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Gauge invariance and renormalization schemes in quantum chromodynamics

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Abstract. A general analysis of the Slavnov-Taylor identity connecting the triple gluon and ghost-ghost-gluon vertices and its consequences for two momentum subtraction (symmetric and asymmetric) renormalization schemes are given. It is shown that in the asymmetric scheme proposed in this paper the relation $Z_3Z_1^{-1} = \tilde{Z}_3\tilde{Z}_1^{-1}$ follows directly from the identity for a simple and natural definition of the renormalization constants. Explicit one-loop expressions for the renormalization constants (Z_1, Z_3, $\tilde{Z}_1$, $\tilde{Z}_3$) in an arbitrary covariant gauge, including quark masses are given in support of the general analysis.

Keywords. Quantum chromodynamics; momentum subtraction; renormalization schemes; gauge invariance.

1. Introduction

The Slavnov-Taylor (S-T) identities (Slavnov 1972; Taylor 1971) for quantum chromodynamics (QCD) are an exact consequence of local gauge symmetry. These identities are extremely important and useful in proving the renormalizability and unitarity of QCD. The identities also imply certain relations connecting the various renormalization constants. In particular, the S-T identity connecting the triple gluon vertex and the ghost-ghost-gluon vertex implies the relation

$$Z_3 Z_1^{-1} = \tilde{Z}_3\tilde{Z}_1^{-1}, \tag{1}$$

among the four renormalization constants, Z_1 for the triple gluon vertex, $\tilde{Z}_1$ for the ghost-ghost-gluon vertex, Z_3 for the gluon wavefunction and $\tilde{Z}_3$ for the ghost wavefunction. That (1) must be satisfied can also be seen directly from the QCD Lagrangian if both the bare and the renormalized Lagrangians are to have Becchi-Rouet-Stora invariance (de Rafael 1979; Becchi *et al* 1975). Moreover, this relation guarantees that the renormalized coupling constants for the two vertices are the same. However, (1) is not satisfied in all renormalization schemes even when a gauge invariant regularization procedure is used. This problem exists (though is not generally recognized) in quantum electrodynamics (QED) also, where it can be shown that the equality of the photon-electron vertex and the electron wavefunction renormalization constants (using the usual definitions) does not hold in all renormalization schemes.

Of the three renormalization schemes discussed in the literature, the minimal scheme, MS ('t Hooft 1973), and the $\overline{\text{MS}}$ scheme (Bardeen *et al* 1978) satisfy (1). Unfortunately, however, these schemes give rather large higher order corrections in perturbative QCD*. The symmetric momentum subtraction scheme (SMOS) gives small higher order corrections, but (1) is not satisfied in this scheme for a natural definition of the renormalization constants (Celmaster and Gonsalves 1979; Chiu 1981). In this paper we propose a new renormalization scheme for QCD, the asymmetric momentum subtraction scheme (AMOS)** and show that (1) is true to all orders in this scheme for a simple and natural definition of the renormalization constants. We substantiate this result by explicit calculations to one-loop order in an arbitrary covariant gauge including quark masses.

The organization of the paper is as follows. In §2a general and detailed analysis of the relevant S-T identity is carried out. Consequences of this analysis are studied in § 3 where (1) is proved for AMOS. Also, in this section, a discussion for SMOS is given for comparison. Results of one loop calculations for the renormalization constants are presented in § 4, followed by concluding remarks in § 5.

2. General analysis of the S-T identity

The S-T identity satisfied by the proper triple gluon vertex $\Gamma^{ijk}_{\mu\nu\lambda}$ and the associated ghost-ghost-gluon vertex $G^{ijk}_{\mu\sigma}$ is

$$\begin{aligned} - p^{\mu}\, \Gamma^{ijk}_{\mu\nu\lambda}\,(p, q, r)\,[1 + b\,(p^2)] \\ = G^{jik}_{\nu\sigma}\,(q, p, r)\,(r^2 g^{\sigma}_{\lambda} - r^{\sigma} r_{\lambda})\,[1 + \pi(r^2)] \\ + G^{kij}_{\lambda\sigma}\,(r, p, q)\,(q^2 g^{\sigma}_{\nu} - q^{\sigma} q_{\nu})\,[1 + \pi(q^2)]. \end{aligned} \tag{2}$$

