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Nonlinear distribution functions for the Vlasov-Poisson system

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Abstract. The nonlinear distribution function introduced by Allis has been used to investigate the stability of the solution of Vlasov-Poisson's equations. The 'average' Lagrangian is calculated on the basis of this distribution function, and the 'average' variational principle of Whitham is applied to discuss modulational stability. It is found that the distribution function of Allis exactly gives rise to the Lighthill's stability condition of non-linear waves.

Keywords. Nonlinear distribution function; particle trapping; Lagrangian; modulational stability.

1. Introduction

It is well-known that the Vlasov-Poisson's system is nonlinear through the self-consistent electric field in the plasma. The usual technique for linearization is to expand the distribution function in ascending powers of potential and retain only the first power. This simplification leads principally to three different results: (a) real dispersion relation of Vlasov for undamped wave propagation, (b) complex dispersion law of Landau giving the Landau damping, and (c) arbitrary dispersion law of Van Kampen (1955). This multiplicity in the results, according to Allis (1969), is due to the process of linearization, which reduces finite objects to δ-functions and wave amplitudes to zero values. One cannot improve the linear solution by retaining the next term in the series referred to above, which is proportional to the square of the potential, since one ends up with divergence in that situation.

On the other hand, two significant improvements have been made by O'Neil (1965) and Bernstein *et al* (1957) hereafter referred to as (BGK). These are obtained by taking into account, the effect of the particles (electrons and ions) which are trapped into the potential well when amplitudes become large. O'Neil considers the phase mixing of the trapped electrons and calculates the effective period of nonlinear damping of the waves. BGK has shown that by prescribing arbitrary wave-forms, arbitrary dispersion relation can be obtained and the lowest significant power in the expansion of distribution function is not a linear, but a square root of the potential. These results clearly show that the nonlinear improvement on the linear solution should be closely connected with the phenomena of particle trapping. The oscillatory motions of the trapped electrons, after being phase mixed, produce a vortex-like structure in the phase space distribution, leaving an almost empty hole (Drummond *et al* 1970). These are referred to as solitary electron holes studied by Saeki *et al* (1979) experimentally, and their stability investigated by Kako *et al* (1971). Recently Schamel

(1979a, b), Bujarbarua and Schamel (1981) have investigated ion holes in addition to electron holes. Problems like these and other similar problems can be investigated by studying the nonlinear distribution functions.

In this paper we study the stability of the solution of Vlasov-Poisson system, by using the nonlinear distribution functions introduced by Allis (1968, 1969). We show that these produce solutions which are modulationally unstable.

2. The Nonlinear distribution function of Allis

By considering the Vlasov equation in the wave frame, and analysing the energy constant of the individual particle and their phase space trajectories, Allis has prescribed the following nonlinear distribution function for the electrons:

$$f = n_0\left(\frac{\beta}{\pi}\right)^{1/2} \exp\left[-\beta\,(u+c)^2\right] \tag{1}$$

$$\beta = m_e/2kT$$

where $u=\omega/k$, wave phase velocity, $w=v-u$ is the electron velocity in the wave frame, and $\frac{1}{2}\,m_e c^2 = \frac{1}{2} m_e w^2 - eV$ is the total energy of the electron in the wave frame, V being the electrostatic potential. Allis has shown that this distribution is appropriate for a Maxwellian plasma, because:

(i) it tends to a Maxwell-Boltzmann distribution

$$f \to n_0\,(\beta/\pi)^{1/2} \exp\,(\eta - \beta\, v^2)$$

for either $\eta \ll w^2$,

or $u^2 \ll v^2$

with $\eta = eV/\kappa T$ (2)

(ii) it gives exactly the results of the linear theory in the zero amplitude limit, and

(iii) it describes the trapped particle distribution as the analytic continuation of the velocity symmetric part of the untrapped distribution (1):

$$f_t = n_0\,(\beta/\pi)^{1/2} \exp\left[\eta - \beta\,(u^2 + w^2)\right] \cosh 2\beta\, uc \tag{3}$$

(iv) The normalisation of the distribution function (1) is determined by fixing the zero position of the potential. By choosing some intermediate point between the potential minimum η_1 and potential maximum η_2 as the zero of potential, the distribution (1) will become exactly Maxwellian each time the potential η passes through that zero point of potential. In this way, the normalisation is fixed.

Shamel (1979a) has used a similar distribution to discuss the role of trapped particles in an ion-acoustic and Langmuir envelope soliton.

