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Uniqueness of the solution of the Lippmann-Schwinger equation

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Abstract. A few consequences of the uniqueness of the scattering wavefunction in non-break up channel are discussed.

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1. Introduction

Following the work of Lippmann (1956) showing the non-uniqueness of the solution $|\Psi_\alpha^+(E_\alpha)\rangle$ in α-channel of Lippmann Schwinger equation in the presence of rearrangement channels, Gerjuoy (1958) further showed from an independent consideration that $|\Psi_\alpha^+(E_\alpha)\rangle$ for real energy E_α is non-unique anyhow, unless it satisfied a stringent condition of being 'everywhere outgoing'. However, people who computed numbers to 'explain' various experimentally observed nuclear scattering and reaction cross-sections, polarizations etc. had completely ignored this theoretical (and logical) problem of non-uniqueness and treated $|\Psi_\alpha^+(E_\alpha)\rangle$ as unique in their computations, and suggested various working models to compute $|\Psi_\alpha^+(E_\alpha^+)\rangle$ to suit different purposes. It is not clear, however, if their grand list of successes does mean anything to those who worry about the theoretical non-uniqueness of $|\Psi_\alpha^+(E_\alpha)\rangle$. Nevertheless, logical considerations alone are enough to demand a discussion on the said uniqueness problem even if it is purely academic.

We have recently proved (Mukherjee 1981a, b, c) that the earlier conclusions of uniqueness are wrong by showing that the Lippmann's identity is invalid and the scattered part of $|\Psi_\alpha^+(E_\alpha)\rangle$ is automatically outgoing everywhere. In this paper we show a few other consequences of these results involving the equation for components of the Faddeev wavefunctions, BRS-equation, etc.

2. Theory

The scattering state $|\Psi_\alpha^+(E_\alpha)\rangle$ at real energy E_α is given (Mukherjee 1981a) by the following equations ($Z_1 = E_\alpha + i\,\epsilon_1$; $\epsilon_1 > 0$)

$$|\Psi_\alpha^+(E_\alpha)\rangle = \operatorname*{Lt}_{\epsilon_1\to 0} |\Psi_\alpha(Z_1)\rangle, \tag{1}$$

$$(E_a - H)\,|\Psi_a^+(E_a)\rangle = 0, \tag{2}$$

$$|\Psi_a(Z_1)\rangle = i\,\epsilon_1\, G(Z_1)\,|\phi_a(E_a)\rangle, \tag{3}$$

$$= |\phi_a(E_a)\rangle + |\chi_a\;(Z_1)\rangle, \tag{4}$$

$$= |\phi_a(E_a)\rangle + G_a(Z_1)\,\bar{V}_a\,|\Psi_a(Z_1)\rangle + \mathcal{J}\,(G_a(Z_1),\,|\chi_a(Z_1)\rangle), \tag{5}$$

$$= |\phi_a(E_a)\rangle + G(Z_1)\,\bar{V}_a\,|\phi_a(E_a)\rangle + \mathcal{J}\,(G(Z_1),|\chi_a(Z_1)\rangle), \tag{6}$$

where H, H_0 and V are the total Hamiltonian, kinetic and potential energy operators respectively; $H = H_0 + V = H_a + \bar{V}_a$; $V = V_a + \bar{V}_a$; $H_a = H_0 + V_a$; $G_a(Z) = (Z - H_a)^{-1}$; $G(Z) = (Z - H)^{-1}$; $|\phi_a(E_a)\rangle$ is the eigenfunction of the channel Hamiltonian H_a in a-channel. $H_a|\phi_a(E_a)\rangle = E_a|\phi_a(E_a)\rangle$. The Wronskian between Green's function and a state vector $|\eta\rangle$ is defined as

$$\mathcal{J}\,(G_a(Z),|\eta\rangle) = G_a(Z)\,\overrightarrow{H_0}\,|\eta\rangle - G_a(Z)\,\overleftarrow{H_0}\,|\eta\rangle, \tag{7}$$

where the arrow on $\overrightarrow{H_0}$ indicates that H_0 acts on the function next to it in that direction. Equations (5) and (6) are established directly by evaluating their Wronskians, using (7), (2), (3) and (4).

