Pramāṇa, Vol. 18, No. 2, February 1982, pp, 157–173. © Printed in India.

# Collective field in Hamiltonian and Lagrangian (path-integral) approach: An application to O($N$) Heisenberg spin operator

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MS received 17 October 1981

**Abstract.** Collective field approach to O($N$) Heisenberg spin system is discussed. Hamiltonian formulation is reviewed and connection with large $N$ limit is shown. Collective field is then introduced in the Lagrangian path-integral formulation and $1/N$ corrections to various quantities like mass-gap, beta-function are computed.

**Keywords.** Collective field; $1/N$ expansion; non-linear sigma model; mass-gap; beta-function; path integral method.

## 1. Introduction

Instead of perturbative expansion in powers of coupling constant, the large $N$ expansion is a powerful alternative to explore field theories with complicated vacuum structure. It was 't Hooft (1974) who first demonstrated that the two-dimensional U($N$) quantum chromodynamics can be realised as a model for mesons in the limit $N \to \infty$. This is achieved by summing all planar diagrams. Since then, the $1/N$ expansion has been explored in detail in various $N$-component vector models (Abe 1973; Abe and Hikami 1973; Wilson 1973; Coleman *et al* 1974; Guha and Sakita 1981), Gross-Neveu model (Gross and Neveu 1974) and others (Brezin *et al* 1978; Mehta 1979; Itzykson and Zuber 1980; Coleman 1979). However, summing planar diagrams to obtain the larger $N$ limit is technically rather difficult for realistic theories like Yang-Mills theory in four dimensions.

Recently it has been proposed (Sakita 1980; Jevicki and Sakita 1980; Guha and Sakita 1981) that the method of collective field might be useful to achieve the large $N$ limit of the above-mentioned theories. In this article we analyse the O($N$) Heisenberg spin-system (a lattice version of the non-linear sigma model) using the collective field approach. The idea is to construct an effective theory in terms of a collective field such that the large $N$ limit can be obtained by a semi-classical approximation.

In § 2, we briefly review the collective field method in the Hamiltonian approach. Although the material in this section is available in literature (Guha and Sakita 1981), it is useful to have it for comparison with the results obtained by introducing the same collective field in the Lagrangian path-integral approach which we discuss in § 3. $1/N$ corrections to various quantities like the mass-gap and the beta-function are

This research was supported in part by the National Science Foundation Grant No. PHY-78-24888.

explicitly computed using Feynman diagrams. Finally in § 4, our conclusions are presented.

## 2. Hamiltonian formulation (A review)

The Hamiltonian for the O($N$) Heisenberg spin system in $d$ dimensions is given by

$$\mathcal{H} = - K \sum_{\mathbf{m}} \sum_{\boldsymbol{\mu}} \boldsymbol{\sigma}(\mathbf{m}) \cdot \boldsymbol{\sigma}(\mathbf{m} + \boldsymbol{\mu}), \tag{1}$$

with

$$\boldsymbol{\sigma}(\mathbf{m}) \cdot \boldsymbol{\sigma}(\mathbf{m}) = 1, \tag{1a}$$

where $\boldsymbol{\sigma}(\mathbf{m})$ is the $N$-dimensional classical spin vector of unit length at the lattice site $\mathbf{m}$. $\boldsymbol{\mu}$ denotes the unit lattice vector. In the continuum limit (the lattice spacing $a \to 0$), the Hamiltonian $\mathcal{H}$ reduces to the euclidean action of the O($N$) non-linear sigma model:

$$\mathcal{H} \to S_E = \tfrac{1}{2} (a)^{2-d} K \int d^d x \sum_{i=1}^{d} (\partial_i \boldsymbol{\sigma}(\mathbf{x}))^2, \tag{2}$$

apart from an unimportant constant. If we now perform a 'Wick rotation'

$$X_d \to it,$$

$$X_i \to X_i \ (i = 1, 2, \ldots, d-1), \tag{3}$$

we get the Minkowski action

$$S_M = \frac{1}{2g} \int dt\, L, \tag{4a}$$

where $$L = \int dx_1\, dx_2 \ldots dx_{d-1} (\partial_\mu \boldsymbol{\sigma})^2, \tag{4b}$$

and $$K (a)^{2-d} = 1/g. \tag{4c}$$

So instead of discussing the original Hamiltonian $\mathcal{H}$, we can discuss an equivalent physical system described by the Lagrangian $L$ of (4b). The advantage is that the Lagrangian $L$ is defined in a $(d-1)$ dimensional space. In order to obtain a physical quantity like the spin-spin correlation function in the original system ($d$-dimensional), we can calculate the same quantity at 'equal time' using the $(d-1)$ dimensional Lagrangian $L$ and use the property of isotropy of the original system to obtain the spin-spin correlation function at 'unequal time'.

As shown in (Hamer *et al* 1979), the quantum $(d-1)$ dimensional Hamiltonian corresponding to the Lagrangian $L$ is

$$H = \frac{g}{2a} \sum_{\mathbf{m}} \mathbf{J}^2(\mathbf{m}) - \frac{1}{ag} \sum_{\mathbf{m}} \sum_{\boldsymbol{\mu}} \sigma(\mathbf{m}) \cdot \sigma(\mathbf{m}+\boldsymbol{\mu}), \tag{5}$$

where $\mathbf{m}$ and $\boldsymbol{\mu}$ now refer to the $(d-1)$ dimensional ' spatial ' lattice. $\mathbf{J}$ is the angular momentum operator for O($N$).

