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The effect of longitudinal-transverse coupling and medium temperature on Cerenkov radiation

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Abstract. The effect of medium properties on the energy loss suffered by a relativistic charged particle in a dielectric medium generating Cerenkov radiation is discussed. Here we have taken into account the effect of the coupling of the longitudinal (σ) mode with the transverse (λ) mode in the electromagnetic interaction. Calculation shows that the inclusion of $\lambda-\sigma$ coupling in the transverse interaction affects significantly the radiation output. However the modification of the radiation due to the thermal state of the system becomes important at very high temperatures such as one finds in astrophysical situations. This temperature effect is negligible at ordinary temperatures and thus is undetectable in laboratory experiments.

Keywords. Dielectric medium; $\lambda-\sigma$ coupling; temperature effect; response function; Cerenkov threshold; coherence.

1. Introduction

The phenomenon of Cerenkov radiation is due to the excitation of the transverse collective modes in the medium by a relativistic charged particle passing through it. Therefore the various applications of this phenomenon in different branches of physics need to consider the modification of the radiation by the medium because of its thermal and density fluctuations.

Majumdar (1961) considered this problem in a magnetoactive plasma with interactions propagated only in the transverse mode. He pointed out however that the presence of the external magnetic field will couple the transverse and longitudinal modes and thereby the radiation emitted by the particle will be modified. This mode of coupling through the magnetic field however is well-known (Ginzburg 1964). Recently Heintzmann and Nitsch (1979) considered the phenomenon of Cerenkov radiation in a very strongly magnetized plasma, where they have also included a relaxation time in the momentum equation and obtained a dielectric tensor. They have applied this to the pulsar magnetosphere and have shown that the Cerenkov radiation is comparable to the curvature radiation emitted by the same particle gliding along the magnetic field lines.

The introduction of a relativistic charged particle into a quiescent plasma would introduce density fluctuations and these fluctuations will couple the longitudinal and the transverse modes. It is this problem that we are going to consider in our formulation. The questions we would like to answer are:

(1) How the medium temperature affects the phenomenon of Cerenkov radiation?

(2) How much energy is lost by the relativistic particle when we take into account the coupling of the longitudinal mode with the transverse mode?

The coupling of the longitudinal mode with the transverse mode has importance in the transfer of energy to the plasma in the form of heat.

In an earlier microscopic theory developed by Pratap (1967) the effect of this density fluctuation as well as the medium temperature are not taken into account. A closer examination of the relevant diagrams in his work reveals that a longitudinal (σ) mode interacting with a transverse (λ) mode does not appear upto the 4th-order (It is shown in Appendix I, Pratap 1967). But the present analysis shows that λ-σ coupling appears in 6th-order diagrams. The concept of the temperature of the medium appears through the initial distribution function of the medium particles (Pratap 1967); but there the effect of temperature was not considered.

In this paper starting the discussion from the 6th-order diagram we have derived the test-particle distribution function in § 2. Section 3 deals with the calculation of the energy lost by the test particle. Finally we have discussed the result obtained by inclusion of the temperature effect as well as λ-σ coupling in the evolution of correlation in § 4. We have estimated the percentage effect of this coupling and also of the medium temperature. We have shown that while the former is significantly high, the latter becomes significant only at very high temperatures.

2. The test particle distribution function

The general formulation is the same as given in Pratap (1967) and we are not repeating the same for the sake of brevity. The one-particle distribution function is given as (Balescu 1963)

$$f(\mathbf{q}_i, \mathbf{p}_i; t) = \int d\mathbf{q}_1\, d\mathbf{q}_2 \ldots d\mathbf{q}_{i-1}\, d\mathbf{q}_{i+1} \ldots d\mathbf{q}_N\, d\mathbf{p}_1\, d\mathbf{p}_2 \ldots d\mathbf{p}_{i-1}\, d\mathbf{p}_{i+1} \ldots d\mathbf{p}_N \cdot \rho\,(\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N; t) \tag{1}$$

where $\rho = \rho\;(\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N;\; t)$ is the phase space density distribution function for the whole system with $\mathbf{q}_i$ and $\mathbf{p}_i$ as canonical variables.

In evaluating the test particle distribution function with the help of equation (1) and the equations (1.22), (1.23), (1.24) and (2.3) of Pratap (1967) and following the diagrammatic technique explained there, we consider only the diagrams which contribute towards terms of order $e^2(e^2c_l)^r$, $r = 0, 1, 2, \ldots$; where e is the electronic charge, $c_l = (N_l \to \infty)/(v \to \infty) =$ constant, N_l being the number of the medium oscillators and v the volume of the system in the thermodynamic limit. The choice of the diagrams amounts to the self-consistent field approximation. The time ordered integrals appearing in (1.22) of Pratap (1967) show the non-Marcovian character of the distribution function, and hence would include all time scales upto the collision time scale given by e^2c_l/m_l (Balescu 1963).

A closer examination of the relevant diagrams in Pratap (1967) reveals that a σ-mode interacting with a λ-mode does not appear upto the 4th-order. Now the diagram in 6th-order contributing to the transverse interaction leading to Cerenkov

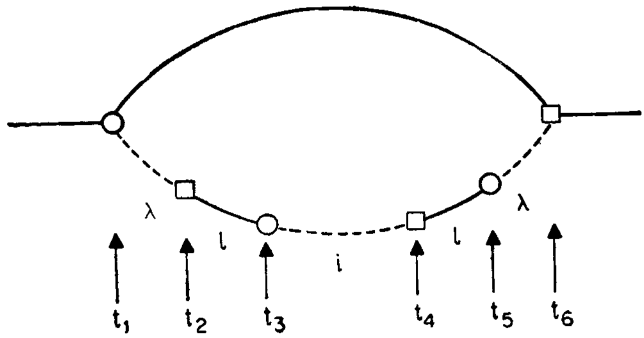


Figure 1. A diagram corresponding to the 6th order irreducible matrix element. '□' represents the creation of correlation vertex and 'O' represents the destruction vertex. Dotted line represents, either photon, phonon or a plasmon and a solid line in lower side represents the propagation of a medium particle while in upper part the propagation of the test particle.

radiation is given in figure 1 where f stands for λ, σ and 0σ. The irreducible matrix element corresponding to the above diagram will involve the product of the operators

$$\mathscr{A}^p_\lambda \mathscr{B}^l_\lambda \mathscr{A}^l_f \mathscr{B}^l_f \mathscr{A}^{l'}_{\lambda''} \mathscr{B}^p_{\lambda''}$$

$$= \mathscr{A}^p_\lambda \mathscr{B}^l_\lambda (\mathscr{A}^l_{\lambda'} \mathscr{B}^{l'}_{\lambda'} + \mathscr{A}^l_\sigma \mathscr{B}^{l'}_\sigma + \mathscr{A}^l_{0\sigma} \mathscr{B}^{l'}_{0\sigma}) \mathscr{A}^{l'}_{\lambda''} \mathscr{B}^p_{\lambda''} \qquad (2)$$

where the form of operators $\mathscr{A}$ and $\mathscr{B}$ for the test particle and medium oscillators are given in Pratap (1967). The first term

$$\mathscr{A}^p_\lambda \mathscr{B}^l_\lambda \mathscr{A}^l_{\lambda'} \mathscr{B}^{l'}_{\lambda'} \mathscr{A}^{l'}_{\lambda''} \mathscr{B}^p_{\lambda''}$$

in (2) is the term considered there. In the present formulation, our interest is to see the effect of the 2nd and 3rd terms *i.e.* the terms involving the products

$$\mathscr{A}^p_\lambda \mathscr{B}^l_\lambda \mathscr{A}^l_\sigma \mathscr{B}^{l'}_\sigma \mathscr{A}^{l'}_{\lambda''} \mathscr{B}^p_{\lambda''}$$

and

$$\mathscr{A}^p_\lambda \mathscr{B}^l_\lambda \mathscr{A}^l_{0\sigma} \mathscr{B}^{l'}_{0\sigma} \mathscr{A}^{l'}_{\lambda''} \mathscr{B}^p_{\lambda''}.$$

