Pramāṇa, Vol. 18, No. 2, February 1982, pp. 215–223. © Printed in India.

Graphical representation of reflection from single and double coating systems on metallic substrate

N BARAKAT*, S MOKHTAR and W OSMAN**
National Institute of Standards, Cairo, Egypt
*Faculty of Science, Ain Shams University, Cairo, Egypt
**Faculty of Science, Cairo University, Cairo, Egypt

MS received 25 April 1981; revised 18 December 1981

Abstract. In the present work a new graphical method is described to represent the resultant of the multiply reflected waves from coating systems consisting of single and double layers on metallic substrate, taking into account the optical phase properties of the films.

Keywords. Reflection coefficient; thin film optics; optical constants; tellurium silicon monoxide films; double coating.

1. Introduction

Reflection from interfaces separating two media may be represented graphically by vectors having inter-angles corresponding to the phase of the reflected amplitudes. Tolansky (1955) described a simple graphical method representing the shape of the multiple-beam transmitted fringe system obeying Airy's formula with no phase consideration of the metallic coating of the interferometer. Barakat and Abou Zakhm (1965) used a graphical method to investigate the intensity distribution in a multiple beam reflected fringe system. Methods based on using computer calculation of amplitude and phase of the components of the reflected waves from a system of multilayers may be used (Cox *et al* 1962; Berthold 1970).

In the present work graphical representation of the reflection from single and double coating systems on metallic substrate is presented taking into consideration the optical phase properties of the films. Clear and simple representation of the phenomena results illustrates the behaviour of the phases and shows their contribution to the resultant amplitudes. Result obtained by this method coincides with the result obtained from computer calculation.

2. Theory

The general formula for reflection of light from k-layers according to Roward's (1937) method from the last film towards the first film in the arrangement is shown in figure 1.

$$A_k \exp i\,\delta_k = \frac{\bar{r}_k + A_{k+1} \exp i\,(\delta_{k+1} + \bar{x}_k)}{1 + \bar{r}_k\, A_{k+1} \exp i\,(\delta_{k+1} + \bar{x}_k)}, \tag{1}$$

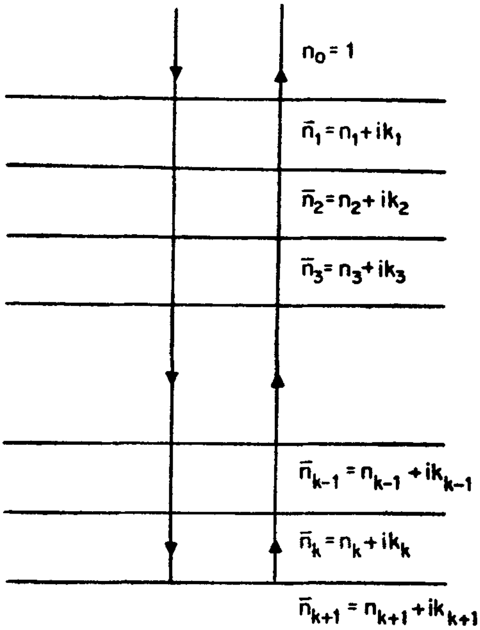


Figure 1. Schematic representation of the reflection of light from K thin films.

taking into account that for applying (1) at the interface bottom layer of order k, substrate of order $k+1$, we put

$$A_{k+1} \exp i\, \delta_{k+1} = \bar{r}_{k+1},$$

where $$\bar{r}_{k+1} = \frac{\bar{n}_k - \bar{n}_{k+1}}{\bar{n}_k + \bar{n}_{k+1}},$$

$$\bar{r}_k = \frac{\bar{n}_{k-1} - \bar{n}_k}{\bar{n}_{k-1} + \bar{n}_k}, \qquad \bar{x}_k = \frac{4\pi}{\lambda} \bar{n}_k\, d_k.$$

By applying (1) for a single layer $k = 1$

$$A_2 \exp i\, \delta_2 = \bar{r}_2,$$

$$A_1 \exp i\, \delta_1 = \frac{\bar{r}_1 + \bar{r}_2 \exp i\, \bar{x}_1}{1 + \bar{r}_1 \bar{r}_2 \exp i\, \bar{x}_1}, \tag{2}$$

where $$\bar{r}_1 = \frac{n_0 - \bar{n}_1}{n_0 + \bar{n}_1}, \quad \bar{r}_2 = \frac{\bar{n}_1 - \bar{n}_2}{\bar{n}_1 + \bar{n}_2}, \quad \bar{x}_1 = \frac{4\pi}{\lambda} \bar{n}_1\, d_1.$$