There are several interesting properties of the Hamiltonian $H$ given by (5). For strong coupling (high temperature), *i.e.* $g \to \infty$, the dominant part is

$$H \to \frac{g}{2a} \sum_{\mathbf{m}} \mathbf{J}^2(\mathbf{m}), \tag{6}$$

which has the invariance group of local O($N$). On the other hand, for $g \to 0$, *i.e.* low temperature.

$$H \to -\frac{1}{ag} \sum_{\mathbf{m}} \sum_{\boldsymbol{\mu}} \sigma(\mathbf{m}) \cdot \sigma(\mathbf{m}+\boldsymbol{\mu}), \tag{7}$$

which is invariant under global O($N$), *i.e.* when all the spin vectors are rotated simultaneously by equal amount. Thus, for any finite $g$, $H$ always has global O($N$) as an invariance subgroup. This gives us a hint that as far as the ground state is concerned, we should look for a collective field which is a singlet under the global O($N$), so that the ground state wave function depends only on the collective field. Such a field is

$$q_{\mathbf{m},\mathbf{n}} = \sigma(\mathbf{m}) \cdot \sigma(\mathbf{n}). \tag{8}$$

From now on, for simplicity we shall work in $d = 2$ dimensions. In terms of the variables $q$, the constraint condition (1a) becomes

$$q_{mm} = 1. \tag{9}$$

We shall also assume periodic boundary condition for the original $\sigma$ field

$$\sigma(M+1) = \sigma(1), \tag{10}$$

where $M$ is the number of lattice sites.

The basic feature of the collective field variables is that they are not all independent. What we have done above is that instead of $M(N-1)$ independent degrees of freedom of the original $\sigma$ variable, we have chosen $M(M-1)/2$ degrees of freedom for $q$. Thus, although the variables $q$ are not independent for finite $N$, they become independent in the limit $N \to \infty$ and $M \to \infty$. Assuming the ground state of the system to be non-degenerate, we can take it to be a function of $q$ only.

$$\psi_0 = \psi_0(q). \tag{11}$$

Using the chain rule of differentiation, the Hamiltonian of (5) can be written as (Guha and Sakita 1981)

$$H = -\frac{g}{2a} \sum_{m} T(m) - \frac{1}{ag} \sum_{m} q_{m,\,m+1}, \tag{12}$$

where

$$T(m) = -4 \sum_{m} \sum_{k,k'}{}' \Omega\,(m, k, k')\, p_{mk}\, p_{mk'} - 2i\,(N-1) \sum_{k}{}' q_{mk}\, p_{mk}$$

$$\Omega\,(m, k, k') = q_{kk'} - q_{mk}\, q_{mk'}$$

$$p_{mk} = (-i)\, \partial / \partial\, q_{mk}, \tag{13}$$

where the prime means that the terms $k = m$ and $k' = m$ are to be omitted. Although the Hamiltonian $H$ of (12) is adequate for calculating the large $N$ limit, it is usually more convenient to use the original Hamiltonian $\mathcal{H}$ of (1) to compute $1/N$ corrections to physical quantities like the ground state energy. However, the collective field variable $q$ has a natural connection with $1/N$ expansion as we shall see in § 3.

Going back to the Hamiltonian $H$ of (12), we see that it is not Hermitian in the ordinary sense, since the scalar product in the Hilbert space of states is defined with a Jacobian $J$. By a similarity transformation (Guha and Sakita 1981), we can absorb the Jacobian into the wave function and get an effective Hamiltonian

$$H_{\text{eff}} = \frac{2g_0}{a} \sum_{m} \sum_{k,k'}{}' \frac{1}{N}\, p_{mk}\, \Omega\,(m, k, k')\, p_{mk'} + V_{\text{eff}}\,(q), \tag{14}$$

$$V_{\text{eff}}\,(q) = N \left[ -\frac{1}{ag_0} \sum_{m} q_{m,\,m+1} + \frac{g_0}{8a} \sum_{m} (q^{-1})_{mm} \right] + \Delta V$$

$$= N V_0\,(q) + \Delta V, \tag{14a}$$

where $g_0 = gN$ and $\Delta V$ is an extra piece containing O(1) and O($1/N$) terms,

$$\Delta V = \frac{g_0}{8a} \left[ -2\,(2M+1) + \frac{1}{N}(2\,M+1)^2 \right] \mathrm{Tr}\,(q^{-1}), \tag{14b}$$

where $q^{-1}$ is the inverse matrix of $q$.

We shall seek the large $N$ limit of the system keeping $g_0 = gN$ finite. By a rescaling of $p$ as

$$\tilde{p}_{mk} = p_{mk}/N, \tag{15}$$

we see that $H_{\text{eff}}$ takes the form

$$H_{\text{eff}} = N\left[\frac{2g_0}{a}\sum_m \sum_{k,k'}{}' \tilde{p}_{mk}\,\Omega\,(m\,k\,k')\,\tilde{p}_{mk'} + V_c(q)\right] + \Delta V$$

$$\equiv N\,\tilde{H}_{\text{eff}}\,(g_0;\tilde{p}, q) + \Delta V. \tag{16}$$

Because of (15) we obtain

$$[\tilde{p}, q] = -\,i/N, \tag{17}$$

so that $1/N$ plays the role of $\hbar$. Thus the $1/N$ expansion is nothing but the semi-classical expansion of the Hamiltonian system of (16). In this systematic expansion in powers of $1/N$, one would obtain divergent higher order terms, *i.e.* terms proportional to $M$, $M^2$, etc. These divergent terms should be cancelled by the divergent terms of $\Delta V$ in (14b). Such cancellation for the ground state energy to lowest order will be demonstrated in appendix. $\Delta V$, therefore, plays the role of counter-terms.