Since the λ-σ-λ matrix element is more dominant than the λ-0σ-λ matrix element; we shall neglect the latter one. One can realise this from the fact that the transverse polarisation can be modified only by a polarization vector and 0σ mode (or the scalar part of the potential) does not have a specific polarization. The longitudinal polarization appears in our formalism only through $\mathbf{A}_\sigma$. Hence a plasmon mode (generated by the scalar potential Φ) can affect the radiation only in a secondary manner through $\mathbf{A}_\sigma$. We therefore neglect the coupling of λ-0σ mode and consider only the modification of polarization by the longitudinal mode.

As shown in Appendix the term

$$\mathscr{A}^p_\lambda \mathscr{B}^l_\lambda \mathscr{A}^l_\sigma \mathscr{B}^{l'}_\sigma \mathscr{A}^{l'}_{\lambda''} \mathscr{B}^p_{\lambda''}$$

leads to the conservation of wave vectors in different mode, *i.e.*

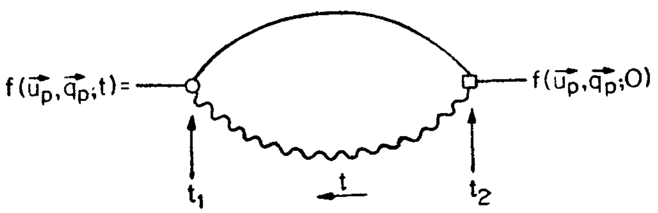
$$\mathbf{k}_\lambda = \mathbf{k}_\sigma, \quad \mathbf{k}_\sigma = \mathbf{k}_{\lambda'}. \tag{3}$$

Here $\mathbf{k}_\lambda$ and $\mathbf{k}_\sigma$ are the wave vectors defined by the conditions

$$\mathbf{e}_\lambda \cdot \mathbf{k}_\lambda = 0 \text{ and } \mathbf{e}_\sigma \times \mathbf{k}_\sigma = 0, \tag{4}$$

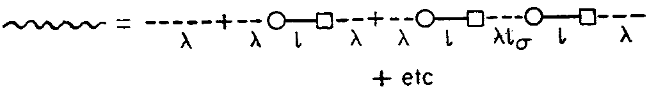
where $\mathbf{e}_\lambda$ and $\mathbf{e}_\sigma$ are the polarization vectors which are mutually perpendicular. The above conditions (4) imply that $\mathbf{k}_\lambda$ is normal to $\mathbf{e}_\lambda$ and $\mathbf{k}_\sigma$ is parallel to $\mathbf{e}_\sigma$ and hence condition (3) can be satisfied. Also to get a non-zero contribution from the term $\mathscr{A}^p_\lambda \, \mathscr{B}^l_\lambda \, \mathscr{A}^l_\sigma \, \mathscr{B}^{l'}_\sigma \, \mathscr{A}^{l'}_{\lambda'} \, \mathscr{B}^p_{\lambda''}$ one has to assume that all the medium oscillators are identical.

The foregoing discussions enable us to write the test-particle distribution function (with the help of (1)) in first Born approximation for transverse interaction (Pratap 1967; Henin 1963) as given in figure 2



(5)

Figure 2. Diagramatic form of equation (5). The diagram is a summation of all irreducible diagrams contributing to the test particle distribution function.



(6)

Figure 3. This diagram represents the sum of interactions of medium particles which contribute towards the response function.

where the wavy line represents the sum given in figure 3.

One can write the above test particle distribution function as

$$f(\mathbf{u}_p\, \mathbf{q}_p;\, t) = e^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \exp\{-i\, \mathscr{L}^p_0 (t - t_1)\}\, \mathscr{A}^p_\lambda$$

$$\exp\{-i\, \mathscr{L}^p_0 (t_1 - t_2)\} \cdot \left[\oint_c \mathrm{d}Z \exp\{-iZ(t_1 - t_2)\}\, U(Z) \right.$$

$$\left. \left(\frac{1 - \epsilon^\sigma_l (Z)}{1 - \epsilon^\sigma_l (Z) - \epsilon^\lambda_l (Z)} \right) \right] \cdot \mathscr{B}^p_{\lambda'} \exp(-i\, \mathscr{L}^p_0 t_2)\, f(\mathbf{U}_p, \mathbf{q}_p;\, 0). \tag{7}$$

Here $\mathscr{A}^p_\lambda$ and $\mathscr{B}^p_\lambda$ are operators only in test-particle variables (Pratap 1967) as the photon field variables have already been integrated out, $\epsilon\,(Z)$ is the response function of the system with superscripts λ and σ corresponding to the transverse interaction and longitudinal interaction and

$$U(Z) = \frac{iZ}{Z^2 - 4\pi^2 \nu_\lambda^2}. \tag{8}$$

As shown in Appendix

$$\epsilon_l^\lambda(Z) = \frac{64\,\pi^4\,\omega_{pl}^2\,\nu_l^2}{3} \cdot \frac{{}_2F_2\,(\frac{1}{2};\,\frac{1}{2};\,1,\,\frac{5}{2};\,-2A)}{(Z^2 - 4\pi^2\,\nu_\lambda^2)\,(Z^2 - 4\pi^2\,\nu_l^2)}, \tag{9}$$

and
$$\epsilon_l^\sigma(Z) = \frac{32\,\pi^4\,\omega_{pl}^2\,\nu_l^2}{3} \cdot \frac{{}_2F_2\,(\frac{1}{2},\,\frac{3}{2};\,1,\,\frac{5}{2};\,-2A)}{(Z^2 - 4\pi^2\,\nu_\sigma^2)\,(Z^2 - 4\pi^2\,\nu_l^2)}, \tag{10}$$

where
$${}_2F_2\,(\alpha,\beta;\,\lambda,\delta;\,-2A) = 1 + \frac{\alpha\cdot\beta}{\lambda\cdot\delta}(-2A) + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\lambda(\lambda+1)\,\,\delta(\delta+1)}\,\frac{(-2A)^2}{2!} + \dots \tag{11}$$

is a hypergeometric function, and

$$A = A(T) = \frac{\nu_\lambda^2}{\nu_l^2}\,\frac{K_B\,T}{m_l\,c^2}. \tag{12}$$

In getting (7) we have summed the infinite series given in figure 3 with the assumption that

$$|\,\epsilon_l^\lambda(Z) + \epsilon_l^\sigma(Z)\,| < 1. \tag{13}$$

But as seen in (9) and (10) the response function is dependent on the density of the medium particles through the plasma frequency

$$\omega_{\mathrm{pl}}^2 = \frac{4\pi\,e_l^2\,c_l}{m_l} \tag{14}$$

where e_l and m_l are the charge and mass of the l-th particle and c_l the concentration. The assumption (13) corresponds to the low density approximation. For $|\,\epsilon_l^\lambda(Z) + \epsilon_l^\sigma(Z)\,| > 1$ one can analytically continue and thereby obtain high density approximation (Balescu 1963).

