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Generalised direct action and absorber theory of radiation

R V KAMAT
Tata Institute of Fundamental Research, Bombay 400 005, India
Permanent Address: St. Xavier's College, Bombay 400 001, India

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Abstract. It is known that corresponding to a field theory of a tensor of arbitrary rank, there exists a direct-action theory under certain conditions. An attempt is therefore made to construct an explicit direct-action theory for a tensor field of rank N in such a form that the absorber theory of radiation may be worked out using the methods made so well known by Wheeler-Feynman and Hoyle-Narlikar. It is possible to recover the familiar electromagnetic theory corresponding to the case of $N = 1$.

Keywords. Direct action; tensor field of arbitrary rank; parallel propagator; absorber theory of radiation.

1. Introduction

It is customary to describe the various interactions in nature in terms of fields and their quantization. Since Faraday, such a formulation has succeeded eminently in describing natural phenomena, particularly the two fundamental interactions of classical physics, namely, electromagnetism and gravitation. These successes have led to a belief that field prescription is the only way to describe these interactions. However, in 1845, Gauss conjectured about a possible description of electrodynamics by direct action between particles, propagating with a finite velocity. This very fruitful idea remained dormant until it was revived by Wheeler and Feynman (1945, 1949), while developing their absorber theory of radiation in the framework of direct-action formalism through the action integral of Schwarzschild and Fokker. The direct action theory has certain advantages over the field theory in that there is no need to construct a model of the electron to derive the formula for the force of radiation reaction on it. This force arises as a result of the reaction of the universe (collectively called the absorber) to the radiation emitted by it. The action of the electron on itself was thereby eliminated in addition to giving a physical basis for the Dirac formula for the radiation reaction. Another feature of the direct action theory is that it is time-symmetric through the description $F = \frac{1}{2} F_{\text{ret}} + \frac{1}{2} F_{\text{adv}}$, obtained with the help of Fokker action, where F_{ret} and F_{adv} are the retarded and the advanced solutions of Maxwell equations. This modifies the situation wherein the advanced solutions of Maxwell equations are rejected on the grounds of the violation of causality. The advanced solutions are incorporated in the direct action theory to explain the radiation damping phenomenon, thus making them as much legitimate as the retarded ones.

Wheeler and Feynman (1945) worked out their theory in flat space-time by including both retarded and advanced solutions. Though the direct action theory is time-symmetric, the observed preference in nature to the retarded field is, according to them, owing to the asymmetry of initial conditions with respect to time. However, the presence of matter in the universe makes space-time curved and it is necessary to examine the absorber theory in curved space-time. Several authors, including Hoyle and Narlikar (1964), working with time-symmetric direct action theory in curved space-time, showed that the curved space-time of the physical universe contributed to fix the arrow of time in the observed direction.

Narlikar (1968) investigated the problem of the existence of a general correspondence between the field theory and the theory of direct interparticle action. He proposed that to a field of an arbitrary rank, there exists a corresponding direct action theory provided certain conditions are satisfied. He showed that there is a similarity in the energy-momentum tensors in the two theories, hence the similarity in the electromagnetic case is not accidental. The electromagnetic case is just a special case of the general field of rank N. If this is so, then the absorber theory, which has been so successfully worked out in the electromagnetic case, should also be possible for the general case of arbitrary rank. There is nothing special about electromagnetism. It is the object of this paper to work out an absorber theory of radiation in the general case of rank N and show that this reduces to the familiar electromagnetic case when $N = 1$. Before we do this, we present in the next section the field theory for the tensor fields of arbitrary rank in the manifestly covariant form and then proceed to construct a direct particle theory analogous to this field theory in the following section. An action integral for this theory is established from which the equations of motion and the 'field' equations are obtained. In order to preserve the general covariance it was found necessary to introduce a tensor of rank $N-1$ describing the 'internal degrees of freedom' of the particle interacting with other particles through direct action. The variation of the action integral by varying particle paths then leads to the equations of motion which are expressed in terms of an antisymmetric second rank tensor $F^{(b)}_{\alpha\beta}$ which serves as the 'field' tensor of the formalism. The potentials of the direct-action satisfy the four-dimensional wave equation in curved space-time with an expression for the 'current' density which is the central object of interest in the paper. When this is worked out, it is found that it has the same form as the electric current density in curved space-time formalism of electrodynamics, except that the charge appearing as the coefficient of the integral is replaced by an expression which contains the coupling constant D and the scalar parameter expressing the 'internal degrees of freedom'. This expression for the 'current' reduces to that of the current in the electromagnetic case provided we identify the coupling constant D with the electric charge e.

When one introduces tensor fields of higher ranks there appear to be two alternatives. These are the potentials (a) which are symmetric in its indices and (b) which are completely antisymmetric in its indices. In this paper, we shall consider both these cases in turn and show that they lead to similar expressions for the 'field' equations and the current, but with the difference that the antisymmetric case cannot be carried to cases $N \geqslant 5$ as a completely antisymmetric tensor of rank $\geqslant 5$ is trivially zero in the four-dimensional space-time. Once the direct action formalism is worked out upto this stage, it is relatively an easy matter to deduce the absorber theory of radiation for the general case following the lead of Wheeler-Feynman and

Hoyle-Narlikar. We may look upon the generalized charge $e_{(N)}$ as some kind of 'inertial moment of order N' which can be considered as the generalisation of the quadrupole moment appearing as the current in the power radiated in the gravitational radiation formula (Landau and Lifschitz (1975)).

In § 2 below, we present the field theory for the fields of arbitrary rank. This is followed by the construction of the direct particle theory of symmetric tensor of rank N in § 3. Section 4 then gives the direct action theory for the antisymmetric case. Finally in § 5, we work out the absorber theory of radiation in the general case.

2. Fields of arbitrary rank: field theory

In this section we shall present the field theory of a tensor field of rank N. The structure of the theory is independent of the fact whether the tensor field is symmetric or antisymmetric. Hence we shall make no mention of the symmetry properties of the potential though one must remember that the antisymmetric tensor of rank $\geqslant 5$ is identically zero in the four-dimensional space-time. Let ϕ denote a tensor field of arbitrary rank N, in interaction with particles. We assume that its properties can be derived from an action integral of the form

$$I = -\int L\,[\phi]\,\sqrt{-g}\,\mathrm{d}^4 x - \sum_a \int F\,[\phi, a]\,\mathrm{d}\,a - \sum_a m_a \int \mathrm{d}a \qquad (1)$$

where (i) $L\,[\phi]$ is a pure field Lagrangian which is assumed to be a bilinear invariant composed of ϕ and its first derivatives. The coefficients in this bilinear form are functions of space-time geometry. $L\,[\phi]$ has the form†

$$L\,[\phi] = B^{\bar{\xi}\,\bar{\eta}\,\zeta\,\chi}\,\phi_{\bar{\xi};\,\zeta}\,\phi_{\bar{\eta};\,\chi} + C^{\bar{\xi}\,\bar{\eta}\,\zeta}\,\phi_{\bar{\xi};\,\zeta}\,\phi_{\bar{\eta}} + A^{\bar{\xi}\,\bar{\eta}}\,\phi_{\bar{\xi}}\,\phi_{\bar{\eta}}, \qquad (2)$$

(ii) the second integral in (1) is the action for the particles interacting with the field. The expression $F\,[\phi, a]$ is of the form

$$D\,\phi_{\bar{\eta}}\,K^{(a)\,\bar{\eta}} \qquad (3)$$

where D is a coupling constant and $K^{(a)\,\bar{\eta}}$ is a tensor of rank N depending entirely on the worldline of particle **a**, (iii) the third integral is the free particle action. In expression (1) we have excluded the purely gravitational term of the form

$$\frac{1}{16\pi G}\int R\,\sqrt{-g}\,\mathrm{d}^4 x$$

† In the expression (2) and what follows we use the following notation:

(a) we use Greek indices ξ, η, ζ, χ as tensor indices at the general point X.
(b) indices α, β, γ, δ, ϵ at the location A of the particle **a**.
(c) λ, μ, ν, ρ, σ at the location B of the particle **b**.
(d) We shall use a short form like, say, $\bar{\xi}$ to indicate that it stands for $\xi_1, \xi_2, \ldots, \xi_N$. At times, when needed, we may explicitly write all the indices.
(e) Summation over repeated indices is assumed.
(f) da is the invariant interval for the particle **a**.

as our theory is worked out in a given or fixed Riemannian space-time.

From the action integral (1), we obtain the field equations by varying the potentials ϕ and the equations of motion by varying the particle paths. Varying ϕ in equation (1), we obtain

$$\int \left[\frac{\partial L}{\partial \phi_{\bar{\eta}}} - \left(\frac{\partial L}{\partial \phi_{\bar{\eta};x}}\right)_{;x}\right] \delta \phi_{\bar{\eta}} \sqrt{-g}\, \mathrm{d}^4 x + \sum_a D \int \delta \phi_{\bar{\eta}} K^{(a)\bar{\eta}}\, \mathrm{d}a = 0. \tag{4}$$

The second term in the above equation can be written as

$$\sum_a D \int \int \mathrm{d}^4 x\, \delta^4 (X, A)\, K^{(a)\bar{\eta}}\, \delta \phi_{\bar{\eta}}\, \mathrm{d}a$$

where $\delta^4(X, A)$ is the four-dimensional δ-function. We now introduce an *Ansatz* which involves a bitensor $\bar{g}^{\bar{\eta}}_{\bar{a}}$ of rank N at each point X and A and known as the generalised parallel propagator of N components to modify the above expression to

$$\sum_a D \int\int \mathrm{d}^4 x \delta^4 (X, A)\, \bar{g}^{\bar{\eta}}_{\bar{a}}\, K^{(a)\bar{a}}\, \delta\phi_{\bar{\eta}}\, \mathrm{d}a.$$

