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Strong coupling expansion in the massive Thirring model

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Abstract. A strong coupling expansion of the Green functions in the massive Thirring model is formulated. Expressions obtained for the fundamental fermion mass and the β function agree with the known qualitative features of these physical quantities.

Keywords. Strong coupling expansion; regularisation schemes; renormalised parameters; Thirring model.

1. Introduction

The study of strong coupling expansion has recently evoked considerable interest (Castoldi and Schomblond 1977, 1978; Bender *et al* 1979; Guerin and Kenway 1980). Its success in simple problems like the one-dimensional anharmonic oscillator is quite encouraging. Some investigations have also been carried out in field theories in two or more dimensions. In this context, both bosonic as well as fermionic theories have been considered. There are some difficulties peculiar to fermionic theories. For example spurious fermionic states appear when strong coupling expansion is performed on the lattice (Susskind 1977; Guerin and Kenway 1980). For gauge field theories like QED or QCD, the implementation of gauge invariance within the strong coupling expansion schemes poses additional problems (Cooper and Kenway 1981).

In the present investigation we examine the predictions of strong coupling expansion schemes in the context of the massive Thirring model (MTM). Since MTM is a purely fermionic theory we can study here the problems in strong coupling schemes peculiar to such theories in isolation from the additional complexity associated with preserving gauge invariance. Lattice regularisation schemes are unsuitable for fermionic theories and we shall therefore work with two other regularisation prescriptions proposed in this context (Parga *et al* 1979; Castoldi and Schomblond 1978). On the other hand in MTM results are available regarding the qualitative behaviour of some physical quantities of interest. For example, it has been suggested that the Callan-Symanzik β function vanishes to all orders in the coupling constant (Mueller and Trueman 1971; Coleman 1975) and that the physical mass of the fermion, in the strong coupling limit, grows linearly with the coupling constant (Coleman 1975; Thacker 1981). Thus, MTM, though not exactly solvable, provides a reasonable ground for testing the predictions of strong coupling expansions.

It may be mentioned here that though the conventional massless Thirring model is exactly soluble, the solution obtained for the renormalised two-point function by Johnson (1961) and Coleman (1975) is defined only in a finite range for the coupling constant and does not permit a strong coupling limit.

The paper is organised as follows: in §2 the final reduction of the generating functional in its strong coupling version is formulated from which the unregulated Green functions are obtained; in § 3 we introduce the regularisation procedures; in § 4 we attain the strong coupling expansion of the renormalised parameters; § 5 summarises our findings.

2. The strong coupling expansion

MTM is the theory of a single Fermi field in one space and one time dimension governed, in the Euclidean metric, by the Lagrangian density

$$\mathscr{L}_E = -\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + g(\bar{\psi}\gamma_\mu\psi)^2, \tag{1}$$

where the γ matrices are represented by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The generating functional for the Green functions in the presence of fermion sources is

$$Z[\eta, \bar{\eta}] = \int D\psi\, D\bar{\psi} \exp \int d^2 x [-\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + \\ + g(\bar{\psi}\gamma_\mu\psi)^2 + \bar{\psi}\eta + \bar{\eta}\psi)]. \tag{2}$$

The integrand in the exponential is simplified *a la* Guerin and Kenway (1980). A σ_μ field is introduced to eliminate the quartic coupling term by plugging a multiplicative factor

$$\int D\sigma_\mu \exp\left[-\frac{1}{g}\int d^2 x (\sigma_\mu + g\bar{\psi}\gamma_\mu\psi)^2 \right].$$

The vacuum functional (2) thus changes to

$$Z[\eta, \bar{\eta}] = \int D\sigma_\mu\, D\psi\, D\bar{\psi} \exp \int d^2 x [-\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - \\ -\frac{\sigma^2}{g} - 2\bar{\psi}\gamma_\mu \sigma_\mu \psi + \bar{\psi}\eta + \bar{\eta}\psi]. \tag{3}$$