The main interest is to calculate the rate of energy loss of the test particle. We do this by evaluating

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathcal{H}^p\,\mathrm{d}\mathbf{u}_p\,\mathrm{d}\mathbf{q}_p$$

and commuting the time derivative with the integral sign and remembering the fact that $\mathrm{d}\mathcal{H}^p/\mathrm{d}t = 0$ we have to evaluate $\mathrm{d}f/\mathrm{d}t$ which is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{e_p^2}{m_p} \frac{16\pi^2}{v} \int_0^t \mathrm{d}t_2 \sum_{\mathbf{k}_\lambda} \frac{\partial}{\partial \mathbf{u}_p} \cdot \mathbf{e}_\lambda (\mathbf{e}_\lambda \cdot \boldsymbol{\beta}) \cos \{\mathbf{k}_\lambda \cdot \boldsymbol{\beta}\, c\, (t - t_2)\}$$

$$\cdot \exp(-i\mathscr{L}_0^p t_2) f(\mathbf{u}_p, \mathbf{q}_p; 0) \left[\oint_C \mathrm{d}Z \exp\{-iZ(t - t_2)\} \right.$$

$$\left. \cdot \frac{iZ(1 - \epsilon_l^\sigma(Z))}{(Z^2 - 4\pi^2 \nu_\lambda^2)(1 - \epsilon_l^\sigma(Z) - \epsilon_l^\lambda(Z))} \right] \tag{15}$$

where $\boldsymbol{\beta} = (\mathbf{u}_p/(1 + u_p^2)^{1/2} = \mathbf{v}/c$; $\mathbf{v}$ being the velocity of the test particle.

3. Energy loss by the test particle

Now the rate of average energy lost by the particle because of the electromagnetic interaction of the test particle with the medium particles is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathcal{H}^p \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}\mathbf{u}_p\, \mathrm{d}\mathbf{q}_p\, \mathcal{H}^p f(\mathbf{u}_p, \mathbf{q}_p; t)$$

$$= -\frac{2e_p^2 c^2}{\pi} \int \mathrm{d}\mathbf{u}_p\, \mathrm{d}\mathbf{q}_p\, \mathrm{d}\mathbf{k}_\lambda \frac{(\mathbf{k}_\lambda \times \boldsymbol{\beta})^2}{k_\lambda^2} \int_0^t \mathrm{d}t_2 \cos\{\mathbf{k}_\lambda \cdot \boldsymbol{\beta}\, c\, (t - t_2)\}$$

$$\cdot \exp(-i\mathscr{L}_0^p t_2) f(\mathbf{u}_p, \mathbf{q}_p; 0) \cdot \left[\oint_C \mathrm{d}Z \exp\{-iZ(t - t_2)\} \right.$$

$$\left. \frac{iZ}{(Z^2 - 4\pi^2 \nu_\lambda^2)} \left(\frac{1 - \epsilon_l^\sigma(Z)}{1 - \epsilon_l^\sigma(Z) - \epsilon_l^\lambda(Z)} \right) \right]. \tag{16}$$

Substituting the expressions (9) and (10) in (16) and using the recurrence relation

$$ {}_2F_2(\tfrac{1}{2}, \tfrac{1}{2}; 1, \tfrac{5}{2}; -2A) = \tfrac{3}{2} [{}_2F_2(\tfrac{1}{2}, \tfrac{1}{2}; 1, \tfrac{3}{2}; -2A)$$

$$- \tfrac{1}{3}\, {}_2F_2(\tfrac{1}{2}, \tfrac{3}{2}; 1, \tfrac{5}{2}; -2A)], \tag{17}$$

and having the inverse Laplace transformation, (16) becomes

$$\frac{dE}{dt} = -\frac{2e_p^2 c^2}{\pi} \int d\mathbf{u}_p \, d\mathbf{q}_p \, d\mathbf{k}_\lambda \frac{(\mathbf{k}_\lambda \times \beta)^2}{k_\lambda^2} \int_0^t dt_2 \cos \{\mathbf{k}_\lambda \cdot \beta \, c \, (t - t_2)\}.$$

$$\exp(-i\mathscr{L}_0^p t_2) f(\mathbf{u}_p, \mathbf{q}_p; 0) \Big[\tfrac{1}{3} \phi_1 (-2A) \cos \nu_\lambda (t - t_2)$$

$$+ \tfrac{1}{3} \phi_2 (-2A) \left(1 + \frac{\nu_\lambda^2 - \nu_l^2}{\chi}\right) \cos \left\{ \left(\frac{\nu_\lambda^2 + \nu_l^2 + \chi}{2}\right)^{1/2} (t - t_2) \right\}$$

$$+ \tfrac{1}{3} \phi_2 (-2A) \left(1 - \frac{\nu_\lambda^2 - \nu_l^2}{\chi}\right) \cos \left\{ \left(\frac{\nu_\lambda^2 + \nu_l^2 - \chi}{2}\right)^{1/2} (t - t_2) \right\} \Big] \tag{18}$$

where
$$\phi_1 (-2A) = \frac{{}_2F_2 (\frac{1}{2}, \frac{3}{2}; 1, \frac{5}{2}; -2A)}{{}_2F_2 (\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -2A)}, \tag{19}$$

$$\phi_2 (-2A) = \frac{{}_2F_2 (\frac{1}{2}, \frac{1}{2}; 1, \frac{5}{2}; -2A)}{{}_2F_2 (\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -2A)}, \tag{20}$$

and the notation χ is given by

$$\chi^2 = (\nu_\lambda^2 - \nu_l^2)^2 + 8\omega_{pl}^2 \, \gamma_l^2 \, {}_2F_2 (\tfrac{1}{2}, \tfrac{1}{2}; 1, \tfrac{3}{2}; -2A) \tag{21}$$

In getting (18) we have redefined ν^2 by absorbing the factor $4\pi^2$. In (18) the condition that the arguments of the cosine terms be real demands

$$\nu_\lambda^2 + \nu_l^2 \pm \chi \geqslant 0. \tag{22}$$

Use of (21) reduces (22) to

$$\nu_\lambda^2 \geqslant 2\omega_{pl}^2 \, {}_2F_2 (\tfrac{1}{2}, \tfrac{1}{2}; 1, \tfrac{3}{2}; -2A). \tag{23}$$

The integrations in (18) can be carried out easily by assuming the test particle to be moving along the z-axis with a velocity u; so that

$$f(\mathbf{u}_p, \mathbf{q}_p; 0) = \delta(u_{px}) \, \delta(u_{py}) \, \delta(u_{pz} - u) \, \delta(\mathbf{q}_p), \tag{24}$$

and so

$$\exp(-i\mathscr{L}_0^p t_2) f(\mathbf{u}_p, \mathbf{q}_p; 0)$$
$$= \delta(u_{px}) \, \delta(u_{py}) \, \delta(u_{pz} - u) \, \delta(\mathbf{q}_p - \beta c t_2 \, \mathbf{z}), \tag{25}$$

where $\mathbf{z}$ is the unit vector along the z-axis of the co-ordinate system.
In the asymptotic limit in time ($t \to \infty$ or $Z \to 0$) and with (25), equation (18) after integrations becomes