The ground state in the limit $N \to \infty$ can be obtained from the classical solution of the Hamiltonian system of $\tilde{H}_{\text{eff}}$. We look for static solutions only. Equations of motion are simply obtained by minimising $V_0\,(q)$ subject to condition (9)

$$-\frac{g_0}{4a}(q^{-2})_{mn} - \frac{1}{a\,g_0}\,(\delta_{m+1,\,n} + \delta_{m,\,n+1}) + \lambda\,(m)\,\delta_{mn} = 0. \tag{18}$$

The solution of (18), denoted by $q^0_{mn}$, can be interpreted as the vacuum expectation value of the corresponding operator in the limit $N \to \infty$. From translational invariance of the vacuum, one would expect that $q^0_{mn}$ depends only on $(m - n)$ and $\lambda\,(m)$ independent of $m$, Then we obtain

$$q^0_{mn} = \langle \sigma\,(\mathbf{m})\cdot\sigma\,(\mathbf{n})\rangle_{\text{vac}}$$

$$= \frac{1}{M}\sum_{l=1}^{M}\frac{\exp\,[2\pi i\,(m-n)\,l/M]}{[4a\lambda/g_0 - 8\cos\,(2\pi l/M)/g_0^2]^{1/2}}, \tag{19}$$

which in the infinite volume limit $(M \to \infty)$ becomes

$$q^0_{mn} = \int_0^{2\pi}\frac{d\theta}{2\pi}\exp\,[i\,(m-n)\,\theta]\;[4a\lambda/g_0 - 8\cos\,\theta/g_0^2]^{-1/2}. \tag{20}$$

The Lagrange multiplier is determined from the following equation which is the result of the constraint (9):

$$K\,(k) = (\pi/2)\;[4a\lambda/g_0 + 8/g_0^2]^{1/2},$$

$$k^2 = (16/g_0^2)\;[4a\lambda/g_0 + 8/g_0^2]^{-1}, \tag{21}$$

where $K(k)$ is the complete elliptic integral of the first kind. This equation was already obtained by Berlin and Kac (1952) by a different method for the spherical model which has been proved (Stanley 1961) to be equivalent to the Heisenberg O ($N$) model in the limit $N \to \infty$.

The mass-gap $\mu$ is defined by

$$\mu = -\lim_{|m-n| \to \infty} \left[\frac{1}{a\,|m-n|} \ln q^0_{mn}\right], \tag{22}$$

and is related to $\lambda$ by

$$a^2 \mu^2 = a \lambda g_0 - 2. \tag{23}$$

The $\beta$-function of the theory will be defined as is done by Hamer *et al* (1979)

$$\beta(g) = a \frac{\mathrm{d}g}{\mathrm{d}a}, \tag{24}$$

which is obtained by demanding that a physical quantity like the mass-gap $\mu$ is independent of $a$

$$\frac{\mathrm{d}\mu}{\mathrm{d}a} = 0. \tag{25}$$

For weak coupling $g_0 \ll 1$,

$$\lambda \simeq \frac{g_0}{4a}\left[\frac{8}{g_0^2} + \frac{256}{g_0^2} \exp(-4\pi/g_0)\right],$$

$$\mu \simeq \frac{8}{a} \exp(-2\pi/g_0),$$

$$\beta(g) \simeq g^2 N/2\pi. \tag{26}$$

For strong coupling $g_0 \gg 1$,

$$\lambda \simeq g_0/4a,$$

$$\mu \simeq g_0/2a,$$

$$\beta(g) \simeq g. \tag{27}$$

In the intermediate coupling region, one can solve (21) numerically. However, a good approximation can be obtained by expanding $\cos\theta$ in (20) upto quadratic part as

$$q^0_{mn} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \exp[i(m-n)\theta]\,[4a\lambda/g_0 - 8(1-\theta^2/2)/g_0^2]^{-1/2}. \tag{28}$$

We shall call this the continuum approximation (CA), since this is the same result as would be obtained from the continuum field theory version. Then,

$$\mu \underset{\text{CA}}{\simeq} \frac{2\pi}{a}\left(\frac{\alpha}{\alpha^2 - 1}\right), \tag{29a}$$

$$\beta\,(g) \underset{\text{CA}}{\simeq} \frac{g^2 N}{2\pi}\left(\frac{\alpha^2 - 1}{\alpha^2 + 1}\right), \tag{29b}$$

$$\alpha = \exp\,(2\pi/g_0). \tag{29c}$$

Equation (29b) has the correct limit given by (26) and (27). The expression for $\mu$ however, differs by a factor of $2\pi/8$ from the weak coupling limit (26). This difference can be attributed to CA. The $\beta$-function also agrees with results found in low temperature expansion (Polyakov 1975; Bardeen *et al* 1976; Brezin and Zinn-Justin 1976 and high temperature expansion (Hamer *et al* 1979; Stanley and Kaplan 1966; Guttman 1978).