The various quantities appearing in (2) have been defined in figure 1. The dependence on the covariant gauge parameter α has not been indicated in (2) for simplicity. The functions b and π are connected with Z_3 and $\tilde{Z}_3$, the precise definition depends on the renormalization scheme used. The relevant dependence on the colour group indices (i, j and k) of the vertices, using charge conjugation invariance, appears through the structure constants f^{ijk}. Without loss of generality one can write†

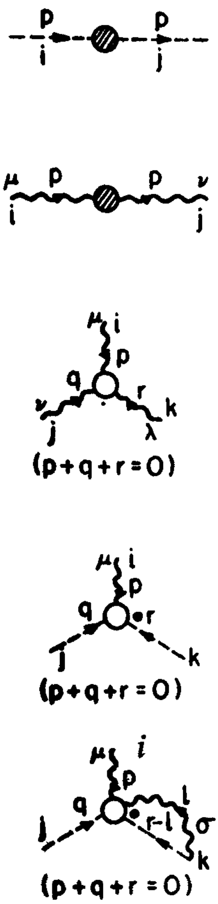
$$\Gamma^{ijk}_{\mu\nu\lambda}\,(p, q, r) = g\, f^{ijk}\;\Gamma_{\mu\nu\lambda}\,(p, q, r), \tag{3a}$$

$$G^{ijk}_{\mu\nu}\,(p, q, r) = g\, f^{ijk}\;G_{\mu\nu}\,(p, q, r), \tag{3b}$$

*For an alternative approach to renormalization scheme dependence of perturbative approximants to physical quantities (which is a problem in any perturbative field theory) see Stevenson (1981a, b).

**The asymmetric scheme for QCD was first proposed by us in a preliminary version of the present paper which was issued as preprint no. TIFR/TH/80-32 in October 1980.

†Equation (3a) can be proved using the formal expression for triple gluon vertex as a vacuum expectation value of the T-product of three gluon fields and charge conjugation invariance. In (3b) terms proportional to d^{ijk} (completely symmetric in i, j and k) which might possibly arise in higher loop orders are not relevant in the present context since (2), due to (3a), does not connect them with the triple gluon vertex.



(a) Full ghost propagator
$G^{ij}(p^2) = \frac{i\delta^{ij}}{p^2}(1 + b(p^2))^{-1}$

(b) Full gluon propagator (transverse (tr) part)
$D^{ij}_{\mu\nu}(p^2)^{\mathrm{tr}} = -\frac{i\delta^{ij}}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)(1 + \pi(p^2))^{-1}$

(c) Proper triple gluon vertex
$\gamma^{ijk}_{\mu\nu\lambda}(p, q, r)$

(d) Proper ghost-ghost-gluon vertex
$G^{ijk}_{\mu}(p, q, r) \equiv r^\sigma G^{ijk}_{\mu\sigma}(p, q, r)$

(e) Associated (proper) ghost-ghost-gluon vertex
$G^{ijk}_{\mu\sigma}(p, q, r)$

Figure 1. Definitions of various quantities appearing in the Slavnov-Taylor identity. $i, j, k, \ldots$ are colour indices; $\mu, \nu, \lambda, \ldots$ are Lorentz indices; $p, q, r, \ldots$ are four-momenta.

where g is the dimensionless unrenormalized coupling constant. In terms of $\Gamma_{\mu\nu\lambda}$ and $G_{\mu\nu}$ defined by (3), the S-T identity in (2) reduces to

$$p^\mu \Gamma^{pqr}_{\mu\nu\lambda} = G^{qpr}_{\nu\sigma}(r^2 g^\sigma_\lambda - r^\sigma r_\lambda) R^{rp} - G^{rpq}_{\lambda\sigma}(q^2 g^\sigma_\nu - q^\sigma q_\nu) R^{qp}, \tag{4}$$

where the definitions

$$\Gamma^{pqr}_{\mu\nu\lambda} \equiv \Gamma_{\mu\nu\lambda}(p, q, r), \tag{5a}$$

$$G^{pqr}_{\mu\nu} \equiv G_{\mu\nu}(p, q, r), \tag{5b}$$

$$R^{qp} \equiv R(q^2, p^2) \equiv \frac{1 + \pi(q^2)}{1 + b(p^2)}, \tag{5c}$$

have been used. R^{rp} is obtained by replacing q by r in (5c). To make further use of (4) we need the general Lorentz structure of the tensors appearing in it.