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2\pi e_p^2 \beta^2 c^2}{3} \int k_\lambda^2 \,\mathrm{d}k_\lambda \int_{-1}^{1} \mathrm{d}x\,(1-x^2) \Big\{ \phi_1(-2A)$$

$$\Big[\delta(x k_\lambda \beta c + \nu_\lambda) + \delta(x k_\lambda \beta c - \nu_\lambda)\Big]$$

$$+ \phi_2(-2A)\left(1 + \frac{\nu_\lambda^2 - \nu_i^2}{\chi}\right) \cdot \left[\delta\left(x k_\lambda \beta c + \left(\frac{\nu_\lambda^2 + \nu_i^2 + \chi}{2}\right)^{1/2}\right)\right.$$

$$\left. + \delta\left(x k_\lambda \beta c - \left(\frac{\nu_\lambda^2 + \nu_i^2 + \chi}{2}\right)^{1/2}\right)\right] + \phi_2(-2A)\left(1 - \frac{\nu_\lambda^2 - \nu_i^2}{\chi}\right)$$

$$\left[\delta\left(x k_\lambda \beta c + \left(\frac{\nu_\lambda^2 + \nu_2 - \chi}{2}\right)^{1/2}\right) + \right.$$

$$\left.\delta\left(x k_\lambda \beta c - \left(\frac{\nu_\lambda^2 + \nu_i^2 - \chi}{2}\right)^{1/2}\right)\right]\Big\}, \tag{26}$$

where $\chi = \cos\theta.$ (27)

Here θ is the angle between the direction of the propagation vector $\mathbf{k}_\lambda$ and that of the test particle.

We shall perform the x-integration using the Dirac δ-function and obtain (26) as

$$\frac{\mathrm{d}E}{\mathrm{d}l} = -\frac{4\pi e_p^2}{3c^2}\int \nu_\lambda \,\mathrm{d}\nu_\lambda \phi_2(-2A)\left(1 - \frac{\nu_\lambda^2 - \nu_i^2}{\chi}\right)\left(1 - \frac{\nu_\lambda^2 + \nu_i^2 - \chi}{2\nu_\lambda^2 \beta^2}\right), \tag{28}$$

where $\mathrm{d}l = \beta c\,\mathrm{d}t$. It may be noted that the range of x viz. $-1 \leqslant x \leqslant 1$ puts limits on the range of ν_λ.

Now under linear approximation in T, the expressions (20) and (21) can be written respectively as

$$\phi_2(-2A) = 1 + \frac{2}{15}\frac{\nu_\lambda^2}{\nu_i^2}\frac{K_B T}{m_i c^2}, \tag{29}$$

and

$$\chi = \left[(\nu_\lambda^2 - \nu_i^2)^2 + 8\,\omega_{pl}^2\,\nu_i^2\left(1 - \frac{1}{3}\frac{\nu_\lambda^2}{\nu_i^2}\frac{K_B T}{m_i c^2}\right)\right]^{1/2}$$

$$= \chi_0 - \frac{4}{3}\omega_{pl}^2 \frac{\nu_\lambda^2}{\chi_0}\frac{K_B T}{m_i c^2}, \tag{30}$$

where $\chi_0 = \chi(T=0) = [(\nu_\lambda^2 - \nu_i^2)^2 + 8\,\omega_{pl}^2\,\nu_i^2]^{1/2}.$ (31)

The ν_λ integration in (28) can now be carried out by defining the variable

$$\omega_0^2 = 2(\nu_\lambda^2 + \nu_i^2 - \chi_0). \tag{32}$$

With (32) we obtain (28) in linear approximation in temperature T as

$$\frac{\mathrm{d}E}{\mathrm{d}l} = -\frac{2\pi\, e_p^2}{3c^2}\left[\frac{\omega_{0M}^2 - \omega_{0m}^2}{2}\left(1 - \frac{1}{\beta^2}\right) + \frac{4\,\omega_{\rm pl}^2\,\nu_l^2}{\beta^2\,(\nu_l^4 + 8\,\omega_{\rm pl}^2\,\nu_l^2)^{1/2}}\cdot\right.$$

$$\left.\ln\left\{1 - \frac{16\,(\omega_{0M}^2 - \omega_{0m}^2)\,(\nu_l^4 + 8\,\omega_{\rm pl}^2\,\nu_l^2)^{1/2}}{(4\nu_l^2 - 2\omega_{0M}^2 + 4(\nu_l^4 + 8\omega_{\rm pl}^2\nu_l^2)^{1/2})\,(4\nu_l^2 - 2\omega_{0m}^2 - 4(\nu_l^4 + 8\omega_{\rm pl}^2\nu_l^2)^{1/2})}\right\}\right]$$

$$-\frac{2\pi\, e_p^2}{3c^2}\cdot\frac{K_B\,T}{m_l\,c^2}\left[\frac{(\omega_{0M}^4 - \omega_{0m}^4)}{120\,\nu_l^2}\left(1 - \frac{1}{\beta^2}\right) + \tfrac{4}{5}\,\omega_{\rm pl}^2\ln\left(\frac{4\nu_l^2 - \omega_{0M}^2}{4\nu_l^2 - \omega_{0m}^2}\right)\right.$$

$$+\frac{\tfrac{4}{3}\,\omega_{\rm pl}^2\left\{\left(1 - \frac{1}{\beta^2}\right)(16\,\nu_l^4 - 4\,\omega_{0M}^2\,\nu_l^2) + 64\,\omega_{\rm pl}^2\,\nu_l^2\left(1 - \frac{1}{2\beta^2}\right)\right\}}{(32\,\omega_{\rm pl}^2\,\nu_l^2 + 16\,\nu_l^4 - 8\,\omega_{0M}^2\,\nu_l^2 + \omega_{0M}^4)}$$

$$\left.-\frac{\tfrac{4}{3}\,\omega_{\rm pl}^2\left\{\left(1 - \frac{1}{\beta^2}\right)(16\,\nu_l^4 - 4\,\omega_{0m}^2\,\nu_l^2) + 64\,\omega_{\rm pl}^2\,\nu_l^2\left(1 - \frac{1}{2\beta^2}\right)\right\}}{(32\,\omega_{\rm pl}^2\,\nu_l^2 + 16\,\nu_l^4 - 8\,\omega_{0m}^2\,\nu_l^2 + \omega_{0m}^4)}\right], \tag{33}$$

where $\omega_{0M} = \omega_0\,|_{\text{maximum}}$ and $\omega_{0m} = \omega_0\,|_{\text{minimum}}$ as a consequence of the limits in x. The terms in the second square bracket are the additional ones due to the consideration of temperature. Expression (33) would have got modified in the absence of the $\lambda - \sigma$ coupling. From (33) we can define a modified temperature as

$$T_m = T_{\text{modified}} = T\left[\frac{(\omega_{0M}^4 - \omega_{0m}^4)}{120\,\nu_l^2\,\omega_{\rm pl}^2}\left(1 - \frac{1}{\beta^2}\right) + \frac{4}{5}\ln\left(\frac{4\,\nu_l^2 - \omega_{0M}^2}{4\,\nu_l^2 - \omega_{0m}^2}\right)\right.$$

$$+\frac{\frac{4}{3}\left\{\left(1 - \frac{1}{\beta^2}\right)(16\,\nu_l^4 - 4\,\omega_{0M}^2\,\nu_l^2) + 64\,\omega_{\rm pl}^2\,\nu_l^2\left(1 - \frac{1}{2\beta^2}\right)\right\}}{(32\,\omega_{\rm pl}^2\,\nu_l^2 + 16\,\nu_l^4 - 8\,\omega_{0M}^2\,\nu_l^2 + \omega_{0M}^4)}$$

$$-\frac{\frac{4}{3}\left\{\left(1 - \frac{1}{\beta^2}\right)(16\,\nu_l^4 - 8\,\omega_{0m}^2\,\nu_l^2) + 64\,\omega_{\rm pl}^2\,\nu_l^2\left(1 - \frac{1}{2\beta^2}\right)\right\}}{(32\,\omega_{\rm pl}^2\,\nu_l^2 + 16\,\nu_l^4 - 8\,\omega_{0m}^2\,\nu_l^2 + \omega_{0m}^4)}. \tag{34}$$