In concluding this section, a few comments are in order. All the results here can be easily generalised to a $d$-dimensional model. For example, the classical solution given by (20) can be generalised to $d = 3$ from which it can be deduced that unlike the two-dimensional model which has a critical coupling constant $g_c = 0$, the three-dimensional model has a non-zero critical coupling constant. The simplification we have achieved by using a collective field $q$ is that all the information about the ground state of the system (for large $N$) is contained in the potential energy term, so that the large $N$ approximation reduces to the ordinary semi-classical approximation ($\hbar \to 0$). However, in the Hamiltonian formulation, one has to face the problem that, the collective fields being not all independent for finite $N$, there might be $1/N$ corrections to the Hamiltonian which are rather difficult to evaluate. In the next section, we introduce the same collective field in the Lagrangian path-integral formulation through $\delta$-function condition in the functional integral so that one can keep track of all $1/N$ corrections. Of course, the lowest order result agrees with that from Hamiltonian approach but, in addition, a systematic $1/N$ expansion can be done.

## 3. Lagrangian formulation

### 3.1 *Path integral and large N limit*

The vacuum to vacuum transition amplitude for the system described by the Lagrangian $L$ of (4b) is

$$Z = \int \prod_m \left[\mathrm{d}\,\sigma\,(m)\,\delta\,(\sigma^2\,(m) - 1)\right]\,\exp\,[\,i\,S\,[\sigma]], \tag{30}$$

with

$$S\,[\sigma] = \int \mathrm{d}t \left[\frac{a}{2g}\sum_m \dot{\sigma}\,(m)\cdot\dot{\sigma}\,(m) + \frac{1}{ag}\sum_m \sigma\,(m)\cdot\sigma\,(m+1)\right]. \tag{31}$$

We shall introduce the collective field $q_{mn}$ in $Z$ as

$$Z = \int \prod_{m} [\mathrm{d}\,\sigma(m)\,\delta(\sigma^2(m)-1)] \prod_{m,\,n} [\mathrm{d}q_{mn}\,\delta\,(q_{mn}-\sigma\,(m)\cdot\sigma(n))]\,\exp\,[iS] \quad (32)$$

Since we want to calculate $\langle q_{mn}\rangle$, we introduce a source for the $\sigma$ field as

$$Z(\mathbf{J}) = \int \mathscr{D}\,\sigma\,\mathscr{D}q\,\delta\,(q_{mn} - \sigma\,(m)\cdot\sigma\,(n))\,\delta\,(\sigma^2 - 1)$$

$$\exp\,[i\,(S\,[\sigma] + \int \mathrm{d}t\,(\sigma\,(m)\cdot\mathbf{J}\,(m))]. \quad (33)$$

Then the required vacuum expectation value of $q_{mn}$ is

$$\langle q_{mn}\,(t)\rangle_{\mathrm{vac}} = \langle\sigma\,(m)\cdot\sigma\,(n)\rangle_{\mathrm{vac}},$$

$$= \frac{1}{i^2}\,\frac{\delta^2\ln Z(\mathbf{J})}{\delta\,J_a\,(m,\,t)\,\delta\,J_a\,(u,\,t)}\bigg|_{\mathbf{J}=0}. \quad (34)$$

We rewrite $Z\,(\mathbf{J})$ as

$$Z\,(\mathbf{J}) = \int \mathscr{D}\,\sigma\,\mathscr{D}\,q\,\delta\,(q_{mn} - \sigma\,(m)\cdot\sigma\,(n))\,\delta\,(q_{mm} - 1)$$

$$\exp\,i\,[(S\,[\sigma] + \int \sigma\cdot\mathbf{J}\,\mathrm{d}t)],$$

$$= \int \mathscr{D}\,q\,\mathscr{D}\,\sigma\,\mathscr{D}\,\gamma\,\delta\,(q_{mm} - 1)\,\exp\left[i\int \mathrm{d}t\,\left\{\frac{a}{2g}\,\dot{\sigma}\,(m)\cdot\dot{\sigma}\,(m)\right.\right.$$

$$\left.\left. + \frac{1}{ag}\,q_{m,\,m+1} + \mathbf{J}\cdot\sigma + \frac{1}{2}\,\gamma_{mn}\,(\sigma\,(m)\cdot\sigma\,(n) - q_{mn})\right\}\right], \quad (35)$$

and integrate out the $\sigma$ field as

$$Z\,(\mathbf{J}) = \int \mathscr{D}\,q\,\mathscr{D}\,\gamma\,\delta\,(q_{mm} - 1)\,\exp\left[-\frac{N}{2}\,\mathrm{tr}\ln O + i\int \mathrm{d}t\,\left(\frac{1}{ag}\,q_{m,\,m+1}\right)\right.$$

$$\left. - i\int \mathrm{d}t\,\left(\frac{1}{2}\,\gamma_{mn}\,q_{mn}\right) - \frac{i}{2}\int \mathrm{d}t\,\mathrm{d}t'\,J_a\,(m,\,t)\,O^{-1}_{mn}(t,\,t')\,J_a\,(n,\,t')\right], \quad (36)$$

where

$$O_{mn}\,(t,\,t') = \left(-\frac{a}{g}\,\partial_t^2\,\delta_{mn} + \gamma_{mn}\,(t)\right)\,\delta\,(t - t'), \quad (37)$$

$$\int \mathrm{d}t'\,\sum_{n} O_{mn}\,(t,\,t')\,O^{-1}_{nk}\,(t'\,t'') = \delta_{mk}\,\delta\,(t - t''). \quad (38)$$