The fermion kinetic term is extracted in the usual way (Bender *et al* 1979)

$$Z[\eta, \bar{\eta}] = K\left[\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] \int D\sigma_\mu\, D\psi\, D\bar{\psi} \exp \int d^2 x$$

$$\left[-\frac{\sigma^2}{g} - 2\bar{\psi}\gamma_\mu\sigma_\mu\psi + \bar{\psi}\eta + \bar{\eta}\psi\right], \tag{4}$$

where $$K\left[\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] = \exp\left[-\int d^2x\, d^2y\, \frac{\delta}{\delta\eta(x)} S_F^{-1}(x-y) \frac{\delta}{\delta\bar{\eta}(y)}\right] \tag{5a}$$

with $$S_F^{-1}(x-y) = (\gamma_\mu\partial_\mu + m)\,\delta(x-y) \tag{5b}$$

being the inverse Fermi propagator.

The remaining functional integral in (4) is a product of one-dimensional integrals at each space time point. We replace the continuum space-time by a discrete two-dimensional square lattice, a unit cell of which has the volume $a^2 = \delta_\Lambda^{-1}(0)$ with a proper choice for the regularisation introducing a cut-off on the δ-function. The regularisation schemes to be used will be discussed in the subsequent section.

Performing the integration over the fermion field variables yields

$$Z[\eta, \bar{\eta}] = K\left[\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] \int D\sigma_\mu \exp\left[-\int d^2x \left(\frac{\sigma^2}{g} - \delta_\Lambda(0)\log 4\sigma^2 - \frac{\bar{\eta}\gamma_\mu\sigma_\mu\eta}{\sigma^2}\right)\right]. \tag{6}$$

In order to work out the final integration over the σ_μ field, we decompose it into a product of two-dimensional integrals at each space time point i, such that $\int d^2x \to \delta_\Lambda^{-1}(0)\sum_i$. Then, making the necessary alterations we obtain

$$Z[\eta, \bar{\eta}] = K\left[\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] \prod_i \int d^2\sigma_i\,(4\sigma_i^2) \times \exp\left(-\frac{\sigma_i^2}{g\delta_\Lambda(0)}\right) \exp\left(\frac{\bar{\eta}_i\gamma_\mu\sigma_{\mu i}\eta_i}{2\delta_\Lambda(0)\sigma_i^2}\right). \tag{7}$$

The integral can be immediately cast into a Gaussian form by realising that the expansion of the second exponential truncates after the first two terms due to the anti-commuting nature of the fermion sources. Thus

$$Z[\eta, \bar{\eta}] = K\left[\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right] \prod_i \int d^2\sigma_i\,(4\sigma_i^2) \exp\left(-\frac{\sigma_i^2}{g\delta_\Lambda(0)}\right) \times \left[1 + \frac{\bar{\eta}_i\gamma_\mu\sigma_{\mu i}\eta_i}{2\delta_\Lambda(0)\sigma_i^2} + \frac{1}{8\delta_\Lambda^2(0)}\left(\frac{\bar{\eta}_i\gamma_\mu\sigma_{\mu i}\eta_i}{\sigma_i^2}\right)^2\right]. \tag{8}$$

The second term vanishes from symmetry arguments while the last term can be simplified by utilising the truncation identity valid in two dimensions

$$(\bar{\eta}\, \gamma_\mu\, \sigma_\mu\, \eta)^2 = \frac{\sigma^2}{2} (\bar{\eta}\, \gamma_\mu\, \eta)^2,$$

$$= \frac{\sigma^2}{2} (\det \gamma_0 + \det \gamma_1)\, (\bar{\eta}\, \eta)^2,$$

$$= -\, \sigma^2\, (\bar{\eta}\, \eta)^2, \tag{9}$$

where the final step follows from the explicit representation of the γ matrices given earlier. Thus

$$Z\,[\eta, \bar{\eta}] = K \left[\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}\right] \prod_{i} \int d^2\, \sigma_i \left[4\, \sigma_i^2 \exp\left(-\frac{\sigma_i^2}{g\, \delta_\Lambda\, (0)}\right) - \right.$$

$$\left. - \frac{(\bar{\eta}_i\, \eta_i)^2}{2\, \delta_\Lambda^2\, (0)} \exp\left(-\frac{\sigma_i^2}{g\, \delta_\Lambda\, (0)}\right)\right]. \tag{10}$$