Thus we can write (33) as

$$\frac{\mathrm{d}E}{\mathrm{d}l} = -\frac{2\pi e_p^2}{3c^2}\left[\frac{\omega_{0M} - \omega_{0m}^2}{2}\left(1 - \frac{1}{\beta^2}\right) + \frac{4\,\omega_{\rm pl}^2\,\nu_l^2}{\beta^2\,(\nu_l^4 + 8\,\omega_{\rm pl}^2\,\nu_l^2)^{1/2}}\right.$$

$$\left.\cdot\ln\left\{1 - \frac{16(\omega_{0M}^2 - \omega_{0m})\,(\nu_l^4 + 8\omega_{\rm pl}^2\,\nu_l^2)^{1/2}}{(4\nu_l^2 - 2\omega_{0M}^2 + 4(\nu_l^4 + 8\omega_{\rm pl}^2\nu_l^2)^{1/2})\,(4\nu_l^2 - 2\omega_{0m}^2 - 4(\nu_l^4 + 8\omega_{\rm pl}^2\nu_l^2)^{1/2})}\right\}\right]$$

$$-\frac{2\pi e_p^2}{3c^2}\,\omega_{\rm pl}^2\,\frac{K_B\,T_m}{m_l\,c^2} \tag{35}$$

In this analysis the effective frequency which is the frequency of the dressed particles of the medium and its temperature dependence (in the linear approximation in T) is defined as

$$\omega_{\text{eff}}^2 = \omega^2(T) = 2(\nu_\lambda^2 + \nu_l^2 - \chi)$$

$$= \omega_0^2 + \frac{8}{3}\frac{\omega_{\text{pl}}^2 \nu_\lambda^2}{\chi_0}\frac{K_B T}{m_l c^2} > \omega_0^2, \quad (36)$$

where ω_0^2 is obtained by setting χ to χ_0 *i.e.* without temperature dependence.

The effective index of the system is now defined as

$$N^2(\omega_0 T) = (4\nu_\lambda^2/\omega_{\text{eff}}^2). \quad (37)$$

or

$$\frac{1}{N^2(\omega_0, T)} \cong \frac{1}{N^2(\omega_0, 0)} + \frac{2}{3}\frac{\omega_{\text{pl}}^2}{\chi_0}\frac{K_B T}{m_l c^2}, \quad (38)$$

where

$$N^2(\omega_0, 0) = 1 + \frac{8\omega_{\text{pl}}^2}{\omega_0^2} + \frac{8\omega_{\text{pl}}^2}{(4\nu_l^2 - \omega_0^2)}. \quad (39)$$

Equation (39) is the same as has been obtained by Pratap (1967) by taking χ_0 instead of χ in the definition of ω_{eff} and the refractive index. Using (36) with χ_0 for χ and remembering the definition

$$\chi_0 = [(\nu_\lambda^2 - \nu_l^2)^2 + 8\omega_{\text{pl}}^2 \nu_l^2]^{1/2}, \quad (40)$$

we can eliminate ν_λ^2 and obtain χ_0 as a function of ω_0 as

$$\chi_0 = \frac{32\omega_{\text{pl}}^2 \nu_l^2 + 16\nu_l^4 - 8\omega_0^2 \nu_l^2 + \omega_0^4}{4(4\nu_l^2 - \omega_0^2)}. \quad (41)$$

Substituting this value of χ_0 in (38), the inverse of the square of the refractive index becomes

$$\frac{1}{N^2(\omega_0, T)} \cong \frac{1}{N^2(\omega_0, 0)} + \frac{8}{3}\frac{\omega_{\text{pl}}^2(4\nu_l^2 - \omega_0^2)}{(32\omega_{\text{pl}}^2 \nu_l^2 + 16\nu_l^4 - 8\omega_0^2 \nu_l^2 + \omega_0^4)}\frac{K_B T}{m_l c^2}. \quad (42)$$

The refractive index given by (39) consists of 3 parts; the first term is that of vacuum, the second is due to the photon gas and the third term is due to the medium of oscillators. The modification of the refractive index due to the thermal state of the system is obtained in (42) which implies that the increase in T reduces the refractive index of the system. Decrease in the refractive index of the system corresponds to absorption of the radiation by the medium. It may however be noted that if the effective frequency resonates with twice the oscillator frequency (*i.e.* $\omega_0 = 2\nu_l$), we do not have any temperature effect on the Cerenkov threshold. A numerical calculation for (33) shows a gradual decrease with the rise in temperature. This absorption is in the lower frequency side. Now the coherence condition for the radiation to come out is obtained from (26) and (27) because of the δ-function as

$$\cos\theta = \left[\frac{\nu_\lambda^2 + \nu_l^2 - \chi}{2\nu_\lambda^2 \beta^2}\right]^{1/2}, \tag{43a}$$

or

$$\cos^2\theta = \frac{\nu_\lambda^2 + \nu_l^2 - \chi}{2\nu_\lambda^2 \beta^2} = \frac{\omega_{\text{eff}}^2}{4\nu_\lambda^2 \beta^2} = \frac{1}{N^2(\omega_0, T)\beta^2}. \tag{43b}$$

Equations (42) and (43) imply that as T increases $\cos\theta$ increases and thus θ decreases. Thus the medium temperature decreases the coherence. From (43) the threshold velocity for the incoming particle is given by

$$\beta_{\min} = \frac{1}{N^2(\omega_0, T)}, \tag{44}$$

which again is increased with the increase in T.

4. Conclusion

In this microscopic theory we have considered the effect of the longitudinal mode on the transverse interaction and studied the role of the medium temperature in the energy output of the Cerenkov radiation. As seen in Heitler (1954), in Lorentz gauge the σ- part of the vector potential corresponds to the polarization dependent part of the Coulomb interaction which is responsible for the density fluctuations in the medium. Since the Cerenkov radiation is due to the polarization of the medium particles, we consider only the effect of σ-mode but not of 0 σ-mode which is the scalar part of the Coulomb interaction. Thus the inclusion of σ-mode will correspond to the coupling of the density fluctuation in the medium particles with the excitation of the transverse collective interaction leading to the Cerenkov radiation. The inclusion of λ-σ coupling appears for the first time in the 6th-order diagram. This leads to the conservation of wave vectors in different modes *i.e.* $\mathbf{k}_\lambda = \mathbf{k}_\sigma$. It is seen that the inclusion of λ-σ coupling in the λ-interaction changes the test particle distribution function as in (7) as compared to the earlier theory.