We shall define a new variable

$$\beta_{mn} = g\,\gamma_{mn}, \tag{39}$$

and write $Z(\mathbf{J})$ as

$$Z(\mathbf{J}) = (\text{constant}) \int \mathscr{D}q\, \mathscr{D}\beta\, \mathscr{D}\gamma \exp\left[ iNS[q, \beta, \lambda] - \frac{i}{2}\int \mathbf{J}\, O^{-1}\, \mathbf{J}\, dt\, dt' \right], \tag{40}$$

where the action $S[q, \beta, \lambda]$ is given by

$$S[q, \beta, \lambda] = \frac{i}{2} \operatorname{tr} \ln(-a\,\partial_t^2\, \delta_{mn} + \beta_{mn}) - \frac{1}{2}\int dt\, \lambda_m (q_{mm} - 1)$$

$$+ \int dt \left( \frac{1}{ag_0} q_{m,\,m+1} - \frac{1}{2g_0} \beta_{mn}\, q_{mn} \right). \tag{41}$$

From now on, we shall drop any J-independent multiplicative constant appearing in front of $Z(\mathbf{J})$. From the original definition of $\langle q_{mn} \rangle$ in (34),

$$\langle q_{mn}(t) \rangle_{\text{vac}} = i\,N\, \langle O^{-1}_{mn}(t, t) \rangle_{q,\,\beta,\,\lambda} \tag{42}$$

where $\langle (\ldots) \rangle_{q,\,\beta,\,\lambda}$ means

$$\langle (\ldots) \rangle_{q,\,\beta,\,\lambda} = \frac{\int \mathscr{D}q\, \mathscr{D}\beta\, \mathscr{D}\lambda\, (\ldots) \exp[i\,N\,S[q, \beta, \lambda]]}{\int \mathscr{D}q\, \mathscr{D}\beta\, \mathscr{D}\lambda \exp[i\,N\,S[q, \beta, \lambda]]}. \tag{43}$$

Since $g_0 = gN$ is kept fixed when large $N$ limit is taken, the action $S[q, \beta, \lambda]$ does not depend on $N$. Because of the explicit presence of $N$ in the exponent, (43) is very suitable for a diagrammatic expansion in powers of $1/N$. The first step is to find the saddle point $(q^0_{mn}, \beta^0_{mn}, \lambda^0)$ of the action $S[q, \beta, \lambda]$:

$$\left. \frac{\delta S[q, \beta, \lambda]}{\delta(q_{mn}, \beta_{mn}, \lambda_m)} \right|_{q^0,\, \beta^0,\, \lambda^0} = 0. \tag{44}$$

If we now expand $S[q, \beta, \lambda]$ around the saddle point given by (44) as

$$(q_{mn}, \beta_{mn}, \lambda_m) = (g^0_{mn}, \beta^0_{mn}, \lambda^0_m) + (\tilde{q}_{mn}, \tilde{\beta}_{mn}, \tilde{\lambda}_m), \tag{45}$$

the trace ln term becomes

$$\operatorname{tr} \ln(-a\partial_t^2\, \delta_{mn} + \beta_{mn}) = \operatorname{tr} \ln G^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr}(G\,\tilde{\beta})^n, \tag{46}$$

where $G$ is given by

$$(-a\partial_t^2\,\delta_{mn} + \beta^0_{mn})\,G_{nk}\,(t, t') = \delta_{mk}\,\delta\,(t - t'). \tag{47}$$

The saddle point is then given by (assuming static solution)

$$q^0_{mn} = 1, \tag{48a}$$

$$\frac{1}{2ag_0}(\delta_{m,\,n+1} + \delta_{n,\,m+1}) - \frac{1}{2g_0}\beta^0_{mn} - \tfrac{1}{2}\,\lambda^0_m\,\delta_{mn} = 0. \tag{48b}$$

$$-\frac{1}{2g_0}q^0_{mn}\,(t) + \frac{i}{2}\,G_{mn}\,(t, t) = 0. \tag{48c}$$

As before, we demand that $q^0_{mn}$ and $\beta^0_{mn}$ are functions of $(m - n)$ only and $\lambda^0_m$ independent of $m$. Then the solutions are

$$\lambda^0_m = \lambda, \tag{49a}$$

$$q^0_{mn} = \frac{1}{M}\sum_{l=1}^{M} \exp\,[2\pi i\,(m - n)\,l/M]\;q(l), \tag{49b}$$

$$q\,(l) = [4a\,\lambda/g_0 - 8\cos\,(2\pi\,l/M)/\,g_0^2]^{-1/2}, \tag{49c}$$

$$\beta^0_{mn} = \frac{1}{M}\sum_{l-1}^{M} \exp\,[2\pi i\,(m - n)\,l/M]\;\beta\,(l), \tag{49d}$$

$$\beta(l) = g_0\,\lambda + \frac{2}{a}\cos\,(2\pi\,l/M), \tag{49e}$$

and $G_{mn}\,(t, t')$ is given by

$$G_{mn}\,(t, t') = \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty}\frac{dw}{2\pi}G\,(w, l)\,\exp\,[2\pi i\,(m - n)\,l/M + i\,w\,(t - t')], \tag{50a}$$

$$G\,(w, l) = [aw^2 + \beta\,(l) + i\,\epsilon]^{-1},\quad \epsilon \to 0^+. \tag{50b}$$

Thus we get the same result as in (19).