Both the integrals are Gaussian and can be trivially evaluated to yield

$$Z\,[\eta, \bar{\eta}] = K \left[\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}\right] \prod_{i} 4\, \pi^2\, g^2\, \delta_\Lambda^2\, (0) \times$$

$$\left[1 - \frac{(\bar{\eta}_i\, \eta_i)^2}{8\, g\, \delta_\Lambda^3\, (0)}\right]. \tag{11}$$

The constant factor independent of the fermion sources drops out when the Green functions of the theory are computed and, hence, is ignored. Re-exponentiating the expression, we obtain,

$$Z\,[\eta, \bar{\eta}] = K \left[\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}\right] \exp \sum_{i} - \frac{a^6}{8g} (\bar{\eta}_i\, \eta_i)^2$$

$$= K \left[\frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}}\right] \exp - \frac{a^4}{8\, g} \int d^2\, x\, (\bar{\eta} \eta)^2, \tag{12}$$

where, in the concluding step, we have reverted back to the continuum language.

Having obtained the cherished expression for the vacuum functional, we now proceed to evaluate the various Green functions by functional differentiation. Here we will explicitly work out the simplest Green functions to two orders of perturbation. The connected two-point function is

$$W_2^{\alpha\beta}(x_1, x_2) = \frac{\delta^2}{\delta\,\bar{\eta}^\alpha(x_1)\,\delta\,\eta^\beta(x_2)} \log Z[\eta, \bar{\eta}]\Big|_{\eta=\bar{\eta}=0}$$

$$= -\frac{a^4}{4g} S_F^{-1}(0)\,\delta_{\alpha\beta}\,\delta(x_1 - x_2)$$

$$-\left(\frac{a^4}{4g}\right)^2 \Big[S_F^{-1}(0) \int d^2x \{S_F^{-1}(x - x_2)\, S_F^{-1}(x_2 - x)\}_{\alpha\beta}\,\delta(x_1 - x_2)$$

$$-\{S_F^{-1}(x_1 - x_2)\, S_F^{-1}(x_2 - x_1)\, S_F^{-1}(x_1 - x_2)\}_{\alpha\beta}$$

$$+ (S_F^{-1}(0))^2 \{S_F^{-1}(x_1 - x_2)\}_{\alpha\beta}\Big] + 0\left(\frac{1}{g^3}\right). \tag{13}$$

Similarly the connected four-point function is given by

$$W_4^{\alpha\beta\gamma\omega}(x_1, x_2, x_3, x_4) = \frac{\delta^4}{\delta\bar{\eta}^\alpha(x_1)\,\delta\bar{\eta}^\beta(x_2)\,\delta\eta^\gamma(x_3)\,\delta\eta^\omega(x_4)} \log Z[\eta, \bar{\eta}]\Big|_{\eta=\bar{\eta}=0}$$

$$= -\frac{a^4}{4g}\,\delta(x_4 - x_1)\,\delta(x_4 - x_2)\,\delta(x_4 - x_3)\,T_{\alpha\beta\gamma\omega}$$

$$+ (a^4/4g)^2\,[X_{\alpha\beta\gamma\omega}] + 0\,(1/g^3), \tag{14}$$

where the abbreviation $T_{\alpha\beta\gamma\omega}$ stands for the tensor operator

$$T_{\alpha\beta\gamma\omega} = \delta_{\alpha\omega}\,\delta_{\beta\gamma} - \delta_{\alpha\gamma}\,\delta_{\beta\omega},$$

and the non leading term $X_{\alpha\beta\gamma\omega}$ has been explicitly written in the appendix.