The electron density is obtained by integrating (1) over all values of w:

$$n_- = \int_{-\infty}^{+\infty} f\, dv$$

$$= n_0\, (\beta/\pi)^{1/2} \exp\, (\eta - \beta\, u^2) \int_{-\infty}^{+\infty} \exp\, (-\,\beta\, w^2) \cosh 2\beta uc\; dw$$

$$= n_0 \sum_{a=0}^{\infty} M_a \frac{\eta^a}{a!}, \tag{4}$$

where the M_a's are confluent hypergeometric functions,

$$M_a = \phi\, (a, \tfrac{1}{2};\, -\,\beta\, u^2) \tag{5}$$

As written above, (4) represents a small potential expansion of the density in powers of the potential. It thus follows from (2) that the kinetic energy kT of the particle must be less than the potential energy eV, in order that the series (4) be valid.

3. Poisson's equation and the average Lagrangian

We consider an electron plasma where ions form only a neutralising background. In general, the average electron density $\langle n_- \rangle$ may not be equal to n_0, the equilibrium density, and for average space charge neutrality we have

$$\langle n_- \rangle = n_+, \text{ the ion density}$$

so we write

$$n_+ = n_0(1 + C_2) \tag{6}$$

where C_2 is a constant which determines the d.c. flow in the plasma caused by trapped as well as resonant electrons (Allis 1969). This can be seen as follows. The electron flow is given by

$$\Gamma = \int_{-\infty}^{+\infty} fv\, dv.$$

If particle trapping by the wave is neglected then it can be shown that

$$\Gamma = (n_- - n_0)u.$$

Γ thus fluctuates with electrons density n_-, and its average value is given by

$$\begin{aligned}\langle \Gamma \rangle &= (\langle n_- \rangle - n_0)u,\\ &= (n_+ - n_0)u, \text{ using (6)},\\ &= n_0 u C_2.\end{aligned}$$

Thus the constant C_2 is proportional to the average electron or d.c. flow.

Using (4) and (6) we write the Poisson's equation in one dimension

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{e^2}{\epsilon_0 kT}(n_- - n_+) = 2\beta\, \omega_0^2 \frac{\mathrm{d}F}{\mathrm{d}\eta} \tag{7}$$

where
$$F = \sum_{a=1}^{\infty} M_a \frac{\eta^{a+1}}{(a+1)!} - C_2 \eta, \tag{8}$$

and $\omega_0^2 = n_0 e^2/\epsilon_0 m_e$ (M.K.S. system of units has been used).
Introducing the phase function

$$\theta = kx - \omega_t$$

to replace the (x,t) variable for periodic solutions, we integrate Poisson's equation, and obtain

$$\frac{k^2}{4\beta\omega_0^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\theta}\right)^2 - F = C_1, \quad \text{constant.}$$

We notice that the function F given by (8) acts as a pseudo-potential, and that the Lagrangian for the system is given by

$$L = \frac{k^2}{4\,\beta\,\omega_0^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\theta}\right)^2 + F. \tag{10}$$

The wave form is obtained by integrating (9) for the phase

$$\frac{\sqrt{\beta}\,\omega_0}{k}\,\theta = \int_{\eta_1}^{\eta \leqslant \eta_2} \frac{\mathrm{d}\eta}{2\,(C_1 + F)^{1/2}}, \tag{11}$$

where $\eta_1 < 0$ and $\eta_2 > 0$ are simple zeros of the function $C_1 + F$ at which points $\mathrm{d}\eta/\mathrm{d}\theta$ is zero, giving the extremum of the η *vs* θ curve. As is well-known, for bounded solutions the zeros of the function $C_1 + F$ must be real, and for periodic solutions, the bounded solutions must oscillate periodically between two zeros of $C_1 + F$. Without any loss of generality, we choose the lowest zero to be at the potential minimum η_1, and the other zero up to which the potential oscillates, is taken as η_2.

$$\eta_1 \leqslant \eta \leqslant \eta_2, \qquad C_1 + F > 0.$$

So the span $\eta_2 - \eta_1$ gives double the oscillation amplitude. Because of our choice of the zero of potential, as some point in between the maximum and minimum, we must take $\eta_1 < 0$ and $\eta_2 > 0$.

Following Whitham (1974), we now construct the average Lagrangian $\mathscr{L}(\omega, k, C_1)$;

$$\mathscr{L}(\omega, k, C_1) = \frac{1}{2\pi} \int_0^{2\pi} L d\theta$$

$$= \frac{1}{2\pi} \frac{k}{\sqrt{\beta}\, \omega_0} \oint (C_1 + F)^{1/2} \, d\eta - C_1, \tag{12}$$

the loop integral being taken over a complete oscillation of η from η_1 to η_2 and back to η_1, with appropriate change in phase in the two parts of the cycle.