### 3.2 $1/N$ expansion and Feynman diagrams

After shifting the fields as in (45), the expression for $Z(\mathbf{J})$ becomes

$$Z(\mathbf{J}) = \exp(iNS_{cl}) \int \mathscr{D}\tilde{q}\ \mathscr{D}\tilde{\beta}\ \mathscr{D}\tilde{\lambda} \exp\Big[iN\Big\{\frac{i}{2}\ \mathrm{tr} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} (G\,\tilde{\beta})^n,$$

$$-\frac{1}{2}\int \mathrm{d}t\ \tilde{\lambda}_m \tilde{q}_{mm} - \frac{1}{2g_0}\int \mathrm{d}t\ \tilde{\beta}_{mn}\ \tilde{q}_{mn}\Big\} - \frac{i}{2}\int \mathrm{d}t\ \mathrm{d}t'\ \mathbf{J}(t)\ \mathrm{O}^{-1}(t,t')\ \mathbf{J}(t'), \tag{51}$$

where $S_{cl}$ is the classical value of $S\,[q, \beta, \lambda]$. So far as a general Green's function of the original $\sigma$ field is concerned, we can integrate out the $\tilde{q}$ and the $\tilde{\lambda}$ fields and get

$$Z(\mathbf{J}) = \int \mathscr{D}\,\tilde{\beta}_m \exp\Big[-\frac{N}{2}\ \mathrm{tr} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} (G\,\tilde{\beta})^n - \frac{i}{2}\int \mathrm{d}t\,\mathrm{d}t'\ \mathbf{J}\ \mathrm{O}^{-1}\ \mathbf{J}\Big], \tag{52}$$

where $\tilde{\beta}_m$ is just the diagonal part of $\tilde{\beta}_{mn}$

$$\tilde{\beta}_m = \tilde{\beta}_{mm}, \tag{53}$$

and everywhere in the exponent of (52) we have to substitute

$$\tilde{\beta}_{mn} = \tilde{\beta}_m\ \delta_{mn}. \tag{54}$$

Then $\langle q_{mn}(t)\rangle$ is given by

$$\langle q_{mn}(t)\rangle = i\,N\,\langle \mathrm{O}^{-1}_{mn}\ (t, t)\rangle_{\tilde{\beta}},$$

$$= i\,N\,\langle \mathrm{O}^{-1}_{mn}\ (t, t)\rangle_{\phi}, \tag{55}$$

where

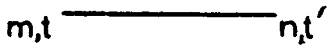
$$\phi_m = \tilde{\beta}_m\,(N)^{1/2}, \tag{56}$$

$$S\,[\phi] = -\frac{1}{2} N^{1-n/2} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}\ \mathrm{tr}\ (G\phi)^n, \tag{57}$$

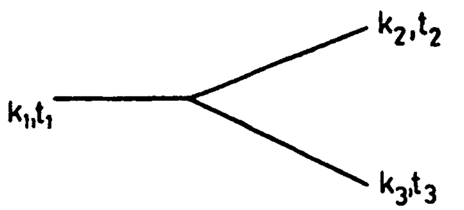
and $\langle(\ldots)\rangle_{\phi}$ is given by

$$\langle(\ldots)\rangle_{\phi} = \frac{\int \mathscr{D}\phi(\ldots) \exp(S[\phi])}{\int \mathscr{D}\phi \exp(S[\phi])}. \tag{58}$$

Equation (58) leads to a natural $1/N$ expansion in terms of Feynman diagrams. From (57) it is easy to see that the $\phi$-propagator is of order (1), the $\phi\phi\phi$-vertex is of order $(N^{-1/2})$, the $\phi\phi\phi\phi$-vertex is of order $(N^{-1})$, and so on. In general a $k$-th order vertex is of order $(N^{1-k/2})$. The Feynman rules are




$= \phi\text{-propagator} = -2D^{-1}_{mn}(t, t'),$




$$= -\tfrac{1}{6} N^{-1/2}\, G_{k1k2}(t_1, t_2)\, G_{k2k3}(t_2, t_3)\, G_{k3k1}(t_3, t_1)$$

and similar rules for other vertices. We have not shown them explicitly since we do not need them to calculate $\langle q_{mn}\rangle$ to order $(1/N)$. The matrix $D_{mn}(t, t')$ is defined by

$$D_{mn}(t, t') = G_{mn}(t, t')\, G_{nm}(t', t), \tag{59a}$$

$$\int \mathrm{d}t' \sum_n D_{mn}(t, t')\, D^{-1}_{nk}(t', t'') = \delta_{mk}\, \delta(t - t''), \tag{59b}$$

Expanding $O^{-1}$ in (55) as

$$O^{-1} = \left[\frac{1}{g}\left(-a\,\partial_t^2\,\delta_{mn} + \beta^0_{mn} + \left(\frac{1}{N}\right)^{1/2} \phi_m\,\delta_{mn}\right)\right]^{-1}$$

$$= g\left[G - \left(\frac{1}{N}\right)^{1/2} G\phi G + \frac{1}{N} G\phi G\phi G + \theta(N^{-3/2})\right], \tag{60}$$

$\langle q_{mn}(t)\rangle$, up to order $(1/N)$, is given by

$$\langle q_{mn}(t)\rangle = ig_0\left[G_{mn}(t, t) - \left(\frac{1}{N}\right)^{1/2} \int \mathrm{d}t'\, G_{mk}(t, t')\, \langle\phi_k(t')\rangle\, G_{kn}(t', t)\right.$$

$$+ \frac{1}{N}\int \mathrm{d}t'\, \mathrm{d}t''\, G_{mk}(t, t')\, G_{kl}(t', t'')\, G_{ln}(t'', t)\, \langle\phi_k(t')\,\phi_l(t'')\rangle$$

$$+ \theta(N^{-3/2}). \tag{61}$$

The tadpole $\langle\phi_k(t)\rangle$ can be calculated (up to order $N^{-1/2}$)

$$\langle\phi_k(t)\rangle = \langle\phi\rangle$$

$=$ 


$$= \frac{-2}{(N)^{1/2}}\frac{1}{M^2} \sum_{l_1, l_2} \int \frac{\mathrm{d}w_1\, \mathrm{d}w_2}{(2\pi)^2} D^{-1}(0, 0)$$