3. Regularisation

The strong coupling expansion of the Green functions achieved in the previous section has an alarming feature in that an expression of the type $S_F^{-1}(0)$ appears which is highly singular. Higher powers of $S_F^{-1}(0)$ appear in higher order terms in the Fourier transform of the Green functions. In order to make such divergent quantities meaningful we need to introduce some regularisation schemes. As has been previously mentioned, lattice regularisation is riddled with ambiguities in fermionic theories and one has to adopt other regularisation prescriptions. Different suggestions have been advocated by various authors (Parga *et al* 1979; Castoldi and Schomblond 1978) and we work with two of these.

In the first such scheme, the Gaussian regularisation, the regularised δ-function is defined by

$$\delta_\Lambda(x) = \int \frac{d^2 k}{(2\pi)^2} \exp(-ik\cdot x) \exp(-k^2/\Lambda^2)$$

$$= \frac{\Lambda^2}{4\pi} \exp(-\Lambda^2 x^2/4) \tag{15a}$$

so that, in momentum space,

$$\delta_\Lambda(k) = \exp(-k^2/\Lambda^2). \tag{15b}$$

The inverse Fermi propagator (5b) then takes the form

$$S_F^{-1}(x, 0) = (\gamma_\mu \partial_\mu + m)\,\delta_\Lambda(x) = \left(-\frac{\Lambda^2}{2}\gamma_\mu x_\mu + m\right)\delta_\Lambda(x). \tag{16}$$

Alternatively, one may introduce a step function in the δ-function.

$$\delta_\Lambda(x) = \int \frac{d^2 k}{(2\pi)^2} \exp(-i\,k\cdot x)\,\theta(\Lambda^2 - k^2)$$

$$= \frac{\Lambda}{2\pi\,|x|} J_1(\Lambda\,|x|). \tag{17a}$$

In momentum space, this is

$$\delta_\Lambda(k) = \theta(\Lambda^2 - k^2). \tag{17b}$$

With this regularisation, the inverse propagator (5b) assumes the form with

$$S_F^{-1}(x, 0) = (\gamma_\mu \partial_\mu + m)\,\delta_\Lambda(x)$$

$$= -\frac{\Lambda^2}{2\pi\,|x|^2}\gamma_\mu x_\mu J_2(\Lambda\,|x|) + \frac{m\,\Lambda}{2\pi\,|x|} J_1(\Lambda\,|x|). \tag{18}$$

It may be observed from (15) and (17) that while both $\delta_\Lambda(x)$ give a Dirac delta function in the limit $\Lambda \to \infty$ only the step regularisation has, for finite Λ, the property of being the identity for the convolution product. That is to say, it satisfies

$$\int d^2 x\,\delta_\Lambda(z - x)\,\delta_\Lambda(x - y) = \delta_\Lambda(z - y), \tag{19a}$$

in co-ordinate space, or

$$\delta_\Lambda(k)\,\delta_\Lambda(k) = \delta_\Lambda(k), \tag{19b}$$

in momentum space.

The two regularisation schemes mentioned here have been extensively studied in the context of the one-dimensional anharmonic oscillator problem (Bender *et al* 1981). Our purpose is to expose both of these approaches to the present model and compare and contrast their salient features.

A straightforward calculation of the Fourier transform of the two point function in its regularised form yields

$$W_2^G(k) = -\frac{\pi}{g}\frac{m}{\Lambda^2} + \left(\frac{\pi}{g}\right)^2 \frac{1}{\Lambda^2}\left[\frac{5}{36}m\left(1 - \frac{42}{5}\frac{m^2}{\Lambda^2}\right)\right.$$
$$\left. + \frac{2}{27}\, i\gamma_\mu k_\mu\left(1 + 12\frac{m^2}{\Lambda^2} - \frac{i}{2}\gamma_\mu k_\mu \frac{m}{\Lambda^2} - \frac{1}{6}\frac{k^2}{\Lambda^2}\right)\right] + 0\left(\frac{1}{g^3}\right), \quad (20a)$$

for the Gaussian regularisation while the corresponding expression obtained by the θ-function regularisation is