The Lagrangian derivatives in the bracket of the first integral in (4) have the form

$$-\left[\left(B^{\bar{\xi}\bar{\eta}\zeta x} + B^{\bar{\eta}\bar{\xi}x\zeta}\right)\phi_{\bar{\xi};\zeta}\right]_{;x} - \left(C^{\bar{\eta}\bar{\xi}x}\phi_{\bar{\xi}}\right)_{;x}$$

$$+ C^{\bar{\xi}\bar{\eta}\zeta}\phi_{\bar{\xi};\zeta} + \left(A^{\bar{\xi}\bar{\eta}} + A^{\bar{\eta}\bar{\xi}}\right)\phi_{\bar{\xi}}.$$

Inserting these results in (4) and in view of $\delta\phi_{\bar{\eta}}$ being arbitrary, we obtain

$$\left[\left(B^{\bar{\xi}\bar{\eta}\zeta x} + B^{\bar{\eta}\bar{\xi}x\zeta}\right)\phi_{\bar{\xi};\zeta}\right]_{;x} + \left(C^{\bar{\eta}\bar{\xi}x}\phi_{\bar{\xi}}\right)_{;x}$$

$$- C^{\bar{\xi}\bar{\eta}\zeta}\phi_{\bar{\xi};\zeta} - \left(A^{\bar{\xi}\bar{\eta}} + A^{\bar{\eta}\bar{\xi}}\right)\phi_{\bar{\xi}}$$

$$= \sum_a D \int \frac{\delta^4 (X, A)}{[-g(X)]^{1/2}}\, \bar{g}^{\bar{\eta}}_{\bar{a}}\, K^{(a)\bar{a}}\, \mathrm{d}a. \tag{5}$$

This is the field equation for the potential $\phi_{\bar{\eta}}$. We note that its form is independent of the symmetry properties of $\phi_{\bar{\eta}}$ except that the coefficients B, C and A will have to be symmetric or antisymmetric in its indices $\bar{\xi}$ and $\bar{\eta}$ according as ϕ is symmetric or antisymmetric in its indices. It is possible to obtain the equation of motion for the particle **a**, if we vary the path of the particle in the action integral (1), but we shall not do this here as, we shall see later, this requires a number of conditions to be imposed on the coefficients B, C and A as well as $K^{(a)}$. We shall be doing this in detail when we deal with the direct action theory in the next section.

3. Direct action theory analogous to *N*th rank tensor field theory: symmetric case

We now construct a direct action theory analogous to the above field theory for the symmetric case. Let $\bar{G}_{\bar{\alpha}\bar{\lambda}}$ be the propagator between points A and B which are the locations of the particles **a** and **b**. $\bar{G}_{\bar{\alpha}\bar{\lambda}}$ is a bitensor with N indices at each end. This is a Green's function satisfying the equation

$$\left[\left(B^{\bar{\alpha}\bar{\beta}\gamma\delta} + B^{\bar{\beta}\bar{\alpha}\delta\gamma}\right)\bar{G}_{\bar{\alpha}\bar{\lambda};\gamma}\right]_{;\delta} + \left[C^{\bar{\beta}\bar{\alpha}\gamma}\,\bar{G}_{\bar{\alpha}\bar{\lambda}}\right]_{;\gamma}$$
$$- C^{\bar{\alpha}\bar{\beta}\gamma}\,\bar{G}_{\bar{\alpha}\bar{\lambda};\gamma} - \left(A^{\bar{\alpha}\bar{\beta}} + A^{\bar{\beta}\bar{\alpha}}\right)\bar{G}_{\bar{\alpha}\bar{\lambda}} = \frac{\delta^4(A, B)}{[-\bar{g}(A, B)]^{1/2}}\,\bar{g}^{\bar{\beta}}_{\bar{\lambda}}. \tag{6}$$

where $\bar{g}(A, B)$ is the determinant of the parallel propagator $\bar{g}_{\alpha\lambda}$ of rank 1 at each point and $\bar{g}^{\bar{\beta}}_{\bar{\lambda}}$ is a parallel propagator of N components. $\bar{G}_{\bar{\alpha}\bar{\lambda}}$ is symmetric with respect to A and B *i.e.*

$$\bar{G}_{\bar{\alpha}\bar{\lambda}} = \bar{G}_{\bar{\lambda}\bar{\alpha}}. \tag{7}$$

We define the ‘ potential ’ of the direct action theory,

$$\phi^{(b)}_{\bar{\alpha}} = D \int \bar{G}_{\bar{\alpha}\bar{\lambda}}\,K^{(b)\bar{\lambda}}\,\mathrm{d}b, \tag{8}$$

where D is a coupling constant and $K^{(b)\bar{\lambda}}$ is a tensor of rank N depending entirely on the worldline of **b**. This gives

$$[(B^{\bar{\alpha}\bar{\beta}\gamma\delta} + B^{\bar{\beta}\bar{\alpha}\delta\gamma})\,\phi^{(b)}_{\bar{\alpha};\gamma}]_{;\delta} + [C^{\bar{\beta}\bar{\alpha}\gamma}\,\phi^{(b)}_{\bar{\alpha}}]_{;\gamma}$$
$$- C^{\bar{\alpha}\bar{\beta}\gamma}\,\phi^{(b)}_{\bar{\alpha};\gamma} - (A^{\bar{\alpha}\bar{\beta}} + A^{\bar{\beta}\bar{\alpha}})\,\phi^{(b)}_{\bar{\alpha}} = J^{(b)\bar{\beta}} \tag{9}$$

where
$$J^{(b)\bar{\beta}} = D\int \frac{\delta^4(A, B)}{[-\bar{g}(A, B)]^{1/2}}\,\bar{g}^{\bar{\beta}}_{\bar{\lambda}}\,K^{(b)\bar{\lambda}}\,\mathrm{d}b. \tag{10}$$

Later we shall express $\bar{g}^{\bar{\beta}}_{\bar{\lambda}}$ by defining it in terms of the usual parallel propagator $\bar{g}^{\beta}_{\lambda}$ in both symmetric and antisymmetric cases. We write (9) as

$$(B^{\bar{\alpha}\bar{\beta}\gamma\delta} + B^{\bar{\beta}\bar{\alpha}\delta\gamma})\,\phi^{(b)}_{\bar{\alpha};\gamma\delta}$$
$$+ [(B^{\bar{\alpha}\bar{\beta}\gamma\delta} + B^{\bar{\beta}\bar{\alpha}\delta\gamma})_{;\delta} - (C^{\bar{\alpha}\bar{\beta}\gamma} - C^{\bar{\beta}\bar{\alpha}\gamma})]\,\phi^{(b)}_{\bar{\alpha};\gamma}$$
$$+ [C^{\bar{\beta}\bar{\alpha}\gamma}_{;\gamma} - (A^{\bar{\alpha}\bar{\beta}} + A^{\bar{\beta}\bar{\alpha}})]\,\phi^{(b)}_{\bar{\alpha}} = J^{(b)\bar{\beta}} \tag{11}$$

In (6), (9) and (11), the coefficients B, C and A are tensors connected with space-time quantities only. In order to give physical meanings to these tensors, we shall have to consider electrodynamics as a special case of the N-rank tensor theory, for $N = 1$, in which the potential $\phi^{(b)}_{a}$ of **b** at A is defined as (Hoyle and Narlikar 1964)

$$\phi^{(b)}_{a} = e \int \bar{G}_{a\lambda} \dot{b}^{\lambda} \, \mathrm{d}\, b, \tag{12}$$

where $\dot{b}^{\lambda} \equiv \mathrm{d}b^{\lambda}/\mathrm{d}b$ and $\bar{G}_{a\lambda}$ is a propagator between points A and B with one index at each end. $\phi^{(b)}_{a}$ is known to satisfy

$$g^{\alpha\beta} g^{\gamma\delta} \phi^{(b)}_{a;\gamma\delta} + R^{\alpha\beta} \phi^{(b)}_{a} = J^{(b)\beta}, \tag{13}$$

where $R^{\alpha\beta}$ is the Ricci tensor evaluated at the point A and

$$J^{(b)\beta} = e \int \frac{\delta^{4}(A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}^{\bar{\beta}}_{\lambda} \mathrm{d}b^{\lambda}, \tag{14}$$

the 4-current. If this correspondence holds, then in the case $N = 1$, we must have

$$B^{\bar{\alpha}\bar{\beta}\gamma\delta} + B^{\bar{\beta}\bar{\alpha}\delta\gamma} \to B^{\alpha\beta\gamma\delta} + B^{\beta\alpha\delta\gamma} = g^{\alpha\beta} g^{\gamma\delta}, \tag{15}$$

so also

$$(B^{\alpha\beta\gamma\delta} + B^{\beta\alpha\delta\gamma})_{;\delta} - (C^{\alpha\beta\gamma} - C^{\beta\alpha\gamma}) = 0. \tag{16}$$

This implies that

$$(B^{\alpha\beta\gamma\delta} + B^{\beta\alpha\delta\gamma})_{;\delta} = 0 \text{ and } C^{\alpha\beta\gamma} = C^{\beta\alpha\gamma}. \tag{17}$$

It can be seen from (15) that the first of (17) is an identity and the second shows that $C^{\alpha\beta\gamma}$ is symmetric in α and β. Finally, we see that $C^{\alpha\beta\gamma}_{;\gamma} - (A^{\alpha\beta} + A^{\beta\alpha})$, which is a symmetric tensor in α and β, must equal the Ricci tensor $R^{\alpha\beta}$. Hence

$$C^{\alpha\beta\gamma}_{;\gamma} - (A^{\alpha\beta} + A^{\beta\alpha}) = R^{\alpha\beta}. \tag{18}$$

The necessary condition that (11) reduces to (13) in the case $N = 1$ of electrodynamics is that the relations (15), (16) and (17) hold. In view of this, we take, in the general case of rank N, the following to hold

$$B^{\bar{\alpha}\bar{\beta}\gamma\delta} + B^{\bar{\beta}\bar{\alpha}\delta\gamma} = g^{\bar{\alpha}\bar{\beta}} g^{\gamma\delta} = g^{\bar{\beta}\bar{\alpha}} g^{\delta\gamma}, \tag{19}$$

$$C^{\bar{\alpha}\bar{\beta}\gamma} = C^{\bar{\beta}\bar{\alpha}\gamma}, \tag{20}$$

$$C^{\bar{\alpha}\bar{\beta}\gamma}_{;\gamma} - (A^{\bar{\alpha}\bar{\beta}} + A^{\bar{\beta}\bar{\alpha}}) = \mathscr{R}^{\bar{\alpha}\bar{\beta}} = \mathscr{R}^{\bar{\beta}\bar{\alpha}}. \tag{21}$$