$$D^{-1}(w_1 - w_2, l_1 - l_2)\, G^2(w_1, l_1)\, G(w_2, l_2). \tag{62}$$

So, finally,

$$\langle g_{mn}(t)\rangle = i\, g_0\,(a+b+c), \tag{63a}$$

$$a = \frac{1}{M}\sum_{l} \exp\,[2\pi\, i\,(m-n)\, l/M] \int \frac{\mathrm{d}w}{2\pi}\, G(w, l), \tag{63b}$$

$$b = -\frac{\langle\phi\rangle}{(N)^{1/2}}\,\frac{1}{M}\sum_{l} \exp\,[2\pi\, i\,(m-n)\, l/M] \int \frac{\mathrm{d}w}{2\pi}\, G^2\,(w, l), \tag{63c}$$

$$c = -\frac{2}{NM^2}\sum_{l_1, l_2} \int \frac{\mathrm{d}w_1\, \mathrm{d}w_2}{(2\pi)^2} \exp\,[2\pi\, i\,(m-n)\, l/M]\; G^2\,(w_1, l_1)\; D^{-1}\,(w_2, l_2)$$

$$G\,(w_1 - w_2,\, l_1 - l_2) \tag{63d}$$

3.3 *Continuum approximation and euclidean rotation*

$G$ is given by (from (50))

$$G_{mn}\,(t, t') = \frac{1}{M}\sum_{l=1}^{M} \int \frac{\mathrm{d}w}{2\pi} \exp\,[2\pi\, i\,(m-n)\, l/M + iw\,(t-t')]\; G(w, l),$$

$$= \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{2\pi}\, \frac{\exp\,[iw\,(t-t') + i\,(m-n)\,\theta]}{a\left[w^2 - \mu^2 - \dfrac{4}{a^2}\sin^2\theta/2 + i\epsilon\right]},$$

$$= \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}k_1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{2\pi}\, \frac{\exp\,[i\,(t-t')\, w + i\, k_1\, a(m-n)]}{w^2 - \dfrac{4}{a^2}\sin^2(\tfrac{1}{2}\, k_1\, a) - \mu^2 + i\,\epsilon}, \tag{64}$$

where we have used (23) for $\mu$. $\theta$ and $k_1$ are defined by

$$\theta = a\, k_1 = 2\pi\, l/M. \tag{65}$$

We now make the continuum approximation (CA)

$$\sin^2(\tfrac{1}{2}\, a\, k_1) = a^2\, k_1^2/4, \tag{66}$$

and get

$$G_{mn}\,(t, t') = \int \frac{\mathrm{d}^2\, k}{(2\pi)^2}\, \frac{\exp\,[ik\cdot(x-y)]}{k^2 - \mu^2 + i\,\epsilon},$$

$$k^2 = k_\mu\, k_\mu = w^2 - k_1^2, \quad x = (ma, t), \quad y = (na, t'). \tag{67}$$

We expect the continuum approximation good for only weak coupling ($g_0 \ll 1$) since strong coupling means large lattice spacing $a$ (see equation (27) for strong coupling $\beta$-function). We now make a " Wick rotation " and finally get

$$\frac{1}{g_0}\langle q_{mn}(t)\rangle = \int \frac{d^2 p \exp\,[i\, p_1\, a\,(m-n)]}{(2\pi)^2\; p^2 + \mu^2 + \Sigma(p)}, \tag{68a}$$

$$\sum(p) = \frac{2}{N}\left[\int \frac{d^2 q}{(2\pi)^2} G(p-q)\, D^{-1}(q) - \int \frac{d^2q\, d^2k}{(2\pi)^4} D^{-1}(0)\, D^{-1}(q)\right.$$
$$\left. G^2(k)\; G(k-q)\right] \tag{68b}$$

where the euclidean functions $G$ and $D$ are given by

$$G(p) = [p^2 + \mu^2]^{-1}, \tag{69a}$$

$$D(p) = \int \frac{d^2 q}{(2\pi)^2} G(q)\, G(q-p)$$
$$= \frac{1}{2\pi\; [p^2\,(p^2 + 4\,\mu^2)]^{1/2}} \ln\left[\frac{(p^2 + 4\,\mu^2)^{1/2} + (p^2)^{1/2}}{(p^2 + 4\,\mu^2)^{1/2} - (p^2)^{1/2}}\right], \tag{69b}$$

$$p^2 = p_1^2 + p_2^2, \tag{69c}$$

Equation (68) was earlier obtained by (Abe 1973; Abe and Hikami 1973) using a slightly different approach. Our approach using Feynman diagrams should be advantageous if one wants to calculate general Green's functions of $\sigma$ field.
To calculate the mass correction to order $(1/N)$, we expand $\Sigma(p)$ around $p^2 = -\mu^2$.

$$\sum(p) = \sum\nolimits_1 + (p^2 + \mu^2)\sum\nolimits_2 + \sum\nolimits_3(p), \tag{70a}$$

where

$$\sum\nolimits_1 = \sum(p^2 = -\,\mu^2), \tag{70b}$$

$$\sum\nolimits_2 = \left.\frac{\partial\, \Sigma(p)}{\partial\, p^2}\right|_{p^2=-\mu^2}. \tag{70c}$$