The Green's function $G(Z_1)$ is uniquely defined everywhere in the complex energy plane and singular on real axis at points comprising the spectrum of H. The square-root cut on the positive real axis gives the meaningful limiting values of $G(Z_1)$, for $\epsilon_1 \to O^+$(or O^-) having outgoing (or incoming) waves, and this is merely reflected in (1) which thus gives, *via* (3), the scattering state $|\Psi_a^+(E_a)\rangle$ with outgoing wave boundary condition, at real (continuum) energy E_a, as a boundary value of the state $|\Psi_a(Z_1)\rangle$ for complex energy Z_1; $Z_1 = E_a + i\epsilon_1$; $\epsilon_1 > 0$. It was shown recently (Mukherjee 1981a, b) that both the Wronskians of (5) and (6) vanish identically for all complex Z_1 *i.e.*

$$\mathcal{J}\,(G_a(Z_1),\;|\chi_a(Z_1)\rangle) = 0, \tag{8}$$

$$\mathcal{J}\,(G(Z_1),\;|\chi_a(Z_1)\rangle) = 0, \tag{9}$$

and these in turn satisfy automatically the Gerjuoy's (1958) requirement that the scattered part of $|\Psi_a^+(E_a)\rangle$ is outgoing everywhere and proves the uniqueness of $|\Psi_a^+(E_a)\rangle$. Equations (5), (6), (8) and (9) give the Lippmann-Schwinger (LS) equation and its Chew-Goldberger (CG) solution at the complex energy $Z_1 = E_a + i\epsilon_1$ given by

$$|\Psi_a(Z_1)\rangle = |\phi_a(E_a)\rangle + G_a(Z_1)\bar{V}_a\,|\Psi_a(Z_1)\rangle, \tag{10}$$

$$|\Psi_a(Z_1)\rangle = |\phi_a(E_a)\rangle + G(Z_1)\bar{V}_a\,|\phi_a(E_a)\rangle, \tag{11}$$

respectively. The LS-equation for $|\Psi_\alpha^+(E_\alpha)\rangle$ for real energy E_α and its CG-solution are usually regarded as those given by (10) and (11) with

$|\Psi_\alpha(Z_1)\rangle$, $G_\alpha(Z_1)$ and $G(Z_1)$ replaced by

$|\Psi_\alpha^+(E_\alpha)\rangle$, $G_\alpha^+(E_\alpha) = \underset{\epsilon_1\to 0}{\text{Lt}}\, G_\alpha(Z_1)$ and $G^+(E_\alpha) = \underset{\epsilon_1\to 0}{\text{Lt}}\, G(Z_1)$

respectively and written as

$$|\Psi_\alpha^+(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha^+(E_\alpha)\,\overline{V}_\alpha\,|\Psi_\alpha^+(E_\alpha)\rangle, \tag{12}$$

$$|\Psi_\alpha^+(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle + G^+(E_\alpha)\,\overline{V}_\alpha\,|\phi_\alpha(E_\alpha)\rangle. \tag{13}$$

The LS-equation (12) for real energy however assumes a stringent requirement that the limit of product of operators is equal to the product of their limits *i.e.*

$$\underset{\epsilon_1\to 0}{\text{Lt}}\; G_\alpha(Z_1)\overline{V}_\alpha\,|\Psi_\alpha(Z_1)\rangle = [\underset{\epsilon_1\to 0}{\text{Lt}}\; G_\alpha(Z_1)].\,\overline{V}_\alpha\cdot\,[\underset{\epsilon_1\to 0}{\text{Lt}}\;|\Psi_\alpha(Z_1)\rangle] \tag{14}$$

$$= G_\alpha^+(E_\alpha)\,.\,\overline{V}_\alpha\,.\,|\Psi_\alpha^+(E_\alpha)\rangle. \tag{15}$$

It is not clear from the literature on scattering in what sense of limit (strong, weak or otherwise) or on what topology, if any, equation (14) actually holds. Apparently for two-body problem, (12) can be directly established (Simon 1971) to be true but the corresponding proof for multichannel case is still lacking. Nevertheless, one treats (12) as valid and derivable from (10) and thus implicitly assumes (14) as valid and this makes the solution of (12) as the limiting value $\epsilon_1 \to 0$ of (10). Recently it was shown (Mukherjee 1981a, b, c) that the solution of the LS-equation (10) is unique and its CG-solution is given by (11) for all Z_1, including those for $\epsilon_1 \to 0$. Expanding $G(Z_1)$ in terms of $G_\beta(Z_1)$ of another channel β we get ($\beta \neq \alpha$; $\beta \neq 0$; $\alpha \neq 0$):

$$|\Psi_\alpha(Z_1)\rangle = |A_{\beta\alpha}(\epsilon_1)\rangle + G_\beta(Z_1)\overline{V}_\beta\,|\Psi_\alpha(Z_1)\rangle, \tag{16}$$

$$|A_{\beta\alpha}(Z_1)\rangle = i\epsilon_1\,G_\beta(Z_1)\,|\phi_\alpha(E_\alpha)\rangle. \tag{17}$$