The expressions (23) and (31) as temperature $T \to 0$ can be written respectively as

$$\nu_\lambda^2 \geqslant 2\,\omega_{\text{pl}}^2,$$

$$\chi(T=0) = \chi_0 = [(\nu_\lambda^2 - \nu_l^2)^2 + 8\,\omega_{\text{pl}}^2\,\nu_l^2]^{1/2}, \tag{45}$$

which in the absence of λ-σ coupling takes the form

$$\nu_\lambda^2 \geqslant \frac{4}{3}\omega_{\text{pl}}^2,$$

and $$\chi_0 = \left[(\nu_\lambda^2 - \nu_l^2)^2 + \frac{16}{3}\omega_{\rm pl}^2\,\nu_l^2\right]^{1/2}. \tag{46}$$

Thus the inclusion of λ-σ coupling in the evaluation of $\mathrm{d}E/\mathrm{d}l$ appears through the factor χ. In (28) which is

$$\frac{\mathrm{d}E}{\mathrm{d}l} = -\frac{4\pi e_p^2}{3c^2}\int \mathrm{d}\nu_\lambda\,\nu_\lambda\,\phi_2\,(-2A)\left(1 - \frac{\nu_\lambda^2 - \nu_l^2}{\chi}\right)\left(1 - \frac{\nu_\lambda^2 + \nu_l^2 - \chi}{2\,\nu_\lambda^2\,\beta^2}\right),$$

it is seen that the increase in χ (comparing (45) and (46)) because of the inclusion of λ-σ coupling, increases the $\mathrm{d}E/\mathrm{d}l$ *i.e.* the energy loss by the test particle per unit path length.

It is also observed (from (30)) for a given ν_λ the $\mathrm{d}E/\mathrm{d}l$ decreases with increase in the temperature of the medium.

Considering the above changes in the test particle distribution function, we have evaluated the energy radiated out from the system (per unit path length of the particle) in this Cerenkov mode, viz. $(\mathrm{d}E/\mathrm{d}l)_\lambda$, $(\mathrm{d}E/\mathrm{d}l)_{\lambda-\sigma}$ and evaluated the percentage ratio

$$\left|\frac{\left(\frac{\mathrm{d}E}{\mathrm{d}l}\right)^{T} - \left(\frac{\mathrm{d}E}{\mathrm{d}l}\right)^{T\to 0}}{\left(\frac{\mathrm{d}E}{\mathrm{d}l}\right)^{T\to 0}}\right|_{\lambda-\sigma} \simeq 7\cdot\frac{K_B\,T}{m_i\,c^2}, \tag{47}$$

which is very small for temperatures up to the order of 10^7 °K. However the ratio

$$\frac{\left(\frac{\mathrm{d}E}{\mathrm{d}l}\right)_{\lambda-\sigma} - \left(\frac{\mathrm{d}E}{\mathrm{d}l}\right)_{\lambda}}{\left(\frac{\mathrm{d}E}{\mathrm{d}l}\right)_{\lambda}} \simeq 60\% \tag{48}$$

Hence the inclusion of the density fluctuation in the evaluation of the Cerenkov radiation becomes significant as we include the transfer of energy from the longitudinal mode to the transverse mode. However, the effect of temperature becomes significant only at high temperatures ($\sim 10^7$ °K). We have also shown that the inclusion of the σ-mode into the radiation problem modifies the effective collective frequency and hence we assert that a new coherence state exists in the medium. This is a new result and has come out essentially because the system is non-equilibrium.

The increase in $(\mathrm{d}E/\mathrm{d}l)$ because of the inclusion of σ-mode in the pure transverse interaction is obvious from the fact that whenever the relativistic charged particle enters the medium it excites both the collective transverse and longitudinal modes. Due to the coupling of the longitudinal mode with the transverse mode, the energy will be transferred from σ-mode to the λ-mode which should ultimately be coming from the moving particle exciting them. This fact is also pointed out in Majumdar (1961).

The effect of the temperature can be seen in the following manner. For a reasonable set of parameters, T_m as defined in (34) becomes negative definite, which in turn makes the last term in (35) positive definite. Thus the energy loss is reduced and this implies that the energy coming out of the system is also reduced. We thus have the phenomenon of Cerenkov radiation which vanishes for a critical temperature and hence the collective modes disappear. This can also be interpreted physically following Bohm and Pines (1951, 1952) that the characteristic interaction length becomes smaller than the Debye length. It may however be noted that this critical temperature is very high and at this temperature our formulation breaks down, since one has to consider the whole system to be constituted by relativistic particles. Hence while the inclusion of the longitudinal mode in the evaluation of the distribution function increases the energy, the thermal state of the system would reduce it and becomes significant only at high temperatures. This is why the temperature effect is not observed in the usual laboratory systems. The temperature effect may however be a significant factor in a very high temperature system such as pulsar atmosphere or in a fusion reactor.

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Appendix

Corresponding to the 2nd-order diagram (Pratap 1967) the matrix element contributing towards the $f(\mathbf{u}_p, \mathbf{q}_p, t)$ can be shown to be

$$W_{pp}(t)\big|_{\pm\lambda\pm\lambda} = \frac{e_p^2}{m_p}\frac{16\pi^2}{v}\int_0^t dt_2 \frac{\partial}{\partial \mathbf{u}_p}\cdot \mathbf{e}_\lambda (\mathbf{e}_\lambda \cdot \beta)\cos\{\mathbf{k}_\lambda\cdot\beta c(t-t_2)\}.$$

$$\cdot \exp(-iL_0^p t_2) f(\mathbf{u}_p, \mathbf{q}_p; 0)\cdot\left[\oint_c dZ \frac{iZ}{(Z^2-4\pi^2\nu_\lambda^2)}\right.$$

$$\left.\cdot\exp(-iZ(t_2-t_1))\right]. \tag{A.1}$$

Now the matrix element corresponding to the 4th-order diagram is

$$W_{plp}(t)\big|_{\lambda\lambda'} = -\left[\frac{e_p^2}{m_p}\frac{16\pi^2}{v}\frac{\nu_\lambda}{(ck_\lambda ck_{\lambda'})^{1/2}}\right]\left[\frac{e_l^2}{m_l}\frac{16\pi}{v}\frac{\nu_l}{(ck_\lambda ck_{\lambda'})^{1/2}}(\mathbf{e}_\lambda\cdot\mathbf{a}_l)(\mathbf{e}_{\lambda'}\cdot\mathbf{a}_l)\right]$$

$$\cdot\int_0^t dt_2\int_{t_2}^t d\tau_1\int_{t_2}^{\tau_1} d\tau_2\, dJ_l\, d\alpha_l \frac{\partial}{\partial\mathbf{u}_p}\cdot\mathbf{e}_\lambda(\mathbf{e}_{\lambda'}\cdot\beta)\cos 2\pi\nu_\lambda(t-\tau_1).$$

$$\cdot \sin 2\pi\nu_{\lambda'}(\tau_2 - t_2) \cos \mathbf{k}_\lambda \cdot \mathbf{q}_p \exp\left\{- i\mathscr{L}_0^p (t - t_2)\right\} \cos \mathbf{k}_{\lambda'} \cdot \mathbf{q}_p$$

$$\cdot \exp(- i\mathscr{L}_0^p t_2) f(\mathbf{u}_p, \mathbf{q}_p; 0) \cdot \Big[\cos \mathbf{k}_{\lambda'} \mathbf{q}_l \cos 2\pi a_l$$

$$\cdot \exp\left\{- i\mathscr{L}_0^l (\tau_1 - \tau_2)\right\} \frac{\partial}{\partial a_l} \cos \mathbf{k}_{\lambda'} \cdot \mathbf{q}_l \cos 2\pi a_l \cdot J_l \frac{\mathrm{d}}{\mathrm{d}J_l} g(J_l)\Big], \qquad (A.2)$$

where J_l and a_l are the action and angle variables for the medium oscillators.