2.1 *Ghost-ghost-gluon vertex*

The most general Lorentz structure of the tensor $G^{pqr}_{\mu\sigma}$ is given by

$$G^{pqr}_{\mu\sigma} = G^{pqr}_1 \, g_{\mu\sigma} + G^{pqr}_2 \, r_\mu \, r_\sigma + G^{pqr}_3 \, r_\mu \, q_\sigma + G^{pqr}_4 \, r_\sigma \, q_\mu + G^{pqr}_5 \, q_\mu \, q_\sigma. \tag{6}$$

In (6) we have used the superscripts to denote the arguments (as in (5c)) of the five invariant functions, so that $G^{pqr}_i \equiv G_i\,(p^2, q^2, r^2)$. It is clear that in general $G^{pqr}_i \neq G^{qrp}_i \neq G^{rpq}_i$, etc. We will use this notation throughout the subsequent analysis. Writing

$$G^{ijk}_\mu\,(p, q, r) = g\, f^{ijk}\, G^{pqr}_\mu = \tilde{G}^{pqr}_1 r_\mu + \tilde{G}^{pqr}_2\, q_\mu, \tag{7}$$

and using the relation in figure (1d) and (6), one obtains

$$\tilde{G}^{pqr}_1 = G^{pqr}_1 + r^2\, G^{pqr}_2 + (q.r)\, G^{pqr}_3, \tag{8a}$$

$$\tilde{G}^{pqr}_2 = r^2\, G^{pqr}_4 + (q.r)\, G^{pqr}_5. \tag{8b}$$

2.2 *Triple gluon vertex*

The most general form of $\Gamma^{pqr}_{\mu\nu\lambda}$ will involve nine terms like $p_\mu g_{\nu\lambda}$, etc., linear in the momenta, and twenty-seven three momenta terms, like $p_\mu p_\nu q_\lambda$, $p_\mu q_\nu r_\lambda$, etc. Each of these will be multiplied by a function of the invariants constructed out of p, q and r. However, the imposition of Bose symmetry, which requires the tensor to be anti-symmetric under the exchange of any pair of momenta and the corresponding Lorentz indices (*i.e.* $\Gamma^{pqr}_{\mu\nu\lambda} = -\Gamma^{qpr}_{\nu\mu\lambda}$, etc.), reduces the number of independent invariant functions to nine. Finally, the use of energy-momentum conservation, $p + q + r = 0$, reduces this number to three. In terms of the two independent momenta q and r, one obtains

$$\begin{aligned}\Gamma^{pqr}_{\mu\nu\lambda} = \Big[&- S^{pqr}_1\, g_{\mu\nu}\, q_\lambda + \Gamma^{qrp}_1\, g_{\nu\lambda}\, q_\mu + \Gamma^{prq}_1\, g_{\lambda\mu}\, q_\nu \\ &+ S^{pqr}_2\, q_\mu\, q_\nu\, q_\lambda + \Gamma^{pqr}_2\, q_\mu\, q_\nu\, r_\lambda \\ &+ \left(S^{pqr}_2 - \Gamma^{qpr}_3\right) q_\nu\, q_\lambda\, r_\mu + \Gamma^{pqr}_3\, q_\lambda\, q_\mu\, r_\nu \Big] \\ &- [q \longleftrightarrow r, \nu \longleftrightarrow \lambda],\end{aligned} \tag{9a}$$

where $$S^{pqr}_{1,2} \equiv \Gamma^{pqr}_{1,2} + \Gamma^{qpr}_{1,2}, \tag{9b}$$

and the functions Γ_2^{pqr} and Γ_3^{pqr} satisfy the relations

$$\Gamma_2^{pqr} + \Gamma_2^{prq} - S_2^{qrp} = \Gamma_3^{qpr} - \Gamma_3^{rqp}, \tag{9c}$$

$$\Gamma_3^{pqr} + \Gamma_3^{prq} = \Gamma_3^{qrp} + \Gamma_3^{qpr}, \tag{9d}$$

and their cyclic permutations. However only four of these are independent. The function Γ^{pqr} has no symmetry properties with respect to the interchange of its arguments.