Following the first of (17), we impose the condition

$$(B^{\bar{\alpha}\,\bar{\beta}\,\gamma\,\delta} + B^{\bar{\beta}\,\bar{\alpha}\,\delta\,\gamma})_{;\,\delta} = 0. \tag{22}$$

In (19), we have introduced a new symbol $g^{\bar{\alpha}\,\bar{\beta}}$ which will be defined presently. In view of this definition, we shall see that the relation (22) becomes an identity. We have also introduced another new quantity $\mathcal{R}^{\bar{\alpha}\,\bar{\beta}}$, which is symmetric in the indices $\bar{\alpha}$ and $\bar{\beta}$ and which we will assume to reduce to the Ricci tensor $R^{\alpha\,\beta}$ in the case $N=1$. We shall further assume that $\mathcal{R}^{\bar{\alpha}\,\bar{\beta}} = 0$ for flat space-time. Later when we work out the explicit form of $\mathcal{R}^{\bar{\alpha}\,\bar{\beta}}$, we shall see that these assumptions are justified. It now remains to define $g^{\bar{\alpha}\,\bar{\beta}} = g^{\bar{\beta}\,\bar{\alpha}}$ in a suitable fashion. We shall do this for the cases $N=2$ and $N=3$. The generalisation to the arbitrary N would then be obvious.

In the case $N = 2$, we define

$$g^{\bar{\alpha}\,\bar{\beta}} \equiv g^{\alpha_1\,\alpha_2\,\beta_1\,\beta_2} = \tfrac{1}{2}\,(g^{\alpha_1\,\beta_1}\,g^{\alpha_2\,\beta_2} + g^{\alpha_1\,\beta_2}\,g^{\alpha_2\,\beta_1}), \tag{23}$$

$g^{\bar{\alpha}\,\bar{\beta}}$ is not only symmetric in $\bar{\alpha}$ and $\bar{\beta}$, but also among α's and β's. Hence

$$\begin{aligned} g^{\bar{\alpha}\,\bar{\beta}}\,\phi^{(b)}_{\bar{\alpha}} &= \tfrac{1}{2}\,(g^{\alpha_1\,\beta_1}\,g^{\alpha_2\,\beta_2} + g^{\alpha_1\,\beta_2}\,g^{\alpha_2\,\beta_1})\,\phi^{(b)}_{\alpha_1\,\alpha_2}, \\ &= \tfrac{1}{2}\,(\phi^{(b)\,\beta_1\,\beta_2} + \phi^{(b)\,\beta_2\,\beta_1}), \\ &= \phi^{(b)\,\beta_1\,\beta_2}, \end{aligned} \tag{24}$$

provided we assume $\phi^{(b)}$ to be symmetric in its indices. Similarly in the case $N = 3$, we define

$$\begin{aligned} g^{\alpha_1\,\alpha_2\,\alpha_3\,\beta_1\,\beta_2\,\beta_3} = \tfrac{1}{6}\,(&g^{\alpha_1\,\beta_1}\,g^{\alpha_2\,\beta_2}\,g^{\alpha_3\,\beta_3} + g^{\alpha_1\,\beta_2}\,g^{\alpha_2\,\beta_3}\,g^{\alpha_3\,\beta_1} \\ &+ g^{\alpha_1\,\beta_3}\,g^{\alpha_2\,\beta_1}\,g^{\alpha_3\,\beta_2} + g^{\alpha_1\,\beta_3}\,g^{\alpha_2\,\beta_2}\,g^{\alpha_3\,\beta_1} \\ &+ g^{\alpha_1\,\beta_2}\,g^{\alpha_2\,\beta_1}\,g^{\alpha_3\,\beta_3} + g^{\alpha_1\,\beta_1}\,g^{\alpha_2\,\beta_3}\,g^{\alpha_3\,\beta_2}). \end{aligned} \tag{25}$$

Hence $\quad g^{\bar{\alpha}\,\bar{\beta}}\,\phi^{(b)}_{\bar{\alpha}} = \phi^{(b)\,\bar{\beta}},$

in this case also, where

$$\begin{aligned} \phi^{(b)\,\bar{\beta}} = \tfrac{1}{6}\,(&\phi^{(b)\,\beta_1\,\beta_2\,\beta_3} + \phi^{(b)\,\beta_2\,\beta_3\,\beta_1} + \phi^{(b)\,\beta_3\,\beta_1\,\beta_2} \\ &+ \phi^{(b)\,\beta_3\,\beta_2\,\beta_1} + \phi^{(b)\,\beta_2\,\beta_1\,\beta_3} + \phi^{(b)\,\beta_1\,\beta_3\,\beta_2}) \end{aligned} \tag{26}$$

again provided $\phi^{(b)\,\beta_1\,\beta_2\,\beta_3}$ is symmetric in β_1, β_2 and β_3. Equation (11) then becomes

$$g^{\bar{\alpha}\,\bar{\beta}}\, g^{\gamma\,\delta}\, \phi^{(b)}_{\bar{\alpha};\,\gamma\,\delta} + \mathcal{R}^{\bar{\alpha}\,\bar{\beta}}\, \phi^{(b)}_{\bar{\alpha}} = J^{(b)\,\bar{\beta}}. \tag{27}$$

We now write the action functional of the direct action theory for the general case of N-rank tensor field as

$$I = -\sum_{a} m_a \int da - \sum_{a<b}\sum D^2 \int\int \bar{G}_{\bar{\alpha}\,\bar{\lambda}}\, K^{(a)\,\bar{\alpha}}\, K^{(b)\,\bar{\lambda}}\, da\, db$$

$$= -\sum_{a} m_a \int da - \sum_{a<b}\sum D \int \phi^{(b)}_{\bar{\alpha}}\, K^{(a)\,\bar{\alpha}}\, da \tag{28}$$

using equation (8) which defines the direct action potential $\phi^{(b)}_{\bar{\alpha}}$ due to the particle **b** at the location of the particle **a**. As before, we note that in this action, we have excluded the purely gravitational term $(1/16\pi G)\int R\sqrt{-g}\, d^4 x$ as we are working in the given or fixed Riemannian space-time. The variation of the worldline of **a** in the first integral in (28) leads to

$$-\delta \sum_{a} m_a \int da = m_a \int g_{\alpha\,\epsilon}\left(\ddot{a}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\, \dot{a}^{\beta}\, \dot{a}^{\gamma}\right) \delta a^{\epsilon}\, da$$

$$= m_a \int g_{\alpha\,\epsilon} \frac{D\dot{a}^{\alpha}}{Da}\, \delta a^{\epsilon}\, da. \tag{29}$$

In the second integral in (28), we put

$$K^{(a)\,\bar{\alpha}} = I^{\alpha_1,\,\ldots,\,\alpha_{N-1}}_{(a)}\, \dot{a}^{\alpha_N}, \tag{30}$$

where the tensor $I^{\alpha_1,\,\ldots,\,\alpha_{N-1}}_{(a)}$ is assumed to describe some kind of 'internal degrees of freedom' of the particle **a**. At present, we leave this concept undefined. The variation of the worldline of **a** in this integral gives

$$\delta \int \phi^{(b)}_{\alpha_1,\,\ldots,\,\alpha_N}\, I^{\alpha_1,\,\ldots,\,\alpha_{N-1}}_{(a)}\, \dot{a}^{\alpha_N} da = \int F^{(b)}_{\epsilon\,\alpha_N}\, \dot{a}^{\alpha_N}\, \delta a^{\epsilon}\, da \tag{31}$$

where
$$F^{(b)}_{\epsilon\alpha_N} = \left(\phi^{(b)}_{\alpha_1,\,\ldots,\,\alpha_N}\, I^{\alpha_1,\,\ldots,\,\alpha_{N-1}}_{(a)}\right)_{;\,\epsilon} - \left(\phi^{(b)}_{\alpha_1,\,\ldots,\,\alpha_{N-1}\,\epsilon}\, I^{\alpha_1,\,\ldots,\,\alpha_{N-1}}_{(a)}\right)_{;\,\alpha_N}$$

$$= -F^{(b)}_{\alpha_N\,\epsilon}. \tag{32}$$

Combining the results of variation, we obtain

$$m_a \frac{D\dot{a}_{\epsilon}}{Da} = D \sum_{b\neq a} F^{(b)}_{\epsilon\,\alpha_N}\, \dot{a}^{\alpha_N}. \tag{33}$$

In the locally flat space-time, the above equation becomes†

$$m_a \ddot{a}_\epsilon = D \sum_{b \neq a} F^{(b)}_{\epsilon\, a_N} \dot{a}^{a_N}. \tag{34}$$

From now on we consider the problem in the locally flat space-time. We shall now try to find the differential equation satisfied by $F^{(b)}_{\epsilon\, a_N}$ for $N=2, 3, 4$ and 5. This is done with a view to obtain an expression for the 'current' in the theory and to generalise it to the case of arbitrary rank N.