Considering the terms in the square bracket; after proper operations by the operator $\partial/\partial a_l$ and the time translation operators $\exp\left\{- i\mathscr{L}_0^l (\tau_1 - \tau_2)\right\}$ and then integrating over all orientations of $\mathbf{q}_{l0}$ where

$$\mathbf{q}_l = \mathbf{q}_{l0} + \mathbf{a}_l (2J_l/m_l \nu_l)^{1/2} \sin 2\pi a_l$$

leading to the conservation of the wave vectors $\mathbf{k}_\lambda = \mathbf{k}_{\lambda'}$ for non-zero contributions, the terms in [....] can be written in terms of the Bessel functions as

$$-\frac{\pi}{2}\Big[\left\{\sin 2\pi (2a_l - \nu_l \tau_{12}) - \sin 2\pi \nu_l \tau_{12}\right\} \Big\{ j_0 (c_l' \delta_l)$$

$$+ 2 \sum_{n=1}^{\infty} \left(j_{2n} (c_l' \delta_l) \cos \overline{4n\pi (a_l + \phi_l)}\right)\Big\} - 2c_l' \left\{\tfrac{1}{2} \cos 2\pi (3a_l - 2\nu_l \tau_{12})\right.$$

$$\left. + \cos 2\pi a_l + \tfrac{1}{2} \cos 2\pi (a_l - 2\nu_l \tau_{12})\right\} \Big\{\sum_{n=0}^{\infty} j_{2n+1} (c_l' \delta_l).$$

$$\sin 2\pi \overline{(2n + 1)(a_l + \phi_l)}\Big\}\Big] J_l \frac{\mathrm{d}}{\mathrm{d}J_l} g(J_l), \qquad (A.3)$$

where $\delta_l = 2 \sin \pi \nu_l \tau_{12}$,

$$\phi_l = \tfrac{1}{4} - \frac{\nu_l \tau_{12}}{2},$$

$$c_l' = \mathbf{k}_\lambda \cdot \mathbf{a}_l (2 J_l/m_l \nu_l)^{1/2},$$

$$\tau_{12} = \tau_1 - \tau_2. \qquad (A.4)$$

Then the a_l-integration with little algebra will lead to the following result

$$\frac{\pi}{2}\Big[j_0 (c_l' \delta_l) \sin 2\pi \nu_l \tau_{12} + \frac{c_l'}{2\delta_l} j_1 (c_l' \delta_l) \left\{\sin 2\pi \nu_l \tau_{12}\right.$$

$$\left. + \sin 4\pi \nu_l \tau_{12}\right\} - \frac{c_l'}{2\delta_l} j_3 (c_l' \delta_l) \sin 2\pi \nu_l \tau_{12}\Big] J_l \frac{\mathrm{d}}{\mathrm{d}J_l} g(J_l). \qquad (A.5)$$

Now with the explicit form of the initial canonical distribution for the medium particles (Pratap 1967) the J_l — integration of (A.5) will give

$$-\frac{\pi}{2}\,[(1-3\gamma_l^2)\sin 2\pi\,\nu_l\,\tau_{12}+3\gamma_l\sin 4\pi\,\nu_l\,\tau_{12}$$

$$+\,\gamma_l^2\sin 6\pi\,\nu_l\,\tau_{12}]\,\exp\,(-\,2\gamma_l+2\gamma_l\cos 2\pi\,\nu_l\,\tau_{12}), \qquad \text{(A.6)}$$

with
$$2\gamma_l=2\left(\frac{\mathbf{k}_\lambda\cdot\mathbf{a}_l}{\nu_l}\right)^2\frac{K_B\,T}{2m_l}=\left(\frac{\mathbf{k}_\lambda\cdot\mathbf{a}_l}{\nu_l}\right)^2\frac{K_B\,T}{m_l}. \qquad \text{(A.7)}$$

Using the general expression

$$\exp\,(2\gamma_l\cos 2\pi\,\nu_l\,\tau_{12})=\left\{I_0\,(2\gamma_l)+2\sum_{n=1}^{\infty}I_n\,(2\gamma_l)\cos 2\pi\,n\,\nu_l\,\tau_{12}\right\} \qquad \text{(A.8)}$$

where $I_n\,(x)$ is the Bessel function of imaginary argument. (A.6) can be written as

$$-\frac{\pi}{2}\Big[\,I_0\,(2\gamma_l)\,\{(1-3\gamma_l^2)\sin 2\pi\,\nu_l\,\tau_{12}+3\lambda_l\sin 4\pi\,\nu_l\,\tau_{12}$$

$$+\,\lambda_l^2\sin 6\pi\,\nu_l\,\tau_{12}\}+\sum_{n=1}^{\infty}I_n\,(2\gamma_l)\,\{(1-3\gamma_l^2)\,(\sin 2\pi\,(n+1)\,\nu_l\,\tau_{12}$$

$$-\,\sin 2\pi\,(n-1)\,\nu_l\,\tau_{12})+3\gamma_l\,(\sin 2\pi\,(n+2)\,\nu_l\,\tau_{12}$$

$$-\,\sin 2\pi\,(n-2)\,\nu_l\,\tau_{12})+\gamma_l^2\,(\sin 2\pi\,(n+3)\,\nu_l\,\tau_{12}$$

$$-\,\sin 2\pi\,(n-3)\,\nu_l\,\tau_{12})\}\Big]\exp\,(-\,2\gamma_l). \qquad \text{(A.9)}$$

Therefore

$$W^{(t)}_{plIp}(t)\Big|_{\lambda\lambda}=\left(\frac{e_p^2}{m_p}\,\frac{16\pi^2}{V}\right)\left(\frac{e_l^2}{m_l}\,\frac{8\pi^2}{V}\,\frac{\nu_l}{\nu_\lambda}\,(\mathbf{e}_\lambda\cdot\mathbf{a}_l)^2\right)\int_0^t \mathrm{d}t_2\int_{t_2}^{t}\mathrm{d}\tau_1\int_{t_2}^{\tau_1}\mathrm{d}\tau_2$$

$$\cdot\left\{\frac{\partial}{\partial\mathbf{u}_p}\cdot\mathbf{e}_\lambda\,(\mathbf{e}_\lambda\cdot\boldsymbol{\beta})\cos\mathbf{k}_\lambda\cdot\mathbf{q}_p\cos\,\{\mathbf{k}_\lambda\cdot(\mathbf{q}_p-\boldsymbol{\beta}\,c\,(t-t_2))\}\right.$$

$$\cdot\exp\,(-\,i\mathscr{L}_0^p\,t_2)\,f\,(\mathbf{u}_p\,\mathbf{q}_p;\,0)\;[\;]\;\exp\,(-\,2\gamma_l)$$

$$\left.\cdot\cos 2\pi\,\nu_\lambda\,(t-\tau_1)\sin 2\pi\,\nu_\lambda\,(\tau_2-t_2)\right\} \qquad \text{(A.10a)}$$

where the terms in [ ] are the terms within [ ] of (A.9). Then by using the convolution theorem in Laplace transformation and including (— λ) mode (A.10a) can be written as

$$W^{(t)}_{plip}(t)\Big|_{\pm\lambda\pm\lambda} = \frac{e_p^2}{m_p}\frac{16\,\pi^2}{V}\int_0^t dt_2 \frac{\partial}{\partial \mathbf{u}_p}\cdot \mathbf{e}_\lambda\,(\mathbf{e}_\lambda\cdot\boldsymbol{\beta})\cos\{\mathbf{k}_\lambda\cdot\boldsymbol{\beta}\,c\,(t-t_2)\}$$

$$\cdot\exp(-\,i\mathscr{L}_0^p t_2)\,f(\mathbf{u}_p, \mathbf{q}_p;\,0)\Big[\oint_C dZ\frac{iZ}{Z^2-4\pi^2\nu_\lambda^2}$$

$$\cdot\,\epsilon_l^\lambda(Z)\exp\{-\,iZ\,(t-t_2)\}\Big]. \qquad \text{(A.10)}$$