2.3 *Relations from the S-T identity*

The general expressions in (6) and (9) are substituted in (4) to obtain relations among the invariant functions appearing in the ghost-ghost-gluon and the triple gluon vertices.

One obtains

$$r^2 R^{rp} G_1^{qpr} - q^2 R^{qp} G_1^{rpq} = (p\cdot q)\, \Gamma_1^{qrp} - (p\cdot r)\, \Gamma_1^{rqp}, \tag{10a}$$

$$R^{qp} G_1^{rpq} + r^2 R^{rp} G_5^{qpr} + (q\cdot r) R^{qp} (G_5^{rpq} - G_3^{rpq})$$

$$= S_1^{pqr} - \Gamma_1^{prq} - p^2 S_2^{pqr} - (p\cdot r)\, \Gamma_3^{qpr}, \tag{10b}$$

$$R^{rp} G_1^{qpr} + q^2 R^{qp} G_5^{rpq} + (q\cdot r) R^{rp} (G_5^{qpr} - G_3^{qpr})$$

$$= S_1^{prq} - \Gamma_1^{pqr} - p^2 S_2^{prq} - (p\cdot q)\, \Gamma_3^{rpq}, \tag{10c}$$

$$(q\cdot r)(R^{qp} G_5^{rpq} - R^{rp} G_5^{qpr})$$

$$= \Gamma_1^{pqr} - \Gamma_1^{prq} + (p\cdot q)\, \Gamma_2^{pqr} - (p\cdot r)\, \Gamma_2^{prq}, \tag{10d}$$

$$(r^2 R^{rp} G_5^{qpr} - q^2 R^{qp} G_5^{rpq}) - (r^2 R^{rp} G_3^{qpr} - q^2 R^{qp} G_3^{rpq})$$

$$= S_1^{pqr} - S_1^{rpq} + (p\cdot q)\, \Gamma_3^{pqr} - (p\cdot r)\, \Gamma_3^{prq}. \tag{10e}$$

Equations (10) and their cyclic permutations constitute a set of fifteen equations which are a consequence of the gauge invariance of the theory. Perturbative calculation using a gauge invariant regularization scheme must respect them to all orders. These relations can be solved for one set of invariant functions in terms of the others, but this is not necessary for the analysis presented below.

3. Analysis of different renormalization schemes

We use dimensional regularization ('t Hooft and Veltman 1972; Bollini and Gambiagi 1972) in which divergences are regulated by continuing loop integrals from 4 to n dimensions. The divergences appear as poles in $\epsilon = 4 - n$ as $\epsilon \to 0^+$. The only superficially divergent quantities are the functions $\pi(p^2)$, $b(p^2)$, Γ_1^{pqr} and G_1^{pqr} or $\tilde{G}_1^{pqr}$ (Lee and Zinn-Justin 1972).

Let H stand for any of the functions $(1+\pi)^{-1}$, $(1+b)^{-1}$, Γ_1 and G_1. The corresponding renormalized function and the renormalization constant will be denoted by H_R and Z_H respectively. We define Z_H by the equation

$$H = Z_H H_R. \tag{11a}$$

The quantities H, H_R and Z_H have the perturbation expansions

$$H = 1 + \sum_{n=1}^{\infty} g^{2n} H^{(n)}, \tag{11b}$$

$$H_R = 1 + \sum_{n=1}^{\infty} g_R^{2n} H_R^{(n)} \tag{11c}$$

and

$$Z_H = 1 + \sum_{n=1}^{\infty} g^{2n} Z_H^{(n)} \tag{11d}$$

where g_R is the renormalized coupling constant and has the expansion

$$g_R = gZ_g = g\left[1 + \sum_{n=1}^{\infty} g^{2n} Z_g^{(n)}\right]. \tag{11e}$$