4. Small-amplitude case

We retain in the series (8) only up to that term in powers of η which produces the first nonlinear term in the Poisson's equation:

$$F = - C_2 \eta + \frac{M_1}{2} \eta^2 + \frac{M_2}{6} \eta^3. \tag{13}$$

Using this expression for F, we calculate the integrals involved in (11) and (12) to obtain wave form and the average Lagrangian in terms of elliptic functions and integrals:

$$\eta = (\eta_2 - \eta_1) sn^2 \Theta + \eta_1 \tag{14}$$

with $\Theta = K\theta/\pi$

and
$$\mathscr{L} = \frac{9\sqrt{2}}{15\pi} \frac{k}{\sqrt{\beta}\omega_0} \frac{(-M_1)^{5/2}}{M_2^2} (1 + 3a)^{5/4} \times \left[\frac{-2 + 3m - m^2}{(1 - m + m^2)^{5/4}} K + \frac{2E}{(1 - m + m^2)^{1/4}} \right] - C_1. \tag{15}$$

Also, the amplitude of oscillation $2A$ is given by the following expression:

$$2A = \eta_2 - \eta_1 = \frac{-3M_1 m}{M_2} [(1 + 3a)/(1 - m + m^2)]^{1/2}. \tag{16}$$

In (14)–(16), K and E are respectively the complete elliptic integrals of the first and second kind, m is the square of the modulus of the elliptic functions (Magnus and Oberhettinger, 1949), and

$$a = \frac{2}{3} \frac{C_2 M_2}{M_2^1} \tag{17}$$

We note here that as F is given by the expression (13), $C_1 + F$ is a cubic. The two zeros η_1 and η_2 are already indicated and their role in eqn. (14) is obvious. The third zero η_3 enters eqn. (14) through the quantity Θ, i.e. through the square modulus m of the elliptic function K. Its relation with m and A is as follows:

$$m = \frac{\eta_2 - \eta_1}{\eta_3 - \eta_1} = \frac{2A}{\eta_3 - \eta_1}.$$

It turns out that the square of the modulus m gives a measure of the relative importance of non-linearity to dispersion: $2A$ is a measure of non-linearity ($A \to 0$ goes over to linear regime) and $\eta_3 - \eta_1$ is thus a measure of the dispersion. It is then obvious that the position of the third zero η_3 of $C_1 + F$, with respect to η_1 determines the competition between non-linearity and dispersion of the waves.

We may also note from eqn. (9) that the constant C_1 is the energy constant of wave motion and thus depends directly on the wave amplitude A. Equation (16) then tells us how the square modulus m, which is the principal determining factor of the nature of solution, depends on M_1, M_2, C_1 (i.e. A) and C_2 (i.e. α); these are the coefficients which occur in the expresion $(C_1 + F)$.

We now apply the average variational principle of Whitham (1974) to the Lagrangian $\mathscr{L}(\omega, k, C_1)$. Varying with respect to δC_1 and $\delta\theta$, we obtain the modulation equations

$$\partial \mathscr{L} / \partial C_1 = 0 \tag{16a}$$

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial k} = 0, \tag{16b}$$

and

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{16c}$$

Equation (16a) is the dispersion relation, which using the value of $\mathscr{L}$ given by (15), becomes

$$\frac{\omega^2}{\omega_0^2} = \frac{\pi^2 \beta u^2}{2} \frac{(-M_1)}{K^2} \frac{(1 + 3\alpha)^{1/2}}{(1 - m + m^2)^{1/2}}. \tag{18}$$

In the above expression M_1 and M_2 are the first and second dispersion functions, the former being related to the real part of the well-known complex plasma dispersion function Z (Fried and Conte 1961):

$$2M_1 = \mathrm{Re}\,[-Z'(\sqrt{\beta}u)],$$

where the prime denotes differentiation with respect to the argument. The function M_1 is negative for $\beta u^2 > 0{\cdot}857$, showing the lower limit of u below which small amplitude wave cannot propagate. The linear results are obtained by taking the limit $m \to 0$, when amplitude goes to zero, as we notice from (16).

We note from (17) and (16) that the nonlinear frequency depends on amplitude through the combination $K^2(1 - m + m^2)^{1/2}$. To arrive at the final form of ω, we simplify by taking $C_2 = 0$ (no d.c. flow) wavelength $\lambda \to \infty$ *i.e.* $u \to \infty$ (so, use the asymptotic form of M_1), and expand the denominator of (17) in powers of m and retain the first significant term. We thus obtain

$$\omega^2 = \omega_L^2 \left(1 - \tfrac{15}{16} m^2\right) \tag{18a}$$

where ω_L is the linear frequency

$$\omega_L^2 = \omega_p^2 + \frac{3k^2}{2\beta}, \quad \omega_p = \frac{n_+ e^2}{\epsilon_0 m_e}. \tag{18b}$$

(As $C_2 \to 0$, $\langle n_- \rangle = n_+ \to n_0$, so $\omega_0 \to \omega_p$, the plasma frequency).