The wave function renormalisation constant $Z_2$ for the original $\sigma$ field is

$$Z_2 = 1 - \sum\nolimits_2, \tag{71}$$

and the mass correction $\delta\mu^2$ is

$$\delta\mu^2 = \sum\nolimits_1 = \sum(p^2 = -\,\mu^2). \tag{72}$$

Since the expression for $\Sigma(p)$ is divergent, we put an euclidean cut-off $\Lambda_E$ and get

$$\delta \mu^2 = -(4\mu^2/N)\int_0^{\lambda} f(x)\,\mathrm{d}x, \tag{73a}$$

$$\lambda = \Lambda_E^2/4\mu^2, \tag{73b}$$

$$f(x) = \frac{1}{2(x+1)} + \frac{(x/(x+1))^{1/2} - 1}{\ln\left[\dfrac{(x+1)^{1/2} + (x)^{1/2}}{(x+1)^{1/2} - (x)^{1/2}}\right]}. \tag{73c}$$

We shall define the euclidean $\beta$-function as

$$\beta_E(g) = -\Lambda_E \frac{dg}{d\Lambda_E}, \tag{74}$$

and evaluate it from the condition

$$\frac{d}{d\Lambda_E}(\mu^2 + \delta\mu^2) = 0. \tag{75}$$

The reader should notice that $\beta_E$ will, in general, be different from the $\beta$-function defined by (24). However, since $g = 0$ is the critical coupling constant, the two definitions should agree for weak coupling. To evaluate $\beta_E$, we have to find $\mu$ in terms of $\Lambda_E$ from (9) written in euclidean space

$$\frac{1}{g_0} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2}(p^2 + \mu^2)^{-1}, \tag{76a}$$

$$\mu^2 = \Lambda_E^2/(a^2 - 1), \tag{76b}$$

$$a = \exp(2\pi/g_0) \tag{76c}$$

Then, to order $(1/N)$, $\beta_E$ is given by

$$\beta_E(g) = -\frac{g_0^2(1-a^2)}{2\pi N a^2}\left[1 - \frac{a^2-1}{N} f(\lambda) + \theta\left(\frac{1}{N^2}\right)\right], \tag{77}$$

which, for weak coupling, reduces to the known result (Polyakov 1975; Bardeen *et al* 1976; Brezin and Zinn-Justin 1976)

$$\beta_E(g) \simeq g^2(N-2)/2\pi + g^3 N/4\pi^2. \tag{78}$$

As mentioned before, (77) should not be used to find $\beta$-function for strong coupling. Rather, the mass correction should be directly obtained from (63).

## 4. Conclusion

We have demonstrated how the collective field method can be employed either in the Hamiltonian or in the Lagrangian formulation to obtain the large $N$ limit. In the case of gauge theories, this method has the added advantage that the problem of Gribov ambiguity (Gribov 1977; Singer 1978) is completely avoided since all the collective field variables are manifestly singlets. This seems to be a promising method in the sense that the large $N$ limit is achieved rather naturally and the technical problem of summing planar diagrams is reduced to solving a classical equation. As we have shown, $1/N$ corrections can also be computed in this framework and application to various other models are presently under study.

## Appendix

*Cancellation of divergence*

As mentioned in § 2, we shall demonstrate cancellation of divergence in the ground state energy by the $\Delta V$ term in (14). The Hamiltonian is

$$H_{\text{eff}} = \frac{2g}{a} \sum_{m} \sum_{k,k'}{}' p_{mk}\, \Omega\,(m\,k\,k')\, p_{mk'} + N\, V_0(q) + \Delta V. \tag{A1}$$

We expand around the classical solution $q^0_{mn}$ as

$$q_{mn} = q^0_{mn} + Q_{mn}. \tag{A2}$$

Then, the potential energy becomes

$$V_0\,(q) = V_0\,(q^0) + \frac{g_0}{8a} \operatorname{Tr}\,(q^{0-1}\, Q\, q^{0-1}\, Q\, q^{0-1}) + \theta\,(Q^3), \tag{A3}$$

and kinetic energy is

$$\text{KE} = \frac{2g}{a} \sum_{m} \sum_{kk'}{}' p_{mk}\, \Omega^{cl}\,(m,k,k')\, p_{mk} + \theta\,(p^2\, Q), \tag{A4}$$

$$\Omega^{cl}\,(m,k,k') = q^0_{kk'} - q^0_{mk}\, q^0_{mk'}. \tag{A5}$$

The quadratic part of the kinetic energy can be diagonalised. Then $H_{eff}$ reduces to a collection of harmonic oscillators and the ground state energy is given by

$$E = E_0 + \Delta E,$$

$$E_0 = NV_0\,(q^0) + \Delta V\,(q^0),$$

$$\Delta E = \frac{g_0}{2a}(M-1) \operatorname{Tr}\,(q^{0-1}). \tag{A6}$$

The divergent part of $\Delta E$ (proportional to $M$) is cancelled by $\Delta V(q^0)$:

$$\Delta V(q^0) = -\frac{g_0}{2a}(M + \tfrac{1}{2}) \operatorname{Tr}(q^{c-1}) + \theta(1/N). \tag{A7}$$

Similarly, the next order divergence (proportional to $M^2$) in order $(1/N)$ should be cancelled by a higher order contribution to ground state energy $E$.

## Note added in proof

Application of collective field method to $CP^{n-1}$ model has recently been done (A Guha, S C Lee and E Gozzi, Stonybrook preprint, SUSB-ITP-82-3). Introduction of an auxiliary variable through delta function condition in the euclidean version of $O(N)$ sigma model has also been discussed by S Wadia in " The large $N$ limit in quantum field theory ", (TIFR lecture notes, unpublished).

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