$H^{(n)}$ is the sum of a divergent part, $H_d^{(n)}$, and a finite part, $H_f^{(n)}$, which have the general form

$$H_d^{(n)} = \sum_{m=1}^{n} H_d^{(n,m)} \epsilon^{-m}, \quad H^{(n)} = \sum_{m=0}^{\infty} H^{(n,m)} \epsilon^{m}. \tag{11f}$$

$H_R^{(n)}$ is finite and has the general form

$$H_R^{(n)} = \sum_{m=0}^{\infty} H_R^{(n,m)} \epsilon^{m}. \tag{11g}$$

In the MS scheme all $Z_H^{(n)}$ and $Z_g^{(n)}$ are defined to contain only the pole terms. Using (10) and (11) it is then easy to verify that (1) is satisfied in this scheme, as is well known.

In a momentum subtraction scheme, apart from the divergent part, the renormalization constants contain a finite part as well. The value of the latter is fixed by choosing definite values for the external momenta (*viz.* the subtraction point). Different choices give different momentum subtraction schemes. In all cases, one chooses H_R [(11a)] to be unity at the subtraction point, so that Z_H is simply given by the value of H at that point.

3.1 *Symmetric momentum subtraction scheme* (SMOS)

In SMOS, one chooses a symmetric subtraction point, $p^2 = q^2 = r^2 = -M^2$, where M is some arbitrary mass. The renormalization constants are then defined by

$$Z_3^{-1} \equiv 1 + \pi(-M^2), \quad \tilde{Z}_3^{-1} \equiv 1 + b(-M^2), \tag{12}$$

$$Z_{1S}^{-1} \equiv \Gamma_1^{pqr}\big|_{p^2 = q^2 = r^2 = -M^2} \tag{13}$$

$$\tilde{Z}_{1S}^{-1} \equiv G_1^{pqr}\big|_{p^2 = q^2 = r^2 = -M^2} \tag{14}$$

The subscript S (for symmetric) is used for Z_1 and $\tilde{Z}_1$ since, unlike Z_3 and $\tilde{Z}_3$, they are different in SMOS and AMOS. In SMOS, (12) provides the most natural definition of Z_1 since at the symmetric point the tensor $\Gamma^{pqr}_{\mu\nu\lambda}$ [see (9)] reduces to the form Γ_1 $(\Gamma_{\mu\nu\lambda})_{\text{bare}}$ + 3 momenta terms. The definition of $\tilde{Z}_1^{-1}$ is different from the customary one. Usually the coefficient of r_μ in $G_\mu^{ijk}(p, q, r)$ (written in terms of the pair (r, q) or (r, p) of independent momenta) is defined to be $\tilde{Z}_1^{-1}$. However, the conclusions of this section remain unchanged regardless of the definition used.

At the symmetric point (10a, d and e) are identically satisfied while, using the above definitions, (10b, or c) yields

$$\tilde{Z}_3\left[\tilde{Z}_{1S}^{-1} - \frac{M^2}{2}(G_3 + G_5)\right] = Z_3\left[Z_{1S}^{-1} + \frac{M^2}{2}(4\,\Gamma_2 - \Gamma_3)\right], \tag{15}$$

where G_i and Γ_i are the values of the corresponding functions G_i^{pqr} and Γ_i^{pqr} at the symmetric point. It is clear that (1) is not valid in this scheme. Furthermore, the constants G_3, G_5, Γ_2 and Γ_3 have dimensions of $(\text{mass})^{-2}$, so they must all be proportional to M^{-2}. It is therefore not obvious that even in the limit $M \to 0$ (1) will emerge. One can, of course, force (1) to hold by redefining Z_1 and $\tilde{Z}_1$ to be the combinations in the square brackets in (15). However, this will merely serve to complicate their computation. Moreover, in QED also SMOS fails to give the usual relation between the renormalization constants in a natural way.