Starting with the case $N=2$, we write

$$F^{(b)}_{\beta\gamma} = \left(\phi^{(b)}_{\alpha\gamma} I^{\alpha}_{(a)}\right)_{,\beta} - \left(\phi^{(b)}_{\alpha\beta} I^{\alpha}_{(a)}\right)_{,\gamma}. \tag{35}$$

As $I^{\alpha}_{(a)}$ describes the *internal* degrees of freedom of a we assume that $I^{\alpha}_{(a)}$ to be independent of its location, hence we put $I^{\alpha}_{(a),\,\beta} = 0$, so that

$$F^{(b)}_{\beta\gamma} = \left(\phi^{(b)}_{\alpha\gamma,\beta} - \phi^{(b)}_{\alpha\beta,\gamma}\right) I^{\alpha}_{(a)}. \tag{36}$$

Taking divergence of (36), we get

$$F^{(b),\beta}_{\beta\gamma} = \left(\phi^{(b)\ \ \beta}_{\alpha\gamma,\beta} - \phi^{(b)\ \ \beta}_{\alpha\beta,\gamma}\right) I^{\alpha}_{(a)}.$$

By definition, we have

$$\phi^{(b)}_{\alpha\beta} = D \int \bar{G}_{\alpha\beta\lambda\mu} I^{\lambda}_{(b)} \dot{b}^{\mu}\, \mathrm{d}b,$$

so that

$$\phi^{(b);\beta}_{\alpha\beta} = D \int \bar{G}_{\alpha\beta\lambda\mu}{}^{;\beta} I^{\lambda}_{(b)} \dot{b}^{\mu}\, \mathrm{d}b.$$

†At this stage, it is pertinent to ask the question: "Is the step $K^{(a)}\bar{a} = I^{a_1,\ldots,a_{N-1}}_{(a)} \dot{a}^{a_N}$ a unique one?" The answer to this question is apparently "no", as we see that a suggested alternative

$$K^{(a)}\bar{a} = \dot{a}^{a_1}\dot{a}^{a_2}\ldots\dot{a}^{a_N}$$

leads to

$$m_a \frac{D\dot{a}_\epsilon}{Da} = D \sum_{b \neq a} \Big\{ \left(\phi^{(b)}_{a_1,\ldots,a_N;\,\epsilon} - N\phi^{(b)}_{\epsilon\, a_2,\ldots,a_N;\, a_1}\right) \dot{a}^{a_1},\ldots,\dot{a}^{a_N}$$
$$- N(N-1)\,\phi^{(b)}_{\epsilon\, a_3\ldots a_N} \ddot{a}^{a_2}\dot{a}^{a_3}\ldots\dot{a}^{a_N}\Big\}$$

which contains the acceleration $\ddot{a}^{a_2}$ on the right. This will present difficulties of initial condition, besides the presence of the product of velocities in the expression.

In analogy with case $N=1$, we put $\phi^{(b);\beta}_{\alpha\beta}=0$ as the gauge condition on $\phi^{(b)}_{\alpha\beta}$. But this requires

$$\bar{G}_{\alpha\beta\lambda\mu}{}^{;\beta} = -\bar{G}_{\alpha\lambda;\mu}, \tag{37}$$

which we assume to hold. This formula, for the case $N=1$, i.e.

$$\bar{G}_{\alpha\lambda}{}^{;\alpha} = -\bar{G}_{;\lambda}$$

has been shown to hold by DeWitt and Brehme (1960). With the assumption of the gauge condition, we get

$$F^{(b),\beta}_{\beta\gamma} = \phi^{(b)}_{\alpha\gamma,\beta}{}^{\beta}\, I^{\alpha}_{(a)} = J^{(b)}_{\alpha\gamma}\, I^{\alpha}_{(a)} = \mathcal{J}^{(b)}_{\gamma}. \tag{38}$$

From (38) we see that $\mathcal{J}^{(b)}_{\gamma}$, and not $J^{(b)}_{\alpha\gamma}$, plays the role of the 'current' in this theory. In arriving at (38) we have used (27) for flat space-time. The vector current is given by the formula.

$$\mathcal{J}^{(b)}_{\gamma} = D\int \frac{\delta^4(A,B)}{[-\bar{g}(A,B)]^{1/2}}\bar{g}_{\beta\gamma\lambda\mu}\, I^{\beta}_{(a)}\, I^{\lambda}_{(b)}\, \dot{b}^{\mu}\, \mathrm{d}b. \tag{39}$$

The undetermined quantities $I^{\beta}_{(a)}$ and $I^{\lambda}_{(b)}$ can now be given a meaning. Let us assume that the particles **a** and **b** are identical and that each has its 'internal degree of freedom' described by a scalar I. Then $I^{\beta}_{(a)}\, I^{\lambda}_{(b)}$ will have the form which will contain a function $f_2\,(I^2)$ and a *bitensor* which is independent of the coordinate system. To this end, we put

$$I^{\beta}_{(a)}\, I^{\lambda}_{(b)} = f_2\,(I^2)\,\bar{g}^{\beta\lambda} \tag{40}$$

so that

$$\mathcal{J}^{(b)}_{\gamma} = f_2\,(I^2)\, D\int \frac{\delta^4(A,B)}{[-\bar{g}(A,B)]^{1/2}}\bar{g}_{\beta\gamma\lambda\mu}\, \bar{g}^{\beta\lambda}\, \dot{b}^{\mu}\, \mathrm{d}b \tag{41}$$

Now we have, by definition of $g_{\beta\gamma\lambda\mu}$

$$\bar{g}^{\beta\lambda}\,\bar{g}_{\beta\gamma\lambda\mu} = \frac{1}{2!}\bar{g}^{\beta\lambda}\,(\bar{g}_{\beta\lambda}\,\bar{g}_{\gamma\mu} + \quad {}_{\beta\mu}\,\bar{g}_{\gamma\lambda})$$

$$= \tfrac{1}{2}\left(\delta^{\lambda}_{\lambda}\,\bar{g}_{\gamma\mu} + \delta^{\lambda}_{\mu}\,\bar{g}_{\gamma\lambda}\right)$$

$$= \tfrac{1}{2}\,(4+1)\,\bar{g}_{\gamma\mu} = \frac{5}{2}\bar{g}_{\gamma\mu}. \tag{42}$$

With this, the expression for $\mathcal{J}_{\gamma}^{(b)}$ becomes

$$\mathcal{J}_{\gamma}^{(b)} = \frac{5}{2} f_2 (I^2)\, D \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\gamma\mu}\, \dot{b}^{\mu}\, \mathrm{d}b. \tag{43}$$

We see that in the expression (38), the 'current' $\mathcal{J}_{\gamma}^{(b)}$ has the same form as that for the electromagnetic case (Hoyle and Narlikar 1964) except that the role of charge e is now played by $e_{(2)} = (5/2) f_2 (I^2) D$.

In the case $N = 3$, in an analogous fashion, we have

$$\mathcal{J}_{\gamma}^{(b)} = D \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\alpha\beta\gamma\lambda\mu\nu}\, \underset{(a)}{I}^{\alpha\beta}\, \underset{(b)}{I}^{\lambda\mu}\, \dot{b}^{\gamma}\, \mathrm{d}b. \tag{44}$$

Here, again, we put

$$\underset{(a)}{I}^{\alpha\beta}\, \underset{(b)}{I}^{\lambda\mu} = f_3 (I^2)\, \bar{g}^{\alpha\beta\lambda\mu} \tag{45}$$

then

$$\bar{g}^{\alpha\beta\lambda\mu}\, \bar{g}_{\alpha\beta\gamma\lambda\mu\nu} = \frac{1}{3!\,2!} (\bar{g}^{\alpha\lambda} \bar{g}^{\beta\mu} + \bar{g}^{\alpha\mu} \bar{g}^{\beta\lambda})$$

$$\times (\bar{g}_{\alpha\lambda} \bar{g}_{\beta\mu} \bar{g}_{\gamma\nu} + \bar{g}_{\alpha\mu} \bar{g}_{\beta\nu} \bar{g}_{\gamma\lambda} + \bar{g}_{\alpha\nu} \bar{g}_{\beta\lambda} \bar{g}_{\gamma\mu}$$

$$+ \bar{g}_{\alpha\nu} \bar{g}_{\beta\mu} \bar{g}_{\gamma\lambda} + \bar{g}_{\alpha\mu} \bar{g}_{\beta\lambda} \bar{g}_{\gamma\nu} + \bar{g}_{\alpha\lambda} \bar{g}_{\beta\nu} \bar{g}_{\gamma\mu})$$

$$= \frac{1}{3!\,2!} \Big(\delta^{\lambda}_{\lambda} \delta^{\mu}_{\mu} \bar{g}_{\gamma\nu} + \delta^{\mu}_{\lambda} \delta^{\lambda}_{\mu} \bar{g}_{\gamma\nu} + \delta^{\lambda}_{\mu} \delta^{\mu}_{\nu} \bar{g}_{\gamma\lambda} + \delta^{\mu}_{\mu} \delta^{\lambda}_{\nu} \bar{g}_{\gamma\lambda}$$

$$+ \delta^{\lambda}_{\nu} \delta^{\mu}_{\lambda} \bar{g}_{\gamma\mu} + \delta^{\mu}_{\nu} \delta^{\lambda}_{\lambda} \bar{g}_{\gamma\mu} + \delta^{\lambda}_{\nu} \delta^{\mu}_{\mu} \bar{g}_{\gamma\lambda} + \delta^{\mu}_{\nu} \delta^{\lambda}_{\mu} \bar{g}_{\gamma\lambda}$$

$$+ \delta^{\lambda}_{\mu} \delta^{\mu}_{\lambda} \bar{g}_{\gamma\nu} + \delta^{\mu}_{\mu} \delta^{\lambda}_{\lambda} \bar{g}_{\gamma\nu} + \delta^{\lambda}_{\lambda} \delta^{\mu}_{\nu} \bar{g}_{\gamma\mu} + \delta^{\mu}_{\lambda} \delta^{\lambda}_{\nu} \bar{g}_{\gamma\mu} \Big)$$

$$= \frac{1}{3!\,2!} [(4 \times 4 + 4) + (1 + 4) + (1 + 4) + (4 + 1)$$

$$+ (4 + 4 \times 4) + (4 + 1)]\, \bar{g}_{\gamma\nu}$$

$$= \frac{1}{3!\,2!} [20 \times 2 + 5 \times 4]\, \bar{g}_{\gamma\nu} = \frac{60}{12} \bar{g}_{\gamma\nu} = 5 \bar{g}_{\gamma\nu}.$$

With this, we write

$$\mathcal{J}_{\gamma}^{(b)} = 5 f_3 (I^2)\, D \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\gamma\nu}\, \dot{b}^{\nu}\, \mathrm{d}b. \tag{46}$$

Here, as we see, again the expression for the current has the same form as in the electromagnetic case, but with the charge replaced by $e_{(3)} = 5\, f_3\, (I^2)\, D$.

In order to arrive at the general formula for in the case of rank N, we may work out the corresponding formulas for the cases $N=4$ and $N=5$. The calculations, though straightforward, are extremely involved. Without going through these calculations here, we state the results for these cases.