$$W_2^\theta(k) = -\frac{\pi}{g}\frac{m}{\Lambda^2} + \left(\frac{\pi}{g}\right)^2 \frac{1}{\Lambda^2}\left\{\frac{8\pi + 9\sqrt{3}}{24\pi}\cdot m\left(1 + a\frac{|k|}{\Lambda} + 0\left(\frac{k^2}{\Lambda^2}\right)\right)\right.$$
$$- \frac{m^3}{\Lambda^2}\cdot\frac{4\pi + 3\sqrt{3}}{4\pi}\left(1 + b\frac{|k|}{\Lambda} + 0\left(\frac{k^2}{\Lambda^2}\right)\right) - i\gamma_\mu k_\mu\left(\frac{33\sqrt{3} + 4\pi}{24\pi}\right)$$
$$\left.\times\left[1 + c\frac{|k|}{\Lambda} + 0\left(\frac{k^2}{\Lambda^2}\right)\right] + 4i\gamma_\mu k_\mu \frac{m^2}{\Lambda^2}\left[1 + d\frac{|k|}{\Lambda} + 0\left(\frac{k^2}{\Lambda^2}\right)\right]\right\}$$
$$+ 0\left(\frac{1}{g^3}\right), \quad (20b)$$

where a, b, c etc. are finite numbers. In appendix 2 we discuss a method for evaluating the integrals involved in the calculation of $W_2^\theta(k)$.

One immediately observes that the leading term is identical in both the regularisation schemes and differences arise only at the non-leading level.

4. Renormalisation and comparison with known results

It may be observed that the connected Green functions in (20) are polynomials in the external momenta and do not exhibit the analyticity structure revealed in usual perturbation theory. This is true to all orders of the calculation and is a deficiency of the method. In order to define the renormalised parameters of the theory we follow the prescription suggested by Castoldi *et al* (1978). Thus from the two-point function, we are able to define the renormalised mass m_r and the wavefunction renormalisation Z.

$$m_r = i\left.\frac{W_2}{\dfrac{dW_2}{d\gamma_\mu k_\mu}}\right|_{k\to 0}, \quad Z = m_r W_2(0). \quad (21)$$

To be able to define the renormalised coupling constant, we first introduce the amputated four-point function.

$$\Gamma_4^{\alpha\beta\gamma\omega}(x_1, x_2, x_3, x_4) = \langle 0 | T(\psi_\alpha(x_1)\psi_\beta(x_2)\bar{\psi}_\gamma(x_3)\bar{\psi}_\omega(x_4)) | 0 \rangle_{\bar{C}}$$

$$= \int d^2 y_1 d^2 y_2 d^2 y_3 d^2 y_4 [W_2^{-1}(x_1 - y_1)]_{\alpha\alpha'}$$

$$[W_2^{-1}(x_2 - y_2)]_{\beta\beta'} \; [W_2^{-1}(y_3 - x_3)]_{\gamma\gamma'} \; [W_2^{-1}(y_4 - x_4)]_{\omega\omega'}$$

$$\langle 0 | T(\psi_{\alpha'}(y_1)\psi_{\beta'}(y_2)\bar{\psi}_{\gamma'}(y_3)\bar{\psi}_{\omega'}(y_4)) | 0 \rangle_C \tag{22}$$

where the alphabet C indicates that only the connected pieces have to be considered. The inverse two point function W_2^{-1} is defined in the usual way

$$\int d^2 z \, W_2^{-1}(x - z) \, W_2(z - y) = \delta(x - y).$$

Using (13) and (14) in (22), Fourier transforming the resulting expression and, finally, setting the external momenta equal to zero, we obtain

$$\Gamma_4^{\alpha\beta\gamma\omega}(k_1 = k_2 = k_3 = k_4 = 0) = \left(\frac{2\Lambda^2}{m^2\pi}\right)^2 g^3 \left[1 + \frac{\pi}{g}\left(1 + \frac{m^2}{\Lambda^2}\right)\right.$$