5. Stability of the solution

To discuss the question of stability we first find out the characteristics of our nonlinear equation. We express $\partial \mathscr{L}/\partial\omega$ and $\partial \mathscr{L}/\partial k$ in terms of phase velocity u and the energy constant C_1, and calculate the differentials $d(\partial \mathscr{L}/\partial\omega)$ and $d(\partial \mathscr{L}/\partial k)$ in terms of du and dC_1. We thus obtain the characteristic velocity using (12), (16a) and (16b).

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}(\partial \mathscr{L}/\partial k)}{\mathrm{d}(\partial \mathscr{L}/\partial \omega)} = -\frac{uPdu - (R - uQ)\,dC_1}{Pdu + QdC_1}, \tag{19}$$

where $$P = 2\frac{\partial^2}{\partial u^2} \oint (C_1 + F)^{1/2}\, \mathrm{d}\eta,$$

$$Q = \frac{\partial}{\partial u} \oint \frac{\mathrm{d}\eta}{(C_1 + F)^{1/2}},$$

$$R = \oint \frac{\mathrm{d}\eta}{(C_1 + F)^{1/2}}. \tag{20}$$

In a similar manner, we express k and ω in terms of u and C_1 using (16a),

$$k = 4\pi\,\omega_0/R \qquad \text{and} \qquad \omega = 4\pi\,\omega_0\, u/R$$

and calculate $\mathrm{d}x/\mathrm{d}t$, using relation (16c)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{(R - uQ)du - u\dfrac{\partial R}{\partial C_1}dC_1}{Q\,du + \dfrac{\partial R}{\partial C_1}dC_1}. \tag{20}$$

Eliminating the variable du/dC_1 from (19) and (20), we finally obtain the nonlinear group velocity (characteristic velocity)

$$\frac{dx}{dt} = -\frac{u\left(1 \pm \frac{Q}{P}\left(P \Big/ \frac{\partial R}{\partial C_1}\right)^{\frac{1}{2}}\right) \mp \frac{R}{P}\left(P \Big/ \frac{\partial R}{\partial C_1}\right)^{\frac{1}{2}}}{1 \pm \frac{Q}{P}\left(P \Big/ \frac{\partial R}{\partial C_1}\right)^{\frac{1}{2}}}. \tag{21}$$

The condition of stability follows at once

$$P \Big/ \frac{\partial R}{\partial C_1} > 0, \tag{22}$$

i.e. for stability, the functions P and $\partial R/\partial C_1$ must have the same signs.

Equation (22) is the general result. For the small amplitude case, we can express the condition (22) in terms of the frequencies ω and ω_L as given by (18a, b). The constant C_1 directly depends on the amplitude A, as can be shown. So we expand the integrals $\oint (C_1 + F)^{1/2}\, d\eta$ in powers of C_1 and write

$$\oint \sqrt{C_1 + F}\, d\eta = A_1(u)C_1 + A_2(u)C_1^2 + \ldots,$$

where A_1, A_2, etc are some functions of u. This gives us the following values

$$P = 2A_{1uu}\, C_1 + 2A_{2uu}\, C_1^2 + \ldots$$

$$\frac{\partial R}{\partial C_1} = 4A_2\left[\, 1 + \frac{3A_3}{A_2}\, C_1 + \ldots\ldots\right]$$

with

$$A_{1uu} = \frac{\partial^2 A_1(u)}{\partial u^2}, \text{ etc.}$$

The Lagrangian $\mathscr{L}$ in (12) can be similarly expanded

$$\mathscr{L} = \left(\frac{k}{2\pi\sqrt{\beta}\omega_0} A_1 - 1\right) C_1 + \frac{k}{2\pi\sqrt{\beta}\omega_0} A_2 C_1^2 + \ldots\ldots\ldots$$

Using all these relations and the linear dispersion in the form

$$\frac{k}{2\pi\sqrt{\beta}\omega_0} A_1(u) - 1 = 0.$$

we obtain, after a little calculation, but in a straightforward manner, the following:

$$P \Big/ \frac{\partial R}{\partial C_1} = \frac{A_1^2}{4A_2^2\,[\omega_L'(k) - u]^2} \frac{\omega_L''(k)\,(\omega - \omega_L(k))}{u}$$
$$+ \text{ terms in } C_1^2.$$

Hence to leading order terms, the stability condition is given by

$$\omega_L''(k)\,(\omega - \omega_L(k)) > 0,$$

which is exactly the Lighthill's criterion (Lighthill 1965) for stability against modulation, expressed only by the sign of the nonlinear frequency shift and the sign of curvature of the linear dispersion curve.

6. Conclusions

We have demonstrated that the nonlinear distribution function (1) of Allis generates solutions which exactly satisfy Lighthill's stability criterion. It gives rise to the correct amplitude and wave forms in the plasma and leads to the known results on linearization. In subsequent papers we shall deal with the question of solitary wave formation and the effect of phase mixing in the formation of phase space 'vortices' using this distribution function.

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