For double layer, (1) is applied twice, first for $k = 2$, $A_3 \exp i\, \delta = r_3$ and then for $k = 1$

$$A_2 \exp i\,\delta_2 = \frac{\bar{r}_2 + \bar{r}_3 \exp i\,\bar{x}_2}{1 + \bar{r}_2\,\bar{r}_3 \exp i\,\bar{x}_2}, \tag{3}$$

$$A_2 \exp i\,\delta_1 = \frac{r_1 + A_2 \exp i\,(\delta_2 + \bar{x}_1)}{1 + \bar{r}_1\,A_2 \exp i\,(\delta_2 + \bar{x}_1)}, \tag{4}$$

where $\bar{r}_1$, $\bar{r}_2$ and $\bar{x}_1$ are the same as before and

$$\bar{r}_3 = \frac{\bar{n}_2 - \bar{n}_3}{\bar{n}_2 + \bar{n}_3}, \quad \bar{x}_2 = \frac{4\pi}{\lambda}\bar{n}_2\,d_2.$$

3. Graphical representation of reflectivity of a single layer

For representing the formula of reflection of light by a single layer on a metallic substrate graphically, taking into account the optical phase properties of the film, (2) is applied and is put in the form

$$A_1 \exp i\,\delta_1 = \frac{\bar{r}_1^2 + \bar{r}_1\,\bar{r}_2 \exp i\,\bar{x}_1}{\bar{r}_1\,(1 + \bar{r}_1\,\bar{r}_2 \exp i\,\bar{x}_1)}.$$

Since $\quad \bar{t}_1\,\bar{t}_1' = 1 - \bar{r}_1^2$,

$$A_1 = \exp i\,\delta_1 = \frac{1}{\bar{r}_1} - \frac{\bar{t}_1\,\bar{t}_1'}{\bar{r}_1\,(1 + \bar{r}_1\,\bar{r}_2 \exp i\,\bar{x}_1)},$$

Putting

$$\bar{r}_1 = \rho_1 \exp i\,\phi_1,\ \bar{r}_2 = \rho_2 \exp i\,\phi_2;\quad \bar{t}_1 = \tau_1 \exp i\,\psi_1,\ \bar{t}_1' = \tau_1' \exp i\,\psi_1',$$

$$\bar{x}_1 = \frac{4\pi}{\lambda}(n_1 + i\,k_1)\,d_1 = N_1 + i\,k_1,\ \Delta_1 = N_1 + \phi_1\,\phi_2.$$

Then

$$A_1 \exp i\,\delta_1 = \exp\,[i\,(\pi + \psi_1 + \psi_1' + \phi_1)]\left[\frac{1}{\rho_1}\exp i\,(\pi - \psi_1 - \psi_1')\right.$$
$$\left. + \frac{\tau_1\,\tau_1'}{\rho_1\,(1 + \rho_1\,\rho_2) \exp\,(-k_1 + i\,\Delta_1)}\right]$$

The representation of the term

$$\frac{\tau_1\,\tau_1}{\rho\,(1 + \rho_1\,\rho_2 \exp\,(-k_1 + i\,\Delta_1)}$$

graphically is as follows:
In figure 2, C is a fixed point on CX axis, O is distant point from C such that

$$\overline{CO} = \frac{\tau_1\,\tau_2}{\rho_1(1 - \rho_1^2\,\rho_2^2 \exp - 2k_1)}.$$

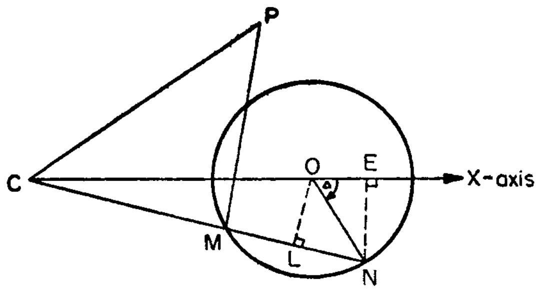


Figure 2. Graphical representation of reflectivity of a single layer.

Then from O a circle is drawn of radius $\rho_1 \rho_2 \exp -k_1 \overline{CO}$ making an angle $-\Delta_1$ with $\overline{OX}$, then the resultant vector of this term is $\overline{\mathbf{XM}}$.