Case $N=4$

$$\mathcal{J}^{(b)}_{\delta} = D \int \frac{\delta^4\,(A,\, B)}{[-\bar{g}\,(A,\, B)]^{1/2}}\, \bar{g}_{\alpha\,\beta\,\gamma\,\delta\,\lambda\,\mu\,\nu\,\rho}\; I^{\alpha\,\beta\,\gamma}_{(a)}\; I^{\lambda\,\mu\,\nu}_{(b)}\; \dot{b}^{\rho}\; \mathrm{d}b. \tag{47}$$

Here $\quad I^{\alpha\,\beta\,\gamma}_{(a)}\; I^{\lambda\,\mu\,\nu}_{(b)} = f_4\,(I^2)\,\bar{g}^{\alpha\,\beta\,\gamma\,\lambda\,\mu\,\nu}$

and

$$\bar{g}_{\alpha\,\beta\,\gamma\,\delta\,\lambda\,\mu\,\nu\,\rho}\; \bar{g}^{\alpha\,\beta\,\gamma\,\lambda\,\mu\,\nu} = \frac{1}{4!\,3!}\,[120\times 6 + 30\times 18]\; \bar{g}_{\delta\rho} = (35/4)\,\bar{g}_{\delta\rho}.$$

This makes

$$\mathcal{J}^{(b)}_{\delta} = (35/4)\, f_4\,(I^2)\; D \int \frac{\delta^4\,(A,\, B)}{[-\bar{g}\,(A,\, B)]^{1/2}}\, \bar{g}_{\delta\rho}\; \dot{b}^{\rho}\; \mathrm{d}b \tag{48}$$

with $\quad e_{(4)} = (35/4)\, f_4\,(I^2)\; D.$

Case $N = 5$:

$$\mathcal{J}^{(b)}_{\epsilon} = D \int \frac{\delta^4\,(A,\, B)}{[-\bar{g}\,(A,\, B)]^{1/2}}\, \bar{g}_{\alpha\,\beta\,\gamma\,\delta\,\epsilon\,\lambda\,\mu\,\nu\,\rho\,\sigma}\, I^{\alpha\,\beta\,\gamma\,\delta}_{(a)}\; I^{\lambda\,\mu\,\nu\,\rho}\, \dot{b}^{\sigma}\; \mathrm{d}b \tag{49}$$

As before, here we set

$$I^{\alpha\,\beta\,\gamma\,\delta}_{(a)}\; I^{\lambda\,\mu\,\nu\,\rho}_{(b)} = f_5\,(I^2)\,\bar{g}^{\alpha\,\beta\,\gamma\,\delta\,\lambda\,\mu\,\nu\,\rho}$$

so that

$$\bar{g}_{\alpha\,\beta\,\gamma\,\delta\,\epsilon\,\lambda\,\mu\,\nu\,\rho\,\sigma}\; \bar{g}^{\alpha\,\beta\,\gamma\,\delta\,\lambda\,\mu\,\nu\,\rho} = \frac{1}{5!\,4!}\,[840 \times 24 + 210 \times 96]\; \bar{g}_{\epsilon\,\sigma}$$

$$= 14\, g_{\epsilon\,\sigma}$$

to get

$$\mathcal{J}^{(b)}_{\epsilon} = 14\, f_5\,(I^2)\; D \int \frac{\delta^4\,(A,\, B)}{[-\bar{g}\,(A,\, B)]^{1/2}}\, \bar{g}_{\epsilon\,\sigma}\; \dot{b}^{\sigma}\; \mathrm{d}b. \tag{50}$$

From the above calculations, we see a pattern emerging. From the results obtained for the cases $N=2$, 3, 4 and 5, one can easily show that in the general case of rank N,

$$\sigma_{\epsilon}^{(b)} = e_{(N)} \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\epsilon\sigma} \dot{b}^{\sigma} \, \mathrm{d}b \tag{51}$$

where
$$e_{(N)} = \frac{(N+3)!}{4!\,N!} f_N (I^2)\, D. \tag{52}$$

From our general formalism, it should be possible to obtain the familiar case of electrodynamics for $N=1$. We note that this means

$$e_{(N)} \xrightarrow[N=1]{} e_{(1)} = e \text{ (electric charge).} \tag{53}$$

Indeed $(N+3)!/4!N!$ equals 1 when $N=1$ and we would, in addition, need $f_N (I^2)$ to go to 1 for $N=1$. There are a number of possible forms of f_N satisfying the above requirement, the simplest of which is, perhaps, $f_N (I^2)=I^{2(N-1)}$. We shall, however, for the present, leave this choice undecided. The requirement (53) compels us to identify the coupling constant D with the electric charge e. Hence we write

$$e_{(N)} = \frac{(N+3)!}{4!N!} f_N (I^2)\, e. \tag{54}$$

as the entity which plays the role of charge in the direct action theory for a tensor field of rank N. One may look upon this generalised charge $e_{(N)}$ of (54) as some kind of 'inertial moment of order N' which can be considered as the generalisation of the quadrupole moment appearing as the current in the power radiated in the gavitational radiation formula (Landau and Lifschitz 1975).

In the most general case of rank N, we have

$$F^{(b);\,\alpha}_{\alpha\beta} = \mathcal{J}^{(b)}_{\beta} \tag{55}$$

as the 'field' equation satisfied by the antisymmetric tensor $F^{(b)}_{\alpha\beta}$ given by (32). The 'current' on the right side of (55) is given by the expression (51).

Inserting the expression (32) for $F^{(b)}_{\alpha\beta}$ in (55) and imposing the gauge condition on ϕ in the form

$$\phi^{(b)}_{\alpha_1, \ldots, \alpha_{N-1}}{}^{\alpha_N}{}_{;\alpha_N} = 0, \tag{56}$$

one can obtain the wave equation (27) satisfied by $\phi^{(b)}_{\bar{\alpha}}$. This also yields an explicit expression for $\mathcal{R}^{\bar{\alpha}\bar{\beta}}$. The expression, which we shall prove in the appendix, is

$$\mathcal{R}^{\alpha_1, \ldots, \alpha_N \beta_1, \ldots, \beta_N} = R^{\alpha_N \beta_N} g^{\alpha_1 \beta_1}, \ldots, g^{\alpha_{N-1} \beta_{N-1}}$$
$$+ \sum_{r=1}^{N-1} R^{\alpha_r \beta_r \alpha_N \beta_N} g^{\alpha_1 \beta_1} \ldots g^{\alpha_{r-1} \beta_{r-1}} g^{\alpha_{r+1} \beta_{r+1}} \ldots g^{\alpha_{N-1} \beta_{N-1}} \tag{57}$$

In the above expression, the first term contains the Ricci tensor $R^{\alpha_N \beta_N}$ and the remaining terms contain the full Riemann tensors. As pointed out earlier, $\mathcal{R}^{\bar{\alpha}\bar{\beta}}$ satisfies $\mathcal{R}^{\bar{\alpha}\bar{\beta}} = \mathcal{R}^{\bar{\beta}\bar{\alpha}}$. It is not completely symmetric in all α's and β's. This is to be expected as we have singled out α_N in defining $K^{(a)\bar{\alpha}}$ by putting $K^{(a)\bar{\alpha}} = I_{(a)}^{\alpha_1, \ldots, \alpha_{N-1}} \dot{a}^{\alpha_N}$, but α_N could be any of the α's from α_1 to α_N hence to this extent $\mathcal{R}^{\bar{\alpha}\bar{\beta}}$ can be said to be symmetric in its indices. For $N = 1$, $\mathcal{R}^{\bar{\alpha}\bar{\beta}} \to R^{\alpha\beta}$ as can be seen from (57). Finally $\mathcal{R}^{\bar{\alpha}\bar{\beta}}$ vanishes in flat space-time as Ricci and Riemann tensors vanish under these conditions, so that in the locally flat space-time the wave equation (27) reduces to

$$g^{\bar{\alpha}\bar{\beta}} g^{\gamma\delta} \phi^{(b)}_{\bar{\alpha};\gamma\delta} = J^{(b)\bar{\beta}}.$$

4. Direct action theory: antisymmetric case*

In this section, we shall consider the direct action theory for a completely antisymmetric tensor field of rank N. We now consider an antisymmetric tensor potential $\phi^{(b)}_{\alpha_1 \ldots \alpha_N}$. We can formally obtain a wave equation satisfied by it without imposing the gauge condition. If we start with the well known expression for the potential $\phi^{\alpha}_{(b)}$ of rank 1, namely

$$\Box \phi^{\alpha}_{(b)} + R^{\alpha}_{\beta} \phi^{\beta}_{(b)} = \text{source} = J^{\alpha}_{(b)}, \tag{58}$$

then we require the wave operator on a second rank tensor potential $\phi^{\alpha\beta}_{(b)}$ to be able to give the above rank 1 wave operator. Let

$$\phi^{\alpha}_{(b)} = \phi^{\alpha\beta}_{(b);\beta}. \tag{59}$$

We now write

$$\Box \phi^{\alpha}_{(b)} + R^{\alpha}_{\beta} \phi^{\beta}_{(b)} = \Box \phi^{\alpha\beta}_{(b);\beta} + R^{\alpha}_{\beta} \phi^{\beta\gamma}_{(b);\gamma}, \tag{60}$$

The right side of (60) can be put in the form

$$\phi^{\alpha\beta\;\;\gamma}_{(b);\beta\gamma} + R^{\alpha}_{\beta} \phi^{\beta\gamma}_{(b);\gamma}$$

$$= \left(\phi^{\alpha\beta\;\;\gamma}_{(b);\beta} + R^{\alpha\gamma}_{\delta\beta} \phi^{\delta\beta}_{(b)} + R^{\beta\gamma}_{\delta\beta} \phi^{\alpha\delta}_{(b)}\right)_{;\gamma} + R^{\alpha}_{\beta} \phi^{\beta\gamma}_{(b);\gamma}$$

*The author thanks Professor J. V. Narlikar for suggesting this alternative.