It was shown (Mukherjee 1981c) that

$$|A_{\beta\alpha}(0)\rangle = \underset{\epsilon_1\to 0}{\text{Lt}}\,|A_{\beta\alpha}(\epsilon_1)\rangle = \mathscr{P}_\beta(E_\alpha)\,|\phi_\alpha(E_\alpha)\rangle, \tag{18}$$

where $\mathscr{P}_\beta(E_\alpha)$ is a (non-zero) projection operator and this may be contrasted with Lippman's identity which reads as

$$|A_{\beta\alpha}(0)\rangle = \delta_{\beta\alpha}\,|\phi_\alpha(E_\alpha)\rangle, \tag{19}$$

Due to equation (18) the inhomogeneous term of (16) now survives and this thus avoids the homogeneous equation for $|\Psi_\alpha^+(E_\alpha)\rangle$ which results from the use of (19), thereby removing the very source of the nonuniqueness of $|\Psi_\alpha^+(E_\alpha)\rangle$. Equations (8), (9) and (18) thus proves that $|\Psi_\alpha^+(E_\alpha)\rangle$ is unique, contrary to the popular claim (Sandhas 1976) and dispensing with both the categories of arguments in favour of non-unique $|\Psi_\alpha^+(E_\alpha)\rangle$.

Equations like (18) have other consequences, as seen below. For simplicity we take a three-body system, with the initial state $|\phi_\alpha(E_\alpha)\rangle$ containing a bound pair (α) with the third particle free. Since the bound-state wave function satisfies a homogeneous equation we get

$$|A_{0\alpha}(0)\rangle = |\phi_\alpha(E_\alpha)\rangle - G_0^+(E_\alpha)\, V_\alpha\, |\phi_\alpha(E_\alpha)\rangle = 0. \tag{20}$$

Then from (20), (18) and (16) we get the following homogeneous equation for $|\Psi_\alpha^+(E_\alpha)\rangle$:

$$|\Psi_\alpha^+(E_\alpha)\rangle = G_0^+(E_\alpha)\, V\, |\Psi_\alpha^+(E_\alpha)\rangle. \tag{21}$$

This, however, does not lead to the nonuniqueness of $|\Psi_\alpha^+(E_\alpha)\rangle$ ($\alpha \neq 0$) as the Kernels of (12) and (21) are different, but it does make the break-up state $|\Psi_0^+(E_\alpha)\rangle$ non-unique. We write the break-up state $|\Psi_0(Z_1)\rangle$ as

$$|\Psi_0(Z_1)\rangle = i\,\epsilon_1\, G(Z_1)\, |\phi_0(E_\alpha)\rangle, \tag{22}$$

$$= |\phi_0(E_\alpha)\rangle + G_0(Z_1)\, V\, |\Psi_0(Z_1)\rangle, \tag{23}$$

$$= |A_{\alpha 0}(\epsilon_1)\rangle + G_\alpha(Z_1)\, \overline{V}_\alpha\, |\Psi_0(Z_1)\rangle \tag{24}$$

where $|\phi_0(E_\alpha)\rangle$ has all the three particles free. It can be shown (Sandhas 1976) that

$$|A_{\alpha 0}(0)\rangle = |\chi_\alpha^+(\vec{k}_{12}, \vec{K}_3)\rangle, \tag{25}$$

where $|\chi_\alpha^+(\vec{k}_{12})\rangle$ is the two-particle scattering state of the pair (α) with the third particle ($\vec{k}_3$) free. Equations (21) and (23) show that corresponding to the Kernel $G_0^+(E_\alpha)\, V$ there is a homogeneous as well as an inhomogeneous equation, implying that the solution $|\Psi_0^+(E_\alpha)\rangle$ of (23) ($\epsilon_1 \to 0$) is non-unique. But even then the Kernel $G_\alpha^+(E_\alpha)\, \overline{V}_\alpha$ does not anymore admit of a homogeneous equation for $|A_{\alpha 0}(0)\rangle \neq 0$, as shown by (25) and also (18). We thus conclude that the channel state $|\Psi_\alpha^+(E_\alpha)\rangle$ for $\alpha \neq 0$ is unique although the break-up state $|\Psi_0^+(E_\alpha)\rangle$ is not. But that does not come in the way of uniquely computing the break-up cross-sections for one can use the 'post-form' instead of 'prior-form' to calculate the scattering amplitude.