To prove that the vector $\overline{\mathbf{CM}}$ represents the resultant in magnitude and direction, let $\overline{\mathbf{CO}} = B$

$$\overline{\mathbf{CO}} + \overline{\mathbf{ON}} = \overline{\mathbf{CN}}$$

From the triangle $\widehat{CON}$

$$\overline{CN}^2 = \overline{CO}^2 + \overline{ON}^2 + 2\,\overline{CO}\cdot\overline{ON}\cos\Delta_1,$$

$$= B^2 (1 + \rho_1^2 \rho_2^2 \exp - 2k_1 + 2\,\rho_1 \rho_2 \exp -k_1 \cos \Delta_1),$$

$$B^2 \rho_1^2 \rho_2^2 \exp - 2k_1 = \overline{OL}^2 + \overline{LN}^2$$

$$B^2 = \overline{OL}^2 + \overline{LC}^2,$$

$$B^2 (1 - \rho_1^2 \rho_2^2 \exp - 2k_1) = \overline{LC}^2 - \overline{LN}^2,$$

since

$$\overline{LC} = \overline{ML} + \overline{CM} \text{ and } 2\overline{ML} = \overline{MN}.$$

$$\therefore\ B^2 (1 - \rho_1^2 \rho_2^2 \exp - 2k_1) = \overline{CM}\cdot\overline{CN},$$

$$\therefore\ \overline{CM} = \frac{B (1 - \rho_1^2 \rho_2^2 \exp - 2k_1)}{(1 + 2\,\rho_1 \rho_2 \exp(-k_1 \cos \Delta_1) + \rho_1^2 \rho_2^2 (\exp - 2k_1)^{1/2}}$$

$$\overline{CM} = \frac{\tau_1 \tau_1'}{\rho_1 (1 + 2\,\rho_1 \rho_2 \exp - k_1 \cos \Delta_1 + \rho_1^2 \rho_2^2 (\exp - 2k_1)^{1/2}}, \tag{5}$$

$$\tan N\hat{C}E = \frac{\overline{NE}}{\overline{\overline{CE}}} = \frac{-\rho_1 \rho_2 \exp - k_1 \sin \Delta_1\,.}{(1 + \rho_1 \rho_2 \exp - k_1 \cos \Delta_1)}. \tag{6}$$

Put

$$\frac{\tau_1 \tau_1'}{\rho_1 (1 + \rho_1 \rho_2 \exp - k_1 \exp \Delta_1)} = C - id$$

$$\therefore\ (C^2 + d^2)^{1/2} = \frac{\tau_1 \tau_1'}{\rho (1 + 2\,\rho_1 \rho_2 \exp - k_1 \cos \Delta_1 + \rho_1^2 \rho_2^2 \exp - 2k_1)^{1/2}} \tag{7}$$

$\tan\theta_1 = -d/c$

$$= -\frac{\rho_1\rho_2 \exp -k_1 \sin\Delta_1}{1+\rho_1\rho_2\exp - k_1\cos\Delta_1}.$$

The last two equations are similar to (5) and (6) which represent the amplitude and direction of the quantity

$$\frac{\tau_1\tau_1}{\rho_1\left[1+\rho_1\rho_2\exp\left(-k_1+i\,\Delta_1\right)\right.}.$$

From the point M in figure 2, MP is drawn which represents the vector

$$\frac{1}{\rho_1}\exp\left(\pi-\psi_1-\psi_1'\right).$$

$\therefore$ **CP** is the resultant vector of

$$\left[\frac{1}{\rho_1}\exp -i\left(\pi-\psi_1-\psi_1'\right)+\frac{\tau_1\tau_1'}{\rho_1\left(1+\rho_1\rho_2\exp\left(-k_1+i\,\Delta_1\right)\right)}\right],$$

$$A_1=\overline{CP}\text{ and }\delta_1=P\hat{C}A+\pi+\psi_1+\psi_1^{*}+\phi_1.$$

4. Graphical representation of a double layer coating

The reflection of light for a double layer coating on metallic substrate can be represented graphically by means of (3) following the same steps mentioned before, where (3) is written in the form

$$A_2\exp i\delta_2=\exp i\left(\pi+\psi_2+\psi_2'-\phi_2\right)\left[\frac{1}{\rho_2}\exp i(\pi-\psi_2-\psi_2')+\frac{\tau_2\tau_2'}{\rho_1\left(1+\rho_2\rho_3\exp\left(-k_2+i\Delta_2\right)\right)}\right],$$