$$\left. + \, 0\left(\frac{1}{g^2}\right)\right] T_{\alpha\beta\gamma\omega} \tag{23a}$$

for the Gaussian regularisation, and

$$\Gamma_4^{\alpha\beta\gamma\omega}(k_1 = k_2 = k_3 = k_4 = 0) = \left(\frac{2\Lambda^2}{m^2\pi}\right)^2 g^3 \left[1 + \frac{2\pi}{g}\left(1 - \frac{m^2}{\Lambda^2}\right)\right.$$

$$\left. + \, 0\left(\frac{1}{g^2}\right)\right] T_{\alpha\beta\gamma\omega} \tag{23b}$$

in the step function regularisation. We observe that the leading term is, as before, identical and differences occur at the non-leading level. It is interesting to note that the power series type of dependence associated with each factor of the non-leading term in W_2^θ (see 20b) obtained by step regularisation is not manifested in $\Gamma_4(0)$ due to the specific choice of the renormalisation point. The zero of the external momenta automatically demands that, barring the constant term, all other terms drop out. Now, by definition, the renormalised coupling constant g_r is taken to be the coefficient of the tensor $T_{\alpha\beta\gamma\omega}$ in the product $\Gamma_4(0)Z^2$.

After some straightforward algebraic manipulations, we arrive, by help of our definitions at the following set of equations

$$m_r = mg\left[p_0 + \frac{1}{g}\left(q_0 + r_0\frac{m^2}{\Lambda^2}\right) + 0\left(\frac{1}{g^2}\right)\right],$$

$$Z = \frac{m^2}{\Lambda^2}\left[p'_0 + \frac{1}{g}\left(q'_0 + r'_0\frac{m^2}{\Lambda^2}\right) + 0\left(\frac{1}{g^2}\right)\right],$$

$$g_r = g^3\left[p''_0 + \frac{1}{g}\left(q''_0 + r''_0\frac{m^2}{\Lambda^2}\right) + 0\left(\frac{1}{g^2}\right)\right]. \tag{24}$$

p, q, r are finite quantities, their values being characterised by the particular regularisation procedure that has been adopted. This implies that, for all the three parameters the dependence on the regularisation sets in from the leading term itself.

From (24) it is observed that $\partial g_r/\partial m$ goes to zero as the limit of the cut-off parameter is taken to infinity. Hence the Callan Symanzik β function, defined below, vanishes

$$\beta(g_r) = m_r \frac{\partial g_r}{\partial m}\left(\frac{\partial m_r}{\partial m}\right)^{-1} = 0. \tag{25}$$

This is an exact result (Mueller and Trueman 1971; Coleman 1975).

Our next query is addressed to the nature of the fundamental fermion mass. Regardless of the fact that the pole structure of the propagator is not apparent we may, however, define the physical or the fundamental fermion mass M from the zero of the inverse propagator by writing

$$W_2^{-1}(k)\,|_{\gamma_\mu k_\mu \to -iM} = 0. \tag{26}$$

Simplifications using either cut-off schemes yield

$$M = mg\left[p + \frac{q}{g} + 0\left(\frac{1}{g^2}\right)\right], \tag{27}$$

where the cut-off dependence is contained from the leading term itself. The resulting expression demonstrates that, in the leading approximation, the fundamental fermion mass is proportional to the coupling constant. This is the second known feature of the *MTM* that was mentioned earlier (Coleman 1975; Thacker 1981).

Several authors, in particular Coleman (1975), have argued that MTM is equivalent to the quantum sine-Gordon model. It has also been conjectured that the quantum sine-Gordon soliton be regarded as the fundamental fermion of MTM. Exact S matrices in various charge sectors (*e.g.* soliton-soliton, soliton-antisoliton etc) have been calculated in the quantum sine-Gordon model (Zamolodchikov and Zamolodchikov 1979; Karowski 1979). *A priori*, it is doubtful if our strong coupling expansions for the Green functions can be compared with these results for exact S matrix elements.