As a consequence of (18), there will be a change in the Faddeev's equation for the wavefunction components

$$|\Psi_\alpha^+(E_\alpha)\rangle = |\Psi_\alpha^{(1)}\rangle + |\Psi_\alpha^{(2)}\rangle + |\Psi_\alpha^{(3)}\rangle. \tag{26}$$

The equation will now read (α, β, γ all different)

$$\begin{pmatrix} |\Psi_\alpha^{(1)}\rangle \\ |\Psi_\alpha^{(2)}\rangle \\ |\Psi_\alpha^{(3)}\rangle \end{pmatrix} = \begin{pmatrix} |\phi_\alpha(E_\alpha)\rangle \\ \mathscr{P}_\beta(E_\alpha)\,|\phi_\alpha(E_\alpha)\rangle \\ \mathscr{P}_\gamma(E_\alpha)\,|\phi_\alpha(E_\alpha)\rangle \end{pmatrix} + G_0^+(E_\alpha) \begin{pmatrix} 0 & \overset{+}{t}_\alpha(E_\alpha) & \overset{+}{t}_\alpha(E_\alpha) \\ \overset{+}{t}_\beta(E_\alpha) & 0 & \overset{+}{t}_\beta(E_\alpha) \\ \overset{+}{t}_\gamma(E_\alpha) & \overset{+}{t}_\gamma(E_\alpha) & 0 \end{pmatrix} \begin{pmatrix} |\Psi_\alpha^{(1)}\rangle \\ |\Psi_\alpha^{(2)}\rangle \\ |\Psi_\alpha^{(3)}\rangle \end{pmatrix} \tag{27}$$

where the two-body matrices are

$$\overset{+}{t}_\alpha(E_\alpha) = \operatorname*{Lt}_{\epsilon_1 \to 0} t_\alpha(Z_1)$$

where $$t_\alpha(Z_1) = V_\alpha + V_\alpha\, G_0(Z_1)\, t_\alpha(Z_1), \tag{28}$$

so that the change is only in the inhomogeneous term of the equation of wavefunction components which previously read as

$$\begin{pmatrix} |\phi_\alpha(E_\alpha)\rangle \\ 0 \\ 0 \end{pmatrix},$$

the rest of the equation (equation (27)) remaining the same. Here also we use equation like (14) to get the right side of (27). The BRS-equation (Sandhas 1976) for the transition matrix $T_{\beta\alpha}(Z_1)$ is given by

$$T_{\beta\alpha}(Z_1) = \overline{V}_\beta + \overline{V}_\beta\, G(Z_1)\, \overline{V}_\alpha \tag{29}$$

$$= \sum_n C_n V_\beta^n G_n(Z_1)\, G_\alpha^{-1}(Z_1) + \sum_n C_n V_\beta^n G_n(Z_1)\, T_{n\alpha}(Z_1), \tag{30}$$

where $$\overline{V_\beta} = \sum C_n V_\beta^n, \tag{31}$$

$$C_n = (-)^n (n-1)!, \tag{32}$$

and the summation over η in (30) and (31) runs over all partition with two or more clusters, n being the number of clusters in the partition n. BRS uses Lippman's identity of (19) to replace

$$\operatorname*{Lt}_{\epsilon_1 \to 0} G_n(Z_1)\, G_\alpha^{-1}(Z_1)\, |\phi_\alpha(E_\alpha)$$

by $\delta_{n\alpha}\, |\phi_\alpha(E_\alpha) >$ getting the BRS equation

$$T^+_{\beta\alpha}(E_\alpha) = C_\alpha V^\beta_\alpha + \sum_n C_n V^n_\beta G^+_n(E_\alpha) T^+_{n\alpha}(E_\alpha), \tag{33}$$

which thus 'transforms away' a good part of the inhomogeneous term of (30) by virtue of Lippman's identity. But this removal will not take place any more when we use the correct equation (18) in place of (19), and instead of BRS equation (33) we end up with

$$T^+_{\beta\alpha}(E_\alpha) = \sum_n C_n V^n_\beta \mathscr{P}_n(E_\beta) + \sum_n C_n V^n_\beta G^+_n(E_\alpha) T^+_{n\alpha}(E_\alpha). \tag{34}$$

This equation is no longer simple to handle compared to the original equation, (33), for scattering matrix and is not particularly advantageous for starting any approximation scheme.

In the absence of homogeneous equations for channel states

$$|\Psi^+_\alpha(E_\alpha)\rangle, |\Psi^+_\beta(E_\alpha)\rangle$$

etc. ($\alpha \neq \beta \neq \gamma$) much of the exercises done in KLT-theory (Sandhas 1976) *via* different channel-coupling array schemes would be unnecessary now.

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