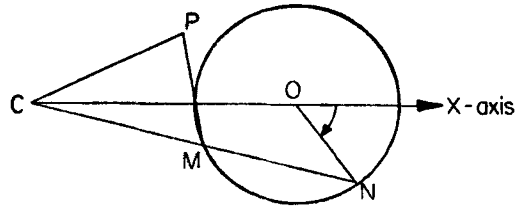
where

$$\bar{r}_2=\rho_2\exp i\phi_2,\ \bar{r}_3=\rho_3\exp i\phi_3\ \bar{t}_2=\tau_2\exp i\,\psi_2,\ t_2'=\tau_2'\exp i\psi_2',$$

$$N_2=\frac{4\pi}{\lambda}n_2d_2,\ k_2=\frac{4\pi}{\lambda}k_2\,d_2,\ \Delta_2=N_2+\phi_2+\phi_3.$$

From figure 3a

$$A_2=\overline{CP}\text{ and }\delta_2=P\hat{C}X+\pi+\pi_2-\psi_2'-\phi_2.$$



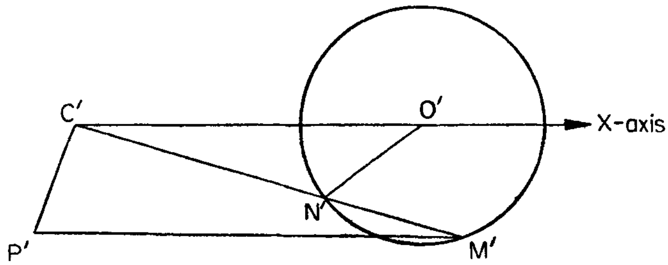


Figure 3. Graphical representation of reflectivity of a double layer.

By substituting A_2 and δ_2 in (4), we get

$$\overline{CP}\exp i\,\delta_1 = \frac{\bar{r}_1 + \overline{CP}\exp i(P\hat{C}x + \pi + \psi_2 - \psi_2' - \phi_2 + \bar{x}_1)}{1 + \bar{r}_1\,\overline{CP}\exp i(P\hat{C}x + \pi + \psi_2 - \psi_2' - \phi_2 + \bar{x}_1)}.$$

This equation can be represented graphically by the same method where the resultant amplitude A_1 is obtained for the double layer.

If the upper layer is a dielectric, *i.e.* $k_1 - 0$

$$\psi_1 = \psi_1' = 0 \text{ and } \phi_1 = \pi$$

$$\text{Then } A_1 \exp i\,\delta_1 = \frac{1}{\rho_1}\exp i\pi + \frac{\tau_1\,\tau_1'}{\rho_1\,(1 + \rho_1\,A_1\exp i\,\delta_2)}, \tag{8}$$

$$\Delta_2 = N_1 + \delta_2 + \pi.$$

Equation (8) is represented graphically as shown in figure 3b where

$$\overline{CO} = \frac{\tau_1\,\tau_1'}{\rho_1\,(1 - \rho_1^2\,A_2^2)},\ \overline{O'N'} = \rho_1\,A_2\,\overline{C'O'}.$$

$\overline{CM}$ represents the resultant amplitude of the second term of (8). Then $\overline{M'P'}$ is drawn which represents the first term $(1/\rho_1)\exp i\pi$. Hence the resultant amplitude of light reflected by two-layers coating is represented by $\overline{C'P'} = A'$. Example of the applicability of the explained graphical method is: calculation of the reflectivity of the selective double coating system Al-Te-SiO (Osman 1979) of Te and SiO thickness 350 Å and 650 Å respectively on Al substrate, in terms of the optical constants, thickness and wavelengths. The reflectance has been calculated at four wavelengths λ_1, $\lambda_2 < \lambda_c$ and λ_3, $\lambda_4 > \lambda_c$ where $\lambda_c = 1{\cdot}3\ \mu$. The values of the optical constants of opaque Al used as a metal substrate are represented in table 1 from previously published data Thun and Hass (1964a). The refractive index of SiO equals 2·0 Thun and Hass (1964b) for the whole wavelength range, since its variation with wavelength is small. The optical constants of Te thickness 350 Å were computed from reflectivity and transmissivity measurements (table 1).