$$= \phi^{\alpha\beta;\gamma}_{(b)\;;\gamma\beta} + R^{\alpha}_{\delta\gamma\beta}\phi^{\delta\beta;\gamma}_{(b)} + R^{\beta}_{\delta\gamma\beta}\phi^{\alpha\delta;\gamma}_{(b)}$$

$$+ R^{\gamma}_{\delta\gamma\beta}\phi^{\alpha\beta;\delta}_{(b)} + \left(R^{\alpha\gamma}_{\delta\beta}\phi^{\delta\beta}_{(b)} + R^{\beta\gamma}_{\delta\beta}\phi^{\alpha\delta}_{(b)}\right)_{;\gamma} + R^{\alpha}_{\beta}\phi^{\beta\gamma}_{(b);\gamma}.$$

The third and fourth terms on the right of the last expression cancel each other to give

$$\phi^{\alpha\beta\;\gamma}_{(b);\beta\gamma} + R^{\alpha}_{\beta}\phi^{\beta\gamma}_{(b);\gamma}$$

$$= \left(\Box\,\phi^{\alpha\beta}_{(b)}\right)_{;\beta} + \left(R^{\alpha\gamma}_{\delta\beta}\phi^{\delta\beta}_{(b)} + R^{\gamma}_{\delta}\phi^{\alpha\delta}_{(b)}\right)_{;\gamma} + R^{\alpha}_{\delta\gamma\beta}\phi^{\delta\beta;\gamma}_{(b)} + R^{\alpha}_{\beta}\phi^{\beta\gamma}_{(b);\gamma}$$

so that

$$\Box\,\phi^{\alpha}_{(b)} + R^{\alpha}_{\beta}\phi^{\beta}_{(b)}$$

$$= \left(\Box\,\phi^{\alpha\beta}_{(b)} + R^{\alpha\beta}_{\delta\gamma}\phi^{\delta\gamma}_{(b)} + R^{\beta}_{\delta}\phi^{\alpha\delta}_{(b)}\right.$$

$$\left. + R^{\alpha\beta}_{\delta\gamma}\phi^{\delta\gamma}_{(b)} + R^{\alpha}_{\gamma}\phi^{\gamma\beta}_{(\delta)}\right)_{;\beta}$$

$$- R^{\alpha\beta}_{\delta\gamma;\beta}\phi^{\delta\gamma}_{(\delta)} - R^{\alpha}_{\gamma;\beta}\phi^{\gamma\beta}_{(b)}.$$

If $\phi^{\alpha\beta}_{(b)} = -\phi^{\beta\alpha}_{(b)}$, it can be shown using Bianchi identities that the terms outside the divergence on the right side vanish.

Thus we get

$$\Box\,\phi^{\alpha}_{(b)} + R^{\alpha}_{\beta}\phi^{\beta}_{(b)}$$

$$= \left(\Box\,\phi^{\alpha\beta}_{(b)} + 2R^{\alpha\beta}_{\delta\gamma}\phi^{\delta\gamma}_{(b)} + R^{\beta}_{\delta}\phi^{\alpha\delta}_{(b)} + R^{\alpha}_{\gamma}\phi^{\gamma\beta}_{(b)}\right)_{;\beta}.$$

Hence the second wave operator is given by

$$\Box\,\phi^{\alpha\beta}_{(b)} + R^{\alpha\beta}_{\gamma\delta}\phi^{\gamma\delta}_{(b)} + R^{\alpha\beta}_{\delta\gamma}\phi^{\delta\gamma}_{(b)} + R^{\beta}_{\delta}\phi^{\alpha\delta}_{(b)} + R^{\alpha}_{\gamma}\phi^{\gamma\beta}_{(b)} = \text{source}. \tag{61}$$

In general, for the tensor potential of rank $N\,(\geqslant 2)$, it can be shown that

$$\Box\,\phi^{\beta_1\ldots\beta_N}_{(b)} + \sum_{r=1}^{N} R^{\beta_r}_{\gamma_r}\phi^{\beta_1\ldots\gamma_r\ldots\beta_N}_{(b)} + \sum\sum_{r\neq s} R^{\beta_r\beta_s}_{\gamma_r\gamma_s}\phi^{\beta_1\ldots\gamma_r\ldots\gamma_s\ldots\beta_N}_{(b)} = J^{\beta_r\ldots\beta_N}_{(b)}. \tag{62}$$

Putting (27) in the form

$$\Box \phi^{\bar{\beta}}_{(b)} + \mathscr{R}^{\bar{\alpha}\bar{\beta}} \phi^{(b)}_{\bar{\alpha}} = J^{\bar{\beta}}_{(b)}$$

we have

$$\mathscr{R}^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_N} \phi^{(b)}_{\alpha_1 \ldots \alpha_N} = \sum_{r=1}^{N} R^{\beta_r}_{\gamma_r} \phi^{\beta_1 \ldots \gamma_r \ldots \beta_N}_{(b)}$$

$$+ \sum_{r \neq s} \sum R^{\beta_r \beta_s}_{\gamma_r \gamma_s} \phi^{\beta_1 \ldots \gamma_r \ldots \gamma_s \ldots \beta_N}_{(b)}. \tag{63}$$

Since, in the above expression $\phi^{(b)}_{\bar{\alpha}}$ is arbitrary, we obtain,

$$\mathscr{R}^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_N} = \sum_{r=1}^{N} R^{\beta_r}_{\gamma_r} g^{[\alpha_1 \beta_1} \ldots g^{\alpha_r \gamma_r} \ldots g^{\alpha_N \beta_N]}$$

$$+ \sum_{r \neq s} \sum R^{\beta_r \beta_s}_{\gamma_r \gamma_s} g^{[\alpha_1 \beta_1} \ldots g^{\alpha_r \gamma_r} \ldots g^{\alpha_s \gamma_s} \ldots g^{\alpha_N \beta_N]} \tag{64}$$

In (64), we have used the symbol [ ] round the indices in g's to indicate the antisymmetrisation of these indices. From (64), one can see that $\mathscr{R}^{\bar{\alpha}\bar{\beta}} = \mathscr{R}^{\bar{\beta}\bar{\alpha}}$ and that $\mathscr{R}^{\bar{\alpha}\bar{\beta}} = 0$ in the flat space-time where the Ricci and Riemann tensors vanish.

The wave equation (27) can also be written as

$$g^{\bar{\alpha}\bar{\beta}} g^{\gamma\delta} \phi^{(b)}_{\bar{\alpha};\gamma\delta} + \mathscr{R}^{\bar{\alpha}\bar{\beta}} \phi^{(b)}_{\bar{\alpha}} = J^{\bar{\beta}}_{(b)}$$

where $g^{\bar{\alpha}\bar{\beta}} = g^{\bar{\beta}\bar{\alpha}}$, but is ıntisymmetric among its indices α and β, or instance, in the case $N = 3$

$$g^{\bar{\alpha}.\bar{\beta}} \equiv g^{\alpha_1 \alpha_2 \alpha_3 \beta_1 \beta_2 \beta_3}$$

$$= \tfrac{1}{6} (g^{\alpha_1 \beta_1} g^{\alpha_2 \beta_2} g^{\alpha_3 \beta_3} + g^{\alpha_1 \beta_2} g^{\alpha_2 \beta_3} g^{\alpha_3 \beta_1}$$

$$+ g^{\alpha_1 \beta_3} g^{\alpha_2 \beta_1} g^{\alpha_3 \beta_2} - g^{\alpha_1 \beta_3} g^{\alpha_2 \beta_2} g^{\alpha_3 \beta_1}$$

$$- g^{\alpha_1 \beta_2} g^{\alpha_2 \beta_1} g^{\alpha_3 \beta_3} - g^{\alpha_1 \beta_1} g^{\alpha_2 \beta_3} g^{\alpha_3 \beta_2}).$$

This gives $\quad g^{\bar{\alpha}\bar{\beta}} \phi^{(b)}_{\bar{\alpha}} = \phi^{\bar{\beta}}_{(b)},$

provided $\phi^{(b)}_{\bar{\alpha}}$ is antisymmetric in all its indices.

The action integral of the direct action theory for the case of Nth-rank tensor field is

$$J = -\sum_{a} ma \int \mathrm{d}a - \sum_{a<b}\sum D^2 \int\int \bar{G}_{\bar{\alpha}\bar{\lambda}} K^{(a)\bar{\alpha}} K^{(b)\bar{\lambda}} \mathrm{d}a\, \mathrm{d}b$$

$$= -\sum_{a} m_a \int \mathrm{d}a - \sum_{a<b}\sum D \int \phi^{(b)}_{\bar{\alpha}} K^{(a)\bar{\alpha}} \mathrm{d}a$$

where $\bar{G}_{\bar{\alpha}\bar{\lambda}} = \bar{G}_{\bar{\lambda}\bar{\alpha}}$ but is antisymmetric in indices α and λ, so also are the tensors $K^{\bar{\alpha}}$. We put, as before, $K^{(a)\bar{\alpha}} = I^{\alpha_1 \dots \alpha_{N-1}}_{(a)} \dot{a}^{\alpha_N}$ where the tensor I describing the ' internal degrees of freedom ' is completely antisymmetric. Varying the action, one gets the equation of motion

$$m_a \ddot{a}_\epsilon = D \sum_{b \neq a} F^{(b)}_{\epsilon\,\alpha_N} \dot{a}^{\alpha_N},$$

where $$F^{(b)}_{\epsilon\,\alpha_N} = \left(\phi^{(b)}_{\alpha_1,\dots,\alpha_N} I^{\alpha_1,\dots,\alpha_{N-1}}\right)_{;\epsilon}$$

$$- \left(\phi^{(b)}_{\alpha_1,\dots,\alpha_{N-1}\epsilon} I^{\alpha_1,\dots,\alpha_{N-1}}\right)_{;\alpha_N}$$

$$= \left[\left(\phi^{(b)}_{\alpha_1,\dots,\alpha_N}\right)_{;\epsilon} - \left(\phi^{(b)}_{\alpha_1,\dots,\alpha_{N-1}\epsilon}\right)_{;\alpha_N}\right] I^{\alpha_1,\dots,\alpha_{N-1}}_{(a)}$$

$$= -F^{(b)}_{\alpha_N\epsilon}$$

We take the divergence of this equation to obtain

$$F^{(b);\epsilon}_{\epsilon\,\alpha_N} = \left(\phi^{(b)}_{\alpha_1,\dots,\alpha_N;\epsilon}{}^{\epsilon} - \phi^{(b)}_{\alpha_1,\dots,\alpha_{N-1}\epsilon;\alpha_N}{}^{\epsilon}\right) I^{\alpha_1,\dots,\alpha_{N-1}}_{(a)}.$$