In obtaining the exact S matrix elements the mass shell limit is already incorporated after which one may perform the strong coupling expansion. On the other hand in obtaining S matrix elements in our strong coupling expansion scheme the mass shell limit of the four-point function will have to be taken later. An inspection of the four-point function reveals that the strong coupling expansion involves polynomials in the external momenta and does not exhibit, order by order, the usual analyticity structure. When we plug this expression in the reduction formula, the contribution to the two-particle scattering amplitude vanishes on imposing the necessary mass shell limit. Absence of pole structure in the Green functions in each order of the calculation makes it unsuitable for the computation of the S matrix elements and is a deficiency of the strong coupling approach. It may be recalled that in defining the physical fermion mass a similar problem occurred and we took recourse to the inverse propagator. Furthermore in quantum sine-Gordon model, the semi-classical approximation (which is the strong coupling limit for MTM) of the soliton-soliton scattering amplitude demonstrates that the leading term goes like $\exp g$ (Zamolodchikov and Zamolodchikov 1979). Since our strong coupling expansion is a series in inverse powers of the coupling constant, it will not be possible to reproduce such a behaviour.

5. Conclusions

Functional methods have been used to set up the strong coupling expansion of the fermion Green functions in inverse powers of the coupling constant in MTM. Explicit forms for the two-point and four-point functions upto the first two orders of perturbation were obtained. This involved, as is usual in strong coupling theories, highly singular quantities which had to be suitably regularised. Two regularisation schemes—the Gaussian and the step function—were used in this context. Our results show that the coefficients of even the leading terms in the strong coupling expansion for the renormalised parameters (e.g. coupling constant, mass) depend on the regularisation schemes. It is, however, interesting to note that the strong coupling method does reproduce the linear grown of the physical mass with the bare coupling constant and the vanishing of the Callan-Symanzik β function, results that are expected to be true for the exact solution of MTM (Mueller and Trueman 1971; Coleman 1975; Thacker 1981).

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Appendix 1

The complete form of the non-leading term in the four point function (14) is given here.

$$X_{\alpha\beta\gamma\omega} = S_F^{-1}(0)\, [\{S_F^{-1}(x_3-x_4)_{\beta\omega}\, \delta_{\alpha\gamma} - S_F^{-1}(x_3-x_4)_{\alpha\omega}\, \delta_{\beta\gamma}\}\, \delta(x_3-x_1)\, \delta(x_3-x_2)$$

$$-\{S_F^{-1}(x_4-x_3)_{\beta\gamma}\, \delta_{\alpha\omega} - S_F^{-1}(x_4-x_3)_{\alpha\gamma}\, \delta_{\beta\omega}\}\, \delta(x_4-x_1)\, \delta(x_4-x_2)$$

$$-\{S_F^{-1}(x_1-x_4)_{\alpha\omega}\, \delta_{\beta\gamma} - S_F^{-1}(x_1-x_4)_{\alpha\gamma}\, \delta_{\beta\omega}\}\, \delta(x_4-x_3)\, \delta(x_4-x_1)$$

$$+\{S_F^{-1}(x_2-x_4)_{\beta\gamma}\, \delta_{\alpha\omega} - S_F^{-1}(x_2-x_4)_{\beta\omega}\, \delta_{\alpha\gamma}\}\, \delta(x_4-x_3)\, \delta(x_4-x_1)]$$

$$-2\{[S_F^{-1}(x_4-x_3)S_F^{-1}(x_3-x_4)]_{\beta\omega}\, \delta_{\alpha\gamma} + [S_F^{-1}(x_3-x_4)\, S_F^{-1}(x_4-x_3)]_{\alpha\gamma}\, \delta_{\beta\omega}\}$$
$$\times\, \delta(x_3-x_1)\, \delta(x_4-x_2)$$

$$+2\{[S_F^{-1}(x_4-x_3)S_F^{-1}(x_3-x_4)]_{\alpha\omega}\, \delta_{\beta\gamma} + [S_F^{-1}(x_3-x_4)S_F^{-1}(x_4-x_3)]_{\beta\gamma}\, \delta_{\alpha\omega}\}$$
$$\times\, \delta(x_4-x_1)\delta(x_3-x_2)$$

