v^1)^2]^{1/2}. \tag{10}$$

We observe here

$$\frac{v^3}{v^0} = \frac{d\Phi}{dt} = \frac{a}{r^2 + a^2}. \tag{11}$$

This gives the rotational velocity of the fluid with respect to an observer at infinity and it goes on increasing as r decreases and tends to a finite limit $= 1/a$ at $r \to 0$.

The angular velocity of a free particle at the horizon of the Kerr-Newman black hole is given by

$$\omega = \frac{a}{2mr_+} = \frac{a}{2m^2 + 2m\,[m^2 - (a^2 + e^2)]^{1/2}}. \tag{12}$$

Comparing this with the rotational velocity of the viscous fluid at the horizon as found from (11) we see that the viscous fluid has a higher angular velocity at $r = r_+$. If $e = 0$, the two angular velocities are equal but this case is not feasible since the viscous fluid interpretation does not hold good here.

We further see from (11) that the angular velocity does not depend on the viscosity parameter η.

A particular case of the viscous fluid motion has been given by Tupper (1981) where the velocity components are given by:

$$v^\mu = \epsilon^{-1/2}\lambda^{-1/2}\,[(r^2 + a^2)(1 + \tau^2)^{1/2},\ \lambda\tau,\ 0,\ a\,(1 + \tau^2)^{1/2}],$$

and $$\eta = e^2\epsilon^{-1/2}\lambda^{1/2}\,[\epsilon\lambda\tau' + (\epsilon r - \epsilon m - 2r\lambda)\tau]^{-1},$$

where τ is an arbitrary function of r.

As λ is negative in the region between $r = r_+$ and $r = r_-$, the velocity components become imaginary in this region. So with the Tupper form, there are in general imaginary velocity components and the viscous fluid interpretation does not work over the entire spacetime. However, this is rather due to Tupper's arbitrary choice of some function as it appears from the following analysis.

When η is constant, from (8) we get

$$v^1 = \frac{e^2}{8\pi\eta a^4\cos^4\theta}\left[r - \frac{1}{3}\frac{r^3}{\epsilon} - \frac{2}{3}\epsilon^{1/2}\right], \tag{13}$$

where the integration constant is suitably chosen such that at $r \to \infty$, $v^1 \to 0$. Equation (13) can also be written in the form

$$v^1 = \frac{-e^2}{8\pi\eta' a^4 \cos^4\theta} \, \frac{(r - \epsilon^{1/2})^2 (r + 2\,\epsilon^{1/2})}{3\epsilon}. \tag{14}$$

v^3 and v^0 are thus obtained from (9) and (10). The reality condition for the velocity components as evident from (9) and (10) is

$$\lambda + \epsilon\,(v^1)^2 \geqslant 0, \tag{15}$$

and this must be satisfied with the above value of v^1.

3.1 *Black hole region*

We see that at the horizon ($r = r_+$) v^1 is finite whereas v^0 and v^3 tends to ∞. And within the horizon, $r < r_+$, v^1 still remains finite and v^3 and v^0 are also finite except at $r = r_-$ where it tends to ∞. Since condition (15) is to be satisfied, the velocity components v^0 and v^3 shall remain real within $r = r_+$ and $r = r_-$. Further at $r = 0$ and at the static limit, all the three components of the velocity vector remain finite. At $r \to \infty$, the velocity components tend to zero or the fluid is at rest.

3.2 *Stagnation surfaces*

At the equatorial plane, (7) reduces to

$$\frac{\mathrm{d}v^1}{\mathrm{d}r} - \frac{1}{r}\,v^1 = \frac{1}{8\pi\eta}\,\frac{e^2}{r^4}, \tag{16}$$

which on integration gives

$$v^1 = -\frac{e^2}{32\pi\eta r^3}. \tag{17}$$

Here the integration constant is taken as zero as otherwise v^1 would blow up as $r \to \infty$. With this value of v^1,

$$\frac{\mathrm{d}r}{\mathrm{d}\Phi} = \frac{v^1}{v^3} = \frac{-\,e^2\,\lambda}{a\,[(32\pi\eta)^2\,\lambda r^4 + e^4]^{1/2}}. \tag{18}$$

We find from (18) that $\mathrm{d}r/\mathrm{d}\,\Phi$ as well as all the higher derivatives of r with respect to Φ tend to vanish as $\lambda \to 0$ *i.e.* at the surfaces $r = r_+$ and $r = r_-$. Also $\mathrm{d}r/\mathrm{d}t$ tends to zero at these surfaces. The time co-ordinate t being the proper time for an observer at infinity, we may say that to such an observer the radial motion of the fluid at the two horizons $r = r_\pm$ vanishes. To such an observer it would seem that the fluid approaches the surfaces r_+ and r_- asymptotically and he would imagine these surfaces as stagnation surfaces as far as the radial motion of the fluid is concerned. The fluid on the other hand does cross the surfaces $r = r_{\pm}$ in a finite time in its own reckoning since $\mathrm{d}r/\mathrm{d}s$ (the radial component of the fluid motion) does not vanish at these surfaces as evident from (17). The fluid goes right up to the origin which acts as a sink. This situation seems similar to the case of the Schwarzschild black hole where again a test particle ever approaches but never crosses the horizon ($r = 2m$) as far as an external observer is concerned but it does cross the horizon and reach the singularity $r = 0$ at a finite proper time in its own reckoning.

Acknowledgement

The author's thanks are due to Prof. A K Raychaudhuri for valuable guidance. The author is also thankful to UGC, New Delhi for the award of a Teacher Fellowshid and to Stewart College, Cuttack for granting him leave.

References

Raychaudhuri A K and Saha S K 1981 *J. Math. Phys.* **22** 2237
Tupper B O J 1981 *J. Math. Phys.* **22** 2666