The resultant reflected amplitude of the double coating AL-Te-SiO is represented graphically at λ 0·5 μ (figure 4) after calculating the following parameters:

$$\rho_3 = 0{\cdot}42 \quad \rho_2 = 0{\cdot}414, \quad \tau_2 = 0{\cdot}77, \quad \tau_2' = 1{\cdot}323,$$

$$\rho_1 = 0{\cdot}333, \quad \tau_1 = 0{\cdot}666, \quad N_2 = 141^\circ, \quad k_2 = 1{\cdot}76,$$

$$N_1 = 187{\cdot}2, \quad \overline{CO} = 2{\cdot}46, \quad \overline{CN} = 0{\cdot}368,$$

$$\Delta_2 = N_2 + \phi_2 + \phi_3 = 267{\cdot}6, \quad 1/\rho_2 = 2{\cdot}415,$$

$$\pi - \psi_2 - \psi_2' = 187{\cdot}7^\circ, \quad A_2 = CP = 0{\cdot}34, \quad \delta_2 = 126{\cdot}3.$$

$$\overline{C'O'} = 2{\cdot}7, \quad \overline{O'N'} = 0{\cdot}306, \quad \Delta_1 = 240{\cdot}5 \text{ and } 1/\rho_1 = 3.$$

In our case the magnification is 5 and the resultant reflected amplitude = 0·35.

∴ The reflectance $R = 0{\cdot}12$.

The same process was applied for λ equals 1·0, 1·5 and 2·0 μ respectively as shown in figures 5, 6 and 7.

Table 1. Optical constants n and k of Te and Al films at different wavelengths.

λ in μ	Te n	Te k	Al n	Al k
0·5	2·80	2·00	0·62	3·80
1·0	4·34	1·80	1·65	8·55
1·5	5·10	0·80	1·55	12·50
2·0	3·85	0·25	2·30	16·50

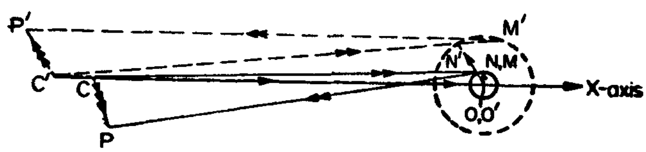


Figure 4. Graph represents the resultant amplitude $C'P'$ of the reflected system Al-Te(350 Å)—SiO (650 Å) for λ=0·5 μ.

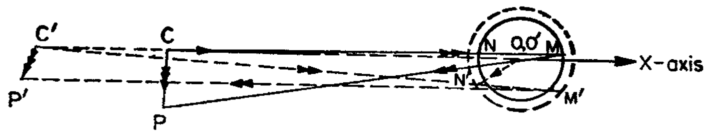


Figure 5. Graph represents the resultant amplitude $\overline{C'P'}$ at $\lambda=1.0\mu$.

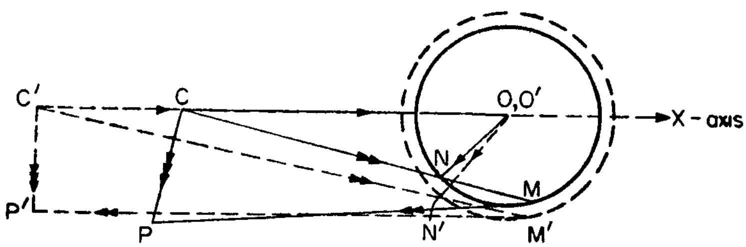


Figure 6. Graph represents the resultant amplitude at $\lambda=1{\cdot}5\ \mu$.

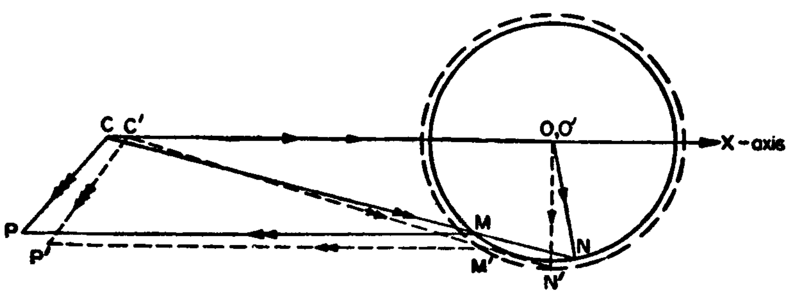


Figure 7. Graph represents the resultant amplitude at $\lambda=2{\cdot}0\ \mu$.

The reflectivity of the Al-Te-SiO coating system has been calculated theoretically by means of a computer program. Table 2 represents the values of reflectivity obtained by graphical and computer programming methods at 0·5, 1·0 and 2·0 μ.
Both results are in good agreement.
The graphical representation gives a more physical meaning to the problem of summing the reflected waves from the double coating system Al-Te-SiO.

Table 2. Computed and graphically obtained reflectivity values of Al-Te-SiO coating system at different wavelengths.

λ in μ	computed reflected values	graphically reflected values
0·5	0·11	0·12
1·0	0·03	0·04
1·5	0·38	0·38
2·0	0·91	0·89

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