$$-2[S_F^{-1}(x_2-x_4)_{\alpha\gamma}S_F^{-1}(x_2-x_4)_{\beta\omega} - S_F^{-1}(x_2-x_4)_{\beta\gamma}S_F^{-1}(x_2-x_4)_{\alpha\omega}]\delta(x_2-x_1)\delta(x_4-x_3)$$

$$+2\mathrm{Tr}\, [S_F^{-1}(x_4-x_3)S_F^{-1}(x_3-x_4)]\{\delta_{\beta\omega}\, \delta_{\alpha\gamma}\, \delta(x_4-x_2)\, \delta(x_3-x_1) - \delta_{\beta\gamma}\, \delta_{\alpha\omega}$$
$$\times\, \delta(x_4-x_1)\, \delta(x_3-x_2)\}$$

$$+2[S_F^{-1}(x_4-x_3)_{\beta\gamma}S_F^{-1}(x_3-x_4)_{\alpha\omega}\, \delta(x_4-x_2)\, \delta(x_3-x_1)$$
$$-S_F^{-1}(x_4-x_3)_{\alpha\gamma}\, S_F^{-1}(x_3-x_4)_{\beta\omega}\, \delta(x_3-x_2)\, \delta(x_4-x_1)].$$

Appendix 2

Here we illustrate the technique for handling integrals involving step functions by selecting a typical example that arises in the Fourier transform of the propagator.

$$I = \int \mathrm{d}^2p\, \mathrm{d}^2P\, \theta(\Lambda^2-p^2)\, \theta(\Lambda^2-(P-p)^2)\, \theta(\Lambda^2-(P-k)^2). \tag{A1}$$

Consider first the integration over the p variable,

$$I_P = \int \mathrm{d}^2p\, \theta(\Lambda^2-p^2)\, \theta(\Lambda^2-(P-p)^2)$$
$$= 4\left[\frac{\Lambda^2}{2}\cos^{-1}\frac{P}{2\Lambda} - \frac{P}{4}\left(\Lambda^2-\frac{P^2}{4}\right)^{1/2}\right]. \tag{A2}$$

Substituting the value of (A2) in (A1) we find,

$$I = \int \mathrm{d}^2P\, I_P\, \theta(\Lambda^2-(P-k)^2)$$
$$= 4\int \mathrm{d}^2P\left[\frac{\Lambda^2}{2}\cos^{-1}\frac{P}{2\Lambda} - \frac{P}{4}\left(\Lambda^2-\frac{P^2}{4}\right)^{1/2}\right]\theta(\Lambda^2-(P-k)^2). \tag{A3}$$

Introduce the following scaling

$$P = \Lambda p'. \tag{A4}$$

in (A3) so that it reduces to

$$I = 4\Lambda^4 \int d^2p' \left[\tfrac{1}{2}\cos^{-1}\frac{p'}{2} - \frac{p'}{4}\left(1 - \frac{p'^2}{4}\right)^{1/2}\right] \theta\left(1 - p'^2 - \frac{k^2}{\Lambda^2} + \frac{2p'.k}{\Lambda}\right) \tag{A5}$$

Choosing k along the polar axis, we obtain

$$I = 4\Lambda^4 \int_0^\infty dp'p' \left[\tfrac{1}{2}\cos^{-1}\frac{p'}{2} - \frac{p'}{4}\left(1 - \frac{p'^2}{4}\right)^{1/2}\right] \int_0^{2\pi} d\phi\, \theta\left(1 - p'^2 - \frac{k^2}{\Lambda^2} + \frac{2p'k\cos\phi}{\Lambda}\right)$$

$$= \pi\Lambda^4\left(\pi - \frac{3\sqrt{3}}{4}\right)\left[1 + 0\left(\frac{k}{\Lambda}\right)\right]. \tag{A6}$$

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