Similarly it can be shown that for sixth order diagrams the important contributions for the phenomenon concerned are.

$$W_{pllllp}(t)\Big|_{\pm\lambda\pm\lambda\pm\lambda} = \frac{e_p^2}{m_p}\frac{16\pi^2}{V}\int_0^t dt_2 \frac{\partial}{\partial \mathbf{u}_p}\cdot \mathbf{e}_\lambda\,(\mathbf{e}_\lambda\cdot\boldsymbol{\beta})\cos\{\mathbf{k}_\lambda\cdot\boldsymbol{\beta}\,c\,(t-t_2)\}$$

$$\cdot\exp(-\,i\mathscr{L}_0^p t_2)\,f(\mathbf{u}_p, \mathbf{q}_p;\,0)\Big[\oint_C dZ\frac{iZ}{Z^2-4\pi^2\nu_\lambda^2}$$

$$\cdot\,\epsilon_{ll}^{\lambda\lambda}(Z)\exp\{-\,iZ\,(t-t_2)\}\Big], \qquad \text{(A.11)}$$

and

$$W_{plllp}(t)\Big|_{\pm\lambda\pm\sigma\pm\lambda} = \frac{e_p^2}{m_p}\frac{16\pi^2}{V}\int_0^t dt_2 \frac{\partial}{\partial \mathbf{u}_p}\cdot \mathbf{e}_\lambda\,(\mathbf{e}_\lambda\cdot\boldsymbol{\beta})\cos\{\mathbf{k}_\lambda\cdot\boldsymbol{\beta}\,c\,(t-t_2)\}$$

$$\cdot\exp(-\,i\mathscr{L}_0^p t_2)\,f(\mathbf{u}_p, \mathbf{q}_p;\,0)\Big[\oint_C dZ\frac{iZ}{Z^2-4\pi^2\nu_\lambda^2}$$

$$\cdot\,\epsilon_{ll}^{\lambda\sigma}(Z)\exp\{-\,iZ\,(t-t_2)\}\Big]. \qquad \text{(A.12)}$$

In getting (A.12) one comes across the conservation of wave vectors in different modes, *i. e.*

$$\mathbf{k}_\lambda = \mathbf{k}_\sigma, \qquad \text{(A.13)}$$

Here $\epsilon_{ll}^{\lambda\lambda}(Z) = (\epsilon_l^\lambda(Z))^2,$

$$\epsilon_{ll}^{\lambda\sigma}(Z) = \epsilon_l^\lambda(Z)\,\epsilon_l^\sigma(Z), \qquad \text{(A.14)}$$

where

$$\epsilon_l^\lambda(Z) = \frac{e_l^2}{m_l}\frac{32\pi^4}{V}(\mathbf{e}_\lambda \cdot \mathbf{a}_l)^2 \frac{\nu_l^2}{(Z^2 - 4\pi^2\nu_\lambda^2)}\Bigg[I_0(2\gamma_l)\Bigg\{\frac{1-3\gamma_l^2}{Z^2 - 4\pi^2\nu_l^2}$$

$$+ \frac{6\gamma_l}{Z^2 - 16\pi^2\nu_l^2} + \frac{3\gamma_l^2}{Z^2 - 36\pi^2\nu_l^2}\Bigg\} + \sum_{n=1}^{\infty} I_n(2\gamma_l)\Bigg\{(1-3\gamma_l^2)$$

$$\cdot\left(\frac{n+1}{Z^2 - 4\pi^2(n+1)^2\nu_l^2} - \frac{n-1}{Z^2 - 4\pi^2(n-1)^2\nu_l^2}\right)$$

$$+ 3\gamma_l\left(\frac{n+2}{Z^2 - 4\pi^2(n+2)^2\nu_l^2} - \frac{n-2}{Z^2 - 4\pi^2(n-2)^2\nu_l^2}\right)$$

$$+ \gamma_l^2\left(\frac{n+3}{Z^2 - 4\pi^2(n+3)^2\nu_l^2} - \frac{n-3}{Z^2 - 4\pi^2(n-3)^2\nu_l^2}\right)\Bigg\}\Bigg]$$

$$\cdot \exp(-2\gamma_l). \tag{A.15}$$

The expression for $\epsilon_l^\sigma(Z)$ is same as (A.15) except for λ replaced by σ (with keeping in mind that $\nu_\lambda = \nu_\sigma$ because of (A.13)). Hence

$$2\gamma_l = \left(\frac{\mathbf{k}_\lambda \cdot \mathbf{a}_l}{\nu_l}\right)^2 \frac{K_B T}{m_l} = (\mathbf{e}_\sigma \cdot \mathbf{a}_l)^2 A(T), \tag{A.16}$$

where $$A(T) = \frac{\nu_\lambda^2}{\nu_l^2}\frac{K_B T}{m_l c^2}. \tag{A.17}$$

As such the expression for $\epsilon_l^\lambda(Z)$ looks very complicated. But the assumption of the non-relativistic nature of the medium particles allows one to take an approximation

$$2\gamma_l \ll 1. \tag{A.18}$$

which will be a good approximation for considering the terms linear in T. Because of this approximation, there will be a cut-off in the upper limit for ν_λ. Under such an approximation

$$\epsilon_l^\lambda(Z) = \frac{e_l^2}{m_l}\frac{32\pi^4}{V}(\mathbf{e}_\lambda \cdot \mathbf{a}_l)^2 \frac{\nu_l^2 I_0(2\gamma_l)\exp(-2\gamma_l)}{(Z^2 - 4\pi^2\nu_\lambda^2)(Z^2 - 4\pi^2\nu_l^2)}. \tag{A.19}$$

With $\mathbf{a}_l$, an unit vector giving the direction of the l-th oscillator; when we sum (A.19) over all oscillators, we get a factor N_l, the number of oscillators, then with $N_l/V = C_l$

$$\epsilon_l^\lambda(Z) = \frac{8\pi^3\omega_{\text{pl}}^2(\mathbf{e}_\lambda \cdot \mathbf{a}_l)^2\nu_l^2 I_0(2\gamma_l)\exp(-2\gamma_l)}{(Z^2 - 4\pi^2\nu_\lambda^2)(Z^2 - 4\pi^2\nu_l^2)} \tag{A.20}$$

where $$\omega_{\rm pl}^2 = \frac{4\pi\, e_l^2\, c_l}{m_l} \tag{A.21}$$

is the plasma frequency associated with the medium oscillators.

After summing over polarization vectors, the integration over all possible orientations reduces (A.20) to the form

$$\epsilon_l^{\lambda}(Z) = \frac{64\pi^4\, \omega_{\rm pl}^2\, \nu_l^2}{3} \cdot \frac{{}_2F_2(\frac{1}{2}, \frac{1}{2}; 1, \frac{5}{2}; -2A)}{(Z^2 - 4\pi^2 \nu_\lambda^2)(Z^2 - 4\pi^2 \nu_l^2)} \tag{A.22}$$

and similarly

$$\epsilon_l^{\sigma}(Z) = \frac{32\pi^4\, \omega_{\rm pl}^2\, \nu_l^2}{3} \, \frac{{}_2F_2(\frac{1}{2}, \frac{3}{2}; 1, \frac{5}{2}; -2A)}{(Z^2 - 4\pi^2 \nu_\sigma^2)(Z^2 - 4\pi^2 \nu_l^2)} \tag{A.23}$$

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