In the locally flat space-time this has the form

$$F^{(b),\epsilon}_{\epsilon\,\alpha_N} = \left(\phi^{(b)}_{\alpha_1,\dots,\alpha_N,\epsilon}{}^{\epsilon} - \phi^{(b)}_{\alpha_1,\dots,\alpha_{N-1}\epsilon}{}^{,\epsilon}{}_{\alpha_N}\right) I^{\alpha_1,\dots,\alpha_{N-1}}_{(a)}.$$

In order to obtain the ' field ' equation involving the tensor $F^{(b)}_{\epsilon\alpha_N}$, we have to impose the gauge condition at this stage, *viz.*

$$\phi^{(b)}_{\alpha_1,\dots,\alpha_{N-1}\epsilon}{}^{,\epsilon} = 0$$

which gives

$$F^{(b),\epsilon}_{\epsilon\alpha_N} = \Box\, \phi^{(b)}_{\alpha_1,\dots,\alpha_N} I^{\alpha_1,\dots,\alpha_{N-1}}_{(a)} = J^{(b)}_{\alpha_1,\dots,\alpha_N} I^{\alpha_1,\dots,N-1}_{(a)} = \mathcal{J}^{(b)}_{\alpha_N}.$$

where $\mathcal{J}^{(b)}_{\alpha_N}$ has the form

$$\mathcal{J}^{(b)}_{\alpha_N} = D \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\alpha_1 \ldots \alpha_N\, \lambda_1 \ldots \lambda_N}\, I^{\alpha_1 \ldots \alpha_{N-1}}_{(a)}\, I^{\lambda_1 \ldots \lambda_{N-1}}_{(b)}\, \dot{b}^{\lambda_N}\, \mathrm{d}b.$$

Here $\bar{g}_{\bar{\alpha}\bar{\lambda}} = \bar{g}_{\bar{\lambda}\bar{\alpha}}$, but is antisymmetric in its indices α and λ separately. In this we put

$$I^{\alpha_1 \ldots \alpha_{N-1}}_{(a)}\, I^{\lambda_1 \ldots \lambda_{N-1}}_{(b)} = f_N (I^2)\, \bar{g}^{\alpha_1 \ldots \alpha_{N-1}\, \lambda_1 \ldots \lambda_{N-1}}.$$

Remembering the antisymmetric nature of $\bar{g}$, we work out the factor

$$g_{\alpha_1 \ldots \alpha_N\, \lambda_1 \ldots \lambda_N}\, \bar{g}^{\alpha_1 \ldots \alpha_{N-1}\, \lambda_1 \ldots \lambda_{N-1}}$$

in the integral for the 'current' $\mathcal{J}^{(b)}_{\alpha_N}$ to get the form

$$\mathcal{J}^{(b)}_{\alpha_N} = e_{(N)} \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\alpha_N\, \lambda_N}\, \dot{b}^{\lambda_N}\, \mathrm{d}b$$

where $e_{(1)} = f_1 (I^2)\, D$, $e_{(2)} = f_2 (I^2)\, (3/2)\, D$, $e_{(3)} = f_3 (I^2)\, D$ and $e_{(4)} = f_4 (I^2)\, (\frac{1}{4})\, D$. It is found that for the cases $N \geqslant 5$ $e_{(N)} = 0$. This result is not entirely unexpected as totally antisymmetric tensors $\phi^{(b)}_{\alpha}$ of rank $N \geqslant 5$ are trivially zero in the four-dimensional space-time and one is not able to carry on to the cases of arbitrarily large N. As in the symmetric case, $e_{(1)} = e$, which means that $f_1 (I^2) = 1$ and $D = e$. $f_N (I^2)$ has to be suitably chosen such that the above condition holds.

5. Absorber theory of radiation for the tensor field of rank N

In the earlier sections, we have developed a formalism for the direct action potential of rank N when the potential is symmetric as well as when it is completely antisymmetric. In both these cases, this has led us to a second rank antisymmetric 'field' tensor $F^{(b)}_{\epsilon\alpha_N}$ defined in terms of the potential $\phi^{(b)}_{\bar{\alpha}}$ and a $(N-1)$ rank tensor $I^{\alpha_1 \ldots \alpha_{N-1}}_{(a)}$ describing some kind of internal degrees of freedom of particles interacting with each other *via* direct interparticle action. The field tensor $F^{(b)}$ satisfies

$$F^{(b);\beta}_{\beta\,\alpha} = \mathcal{J}^{(b)}_{\alpha}, \tag{65}$$

where
$$\mathcal{J}^{(b)}_{\alpha} = e_{(N)} \int \frac{\delta^4 (A, B)}{[-\bar{g}(A, B)]^{1/2}} \bar{g}_{\alpha\,\lambda}\, \dot{b}^{\lambda}\, \mathrm{d}b \tag{66}$$

is the current produced by the particle **b**. The coefficient of the integral in $\mathcal{J}^{(b)}_{\alpha}$, which is the source of the 'field' $F^{(b)}_{\beta\,\alpha}$ is $e_{(N)}$, which has different forms in symmetric and antisymmetric cases. Nevertheless, in both these cases $e_{(N)}$ reduces to the electric charge e, when we put $N = 1$, provided we identify the coupling constant D of rank N tensor field with electric charge e and choose the form of the function $f_N(I^2)$ suitably, so that it reduces to 1, when $N = 1$.

Antisymmetry of $F^{(b)}_{\alpha\,\beta}$ immediately gives

$$F^{(b)}_{\alpha\,\beta;\,\gamma} + F^{(b)}_{\beta\,\gamma;\,\alpha} + F^{(b)}_{\gamma\,\alpha;\,\beta} = 0. \tag{67}$$

Equations (65) and (67) have the same mathematical structure as Maxwell equations in curved space-time and they share with them property of conformal invariance. The expression (66) for the current has also exactly the same form as the electric current in curved space-time (Hoyle and Narlikar 1964). The equations of motion of a particle **a** interacting with all other particles in the universe (the absorber) *via* direct interparticle action, is given by

$$m_a\, \ddot{a}_\beta = D \sum_{b \neq a} F^{(b)}_{\beta\,\alpha}\, \dot{a}^\alpha \tag{68}$$

with D set equal to e. From (65) and (66) we see that $e_{(N)}$, the generalised 'charge' is the source of $F^{(b)}_{\beta\,\alpha} = \frac{1}{2} F^{(b)}_{\beta\,\alpha,\,\mathrm{ret}} + \frac{1}{2} F^{(b)}_{\beta\,\alpha,\,\mathrm{adv}}$ through which all particles in the universe interact with **a**, the coupling constant of interaction being, not $e_{(N)}$ again, but e, as is seen from (68) which is written for locally flat space-time.

We now proceed to give the well-known derivation for the radiation reaction formula in our general direct action formalism using the method of Wheeler and Feynman (1945), derivation I. One writes the full retarded field of a source which is given an acceleration u, as

$$-\,[e_{(N)}\, u/r_k]\, \sin\theta, \tag{69}$$

at the location of the particle **k**. The acceleration of the absorber particle **k** is given by

$$\begin{aligned} u_k &= (e_k/m_k) \times [\text{'field' of the disturbance}] \\ &= -\,(e_k/m_k) \times (e_{(N)}\, u/r_k)\sin\theta. \end{aligned} \tag{70}$$

The radiation reaction force on the source particle along the direction of the acceleration u is

$$\begin{aligned} &-\, e_k\, (e_{(N)}\, u_k/2\, r_k^2)\sin\theta \\ &= (e_k^2\, u/m_k)\,(e_{(N)}^2/2\, r_k^2)\sin^2\theta. \end{aligned} \tag{71}$$

We can Fourier analyse the acceleration and take only one component corresponding to a frequency ω and take into consideration the phase difference between the returned reaction and source acceleration. We then obtain for the contribution to the returned reaction by the absorber particles at a distance r_k from the source and in the volume element $dr_k\ r_k\ d\theta\ r_k \sin\theta\ d\varphi$ where we have chosen the polar axis along the direction of acceleration, the following expression:

$$(e_{(N)}^2\, e_k^2\, u/2\, m_k\, r_k^2) \times r_k^2\, dr_k \sin^3\theta\, d\theta\, d\varphi$$
$$\times N \exp\,[-\, i\, r_k\, 2\pi\, N\, e_k^2/m_k\, \omega], \tag{72}$$

where N is the particle density of the absorber. Integrating overall the possible directions with limits of θ from 0 to π and of φ from 0 to 2π, we get

$$(4\pi\, Nu/3)\, (e_k^2\, e_{(N)}^2/m_k) \int_0^\infty \exp\,[-\, i\, r_k\, 2\pi\, N\, e_k^2/m_k\, \omega]\, dr_k \tag{73}$$

as the magnitude of the radiation reaction along the direction of acceleration on the source, from all the absorber particles in the universe. Carrying out the integration over r_k, the force of radiation damping on the source particle comes out to be

$$(2/3)\, e_{(N)}^2\, (-\, i\, \omega\, k)$$
$$= (2/3)\, e_{(N)}^2\, (du/dt), \tag{74}$$

as we have taken the Fourier component

$$u = u_0 \exp\,(-\, i\, \omega\, t).$$

This differs from the usual electromagnetic case by the factor $(e_{(N)}/e_k)^2$ embodying the contribution of the 'internal degrees of freedom' of the particle **a**. One could similarly carry out calculations for the absorber action on the basis of derivations II, III and IV of Wheeler and Feynman or in the manner Hoyle and Narlikar (1964) have done these calculations for conformally flat space-time. All the conclusions arrived at by Hoyle and Narlikar regarding cosmological models hold when their calculations are extended to apply to the rank N tensor field whose basic mathematical structure is the same as that of classical electrodynamics even in curved space-time except that there is a dichotomy of e and $e_{(N)}$ mentioned earlier. Of course, for $N = 1$, everything reduces to the familiar case of electrodynamics, as $e_{(N)} = e$.

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Appendix

Explicit form of the generalised Ricci tensor $\mathcal{R}^{\bar{\alpha}\bar{\beta}}$: symmetric case.

In curved space-time the antisymmetric 'field' tensor

$$F^{(b)}_{\epsilon\,\alpha_N} = \left(\phi^{(b)}_{\alpha_1 \dots \alpha_N;\,\epsilon} - \phi^{(b)}_{\alpha_1 \dots \alpha_{N-1}\,\epsilon;\,\alpha_N}\right) I^{\alpha_1 \dots \alpha_{N-1}}_{(a)} \tag{A1}$$

satisfies the following equation

$$F^{(b);\,\epsilon}_{\epsilon\,\alpha_N} = \mathcal{J}^{(b)}_{\alpha_N}, \tag{A2}$$

where $$\mathcal{J}^{(b)}_{\alpha_N} = J^{(b)}_{\alpha_1 \dots \alpha_N} I^{\alpha_1 \dots \alpha_{N-1}}_{(a)}. \tag{A3}$$

Inserting (A1) and (A3) in (A2), we obtain

$$\phi^{(b)}_{\alpha_1 \dots \alpha_N;\,\epsilon}{}^{\epsilon} - \phi^{(b)}_{\alpha_1 \dots \alpha_{N-1}\,\epsilon;\,\alpha_N}{}^{\epsilon} = J^{(b)}_{\alpha_1 \dots \alpha_N}. \tag{A4}$$

The first term on the left is the D'Alembertian of $\phi^{(b)}_{\alpha_1 \dots \alpha_N}$, the second term can be written as

$$\begin{aligned}
\phi^{(b)}_{\alpha_1 \dots \alpha_{N-1};\,\alpha_N\,\epsilon}{}^{\epsilon} &= \phi^{(b)}_{\alpha_1 \dots \alpha_{N-1}}{}^{\epsilon}{}_{;\epsilon\,\alpha_N} \\
&\quad + R^{\gamma}_{\alpha_1\,\alpha_N\,\epsilon}\,\phi^{(b)}_{\gamma\,\alpha_2 \dots \alpha_{N-1}}{}^{\epsilon} \\
&\quad + R^{\gamma}_{\alpha_2\,\alpha_N\,\epsilon}\,\phi^{(b)}_{\alpha_1\,\gamma\,\alpha_3 \dots \alpha_{N-1}}{}^{\epsilon} \\
&\quad + \dots \\
&\quad + R^{\gamma}_{\alpha_r\,\alpha_N\,\epsilon}\,\phi^{(b)}_{\alpha_1 \dots \alpha_{r-1}\,\gamma\,\alpha_{r+1} \dots \alpha_{N-1}}{}^{\epsilon} \\
&\quad + \dots + R^{\gamma}_{\alpha_{N-1}\,\alpha_N\,\epsilon}\,\phi^{(b)}_{\alpha_1 \dots \alpha_{N-2}\,\gamma}{}^{\epsilon} \\
&\quad - R^{\epsilon}_{\gamma\,\alpha_N\,\epsilon}\,\phi^{(b)}_{\alpha_1 \dots \alpha_{N-1}}{}^{\gamma},
\end{aligned}$$

where R's on the right side are Riemann tensors. The first term on the right can be written as

$$\left(\phi^{(b)}_{\alpha_1 \dots \alpha_{N-1}}{}^{\epsilon}{}_{;\,\epsilon}\right)_{;\,\alpha_N}.$$

This vanishes if we *impose* the gauge condition on the potentials $\phi^{(b)}_{\bar{a}}$ in the form

$$\phi^{(b)}_{a_1 \ldots a_{N-1}}{}^{\epsilon}{}_{;\epsilon} = 0. \tag{A6}$$

This requires that the Green's function $\overline{G}_{\bar{a}\bar{\lambda}}$ satisfies

$$\overline{G}_{a_1 \ldots a_N \lambda_1 \ldots \lambda_N}{}^{;a_N} = -\overline{G}_{a_1 \ldots a_{N-1} \lambda_1 \ldots \lambda_{N-1}; \lambda_N}. \tag{A7}$$

As pointed out earlier, this is the generalisation of DeWitt-Brehme condition on the Green's function $\overline{G}_{a\lambda}$ namely

$$\overline{G}_{a\lambda;}{}^{a} = -\overline{G}_{;\lambda}.$$

Here (A7) appears as a consequence of the gauge condition (A6) and no independent proof of (A7) based on the properties of space-time is being presented. Its proof, hopefully, will form the subject of a future paper. Taking into account (A6) we have

$$\begin{aligned}
\phi^{(b)}_{a_1 \ldots a_{N-1}; a_N}{}^{\epsilon}{}_{\epsilon} = &- R^{\gamma}_{a_1}{}^{\epsilon}{}_{a_N} \phi^{(b)}_{\gamma a_2 \ldots a_{N-1} \epsilon} \\
&- R^{\gamma}_{a_2}{}^{\epsilon}{}_{a_N} \phi^{(b)}_{a_1 \gamma a_3 \ldots a_{N-1} \epsilon} \\
&- \ldots - R^{\gamma\epsilon}_{a_r}{}_{a_N} \phi^{(b)}_{a_1 \ldots a_{r-1} \gamma a_{r+1} \ldots a_{N-1} \epsilon} \\
&- \ldots - R^{\gamma\epsilon}_{a_{N-1}}{}_{a_N} \phi^{(b)}_{a_1 \ldots a_{N-2} \gamma \epsilon} \\
&- R^{\epsilon\gamma}_{a_N \epsilon} \phi^{(b)}_{a_1 \ldots a_{N-1} \gamma}
\end{aligned}$$

$$\begin{aligned}
= -\Big\{ &R^{\gamma}_{a_N} g_{a_1\beta_1} \cdots g_{a_{N-1}\beta_{N-1}} g_{\gamma\beta_N} \\
&+ R^{\gamma}_{a_1}{}^{\epsilon}{}_{a_N} g_{\gamma\beta_1} g_{a_2\beta_2} \cdots g_{a_{N-1}\beta_{N-1}} g_{\epsilon\beta_N} \\
&+ R^{\gamma}_{a_2}{}^{\epsilon}{}_{a_N} g_{a_1\beta_1} g_{\gamma\beta_2} \cdots g_{a_{N-1}\beta_{N-1}} g_{\epsilon\beta_N} \\
&+ \ldots + R^{\gamma}_{a_r}{}^{\epsilon}{}_{a_N} g_{a_1\beta_1} \cdots g_{a_{r-1}\beta_{r-1}} g_{\gamma\beta_r} g_{a_{r+1}\beta_{r+1}} \cdots g_{a_{N-1}\beta_{N-1}} g_{\epsilon\beta_N} \\
&+ \ldots + R^{\gamma}_{a_{N-1}}{}^{\epsilon}{}_{a_N} g_{a_1\beta_1} \cdots g_{a_{N-2}\beta_{N-2}} g_{\gamma\beta_{N-1}} g_{\epsilon\beta_N} \Big\} \times \phi^{\beta_1 \ldots \beta_N}_{(b)}.
\end{aligned}$$

This last expression can be put in the form

$$\phi^{(b)}_{\alpha_1\ldots\alpha_{N-1};\,\alpha_N\,\epsilon}{}^{\epsilon} = -\Big\{R_{\alpha_N\,\beta_N}g_{\alpha_1\,\beta_1}\cdots g_{\alpha_{N-1}\,\beta_{N-1}}$$

$$+\sum_{r=1}^{N}R_{\alpha_r\,\beta_r\,\alpha_N\,\beta_N}\,g_{\alpha_1\,\beta_1}\cdots g_{\alpha_{r-1}\,\beta_{r-1}}\,g_{\alpha_{r+1}\,\beta_{r+1}}\cdots g_{\alpha_{N-1}\beta_{N-1}}\Big\}$$

$$\times\,\phi^{\beta_1\ldots\beta_N}_{(b)}. \tag{A8}$$

Inserting (A8) in (A4) and using (27) in the form

$$\phi^{(b)}_{\bar{\alpha};\,\epsilon}{}^{\epsilon} + \mathscr{R}_{\bar{\alpha}\,\bar{\beta}}\phi^{\bar{\beta}}_{(b)} = J^{(b)}_{\bar{\alpha}}$$

we obtain the explicit form for the generalised Ricci tensor $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}}$, as

$$\mathscr{R}_{\bar{\alpha}\,\bar{\beta}} = R_{\alpha_N\,\beta_N}\,g_{\alpha_1\,\beta_1}\cdots g_{\alpha_{N-1}\,\beta_{N-1}}$$

$$+\sum_{r=1}^{N}R_{\alpha_r\,\beta_r\,\alpha_N\,\beta_N}\,g_{\alpha_1\,\beta_1}\,\cdots\,g_{\alpha_{r-1}\,\beta_{r-1}}\,g_{\alpha_{r+1}\,\beta_{r+1}}\,\cdots\,g_{\alpha_{N-1}\,\beta_{N-1}}. \tag{A9}$$

The following features about $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}}$ should be noted:

(a) $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}} = \mathscr{R}_{\bar{\beta}\,\bar{\alpha}}$ as asserted before,

(b) $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}} = 0$ in flat space-time.

(c) $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}}$ is symmetric among $\alpha_1\ldots\alpha_{N-1}$ and $\beta_1\ldots\beta_{N-1}$ only, but this is to be expected as we have singled out α_N in defining

$$K^{\alpha_1\,\ldots\,\alpha_N}_{(a)} = I^{\alpha_1\,\ldots\,\alpha_{N-1}}_{(a)}\,\dot{a}^{\alpha_N}$$

but α_N could be any α, from α_1 to α_N, hence to this extent $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}}$ can be said to be symmetric in its indices.

(d) $\mathscr{R}_{\bar{\alpha}\,\bar{\beta}} \to R_{\alpha\,\beta}$ when $N = 1$.

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