3.2 *Asymmetric momentum subtraction scheme* (AMOS)

In this scheme, proposed by us, the vertex renormalization constants are defined by choosing an asymmetric subtraction point in which the four-momentum of one of the external legs is put equal to zero. More precisely, we define

$$Z_{1A}^{-1} = \Gamma_1^{pqr}\big|_{q=0,\, p^2=r^2=-M^2}, \tag{16a}$$

$$\tilde{Z}_{1A}^{-1} = G_1^{pqr}\big|_{p=0,\, q^2=r^2=-M^2}. \tag{16b}$$

The wave function renormalization constants, Z_3 and $\tilde{Z}_3$, are still given by (12). The subscript A stands for the asymmetric prescription used. For these definitions to be useful in practice it is necessary that the functions Γ_1^{pqr} and G_1^{pqr} have no infrared divergences in the limit of q and p going to zero respectively. A power counting proof is available which guarantees this (Poggio and Quinn 1976). Also, our explicit calculations at one loop level (see § 4) do give infrared finite answers for Z_{1A} and $\tilde{Z}_{1A}$ as defined in (16). Here we show that (1) is valid for the above definitions. To see this put $r = 0$ in (10a). This gives

$$R^{pp}\, G_1^{opp} = \Gamma_1^{pop}. \tag{17}$$

Now, setting $p^2 = -M^2$ and using (12) and (16) one obtains $Z_3 Z_{1A}^{-1} = \tilde{Z}_3 \tilde{Z}_{1A}^{-1}$. This means that in AMOS it is possible to satisfy (1) and preserve the S–T identity (during renormalization) to all orders for an arbitrary covariant gauge.

4. One loop calculations

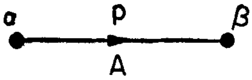
We have performed full one loop calculation of the renormalization constants to substantiate the earlier general discussion. These calculations have been performed using dimensional regularization for an arbitrary covariant gauge parameter a and including quark masses. The Feynman rules used and the diagrams which contribute are shown in figure 2 and figure 3 respectively.

The general expressions for the Z's are of the form

$$\tilde{Z}_3 = 1 + \frac{g^2}{16\pi^2} C_2(G)\left[\left(\frac{3}{2} - \frac{a}{2}\right)\left(\frac{1}{\epsilon} + \bar{\gamma}\right) + \tilde{A}_3 + \tilde{B}_3\, a\right] + O(g^4), \tag{18}$$

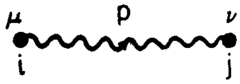
$$\tilde{Z}_1 = 1 + \frac{g^2}{16\pi^2} C_2(G)\left[-a\left(\frac{1}{\epsilon} + \bar{\gamma}\right) + \tilde{A}_1 + \tilde{B}_1\, a + \tilde{C}_1\, a^2\right] + O(g^4), \tag{19}$$

$$Z_3 = 1 + \frac{g^2}{16\pi^2}\left[C_2(G)\left(\frac{13}{3} - a\right) - \frac{8}{3} T(R)\right]\left(\frac{1}{\epsilon} + \bar{\gamma}\right) + \frac{g^2}{16\pi^2}\left[C_2(G)\,(A_3 + B_3\, a + C_3\, a^2) - \frac{8}{3} T(R)\, F_3\right] + O(g^4), \tag{20}$$



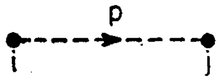
(a) Fermion propagator

$$\frac{i\delta_{\alpha\beta}}{(\gamma.p - m_A)}$$



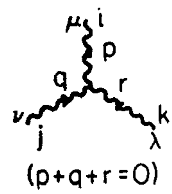
(b) Gluon propagator

$$\frac{-i\delta^{ij}}{p^2}\left[g_{\mu\nu} - (1-\alpha)\frac{p_\mu\, p_\nu}{p^2}\right]$$



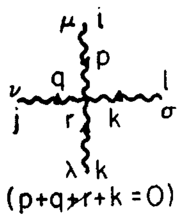
(c) Ghost propagator

$$\frac{i\delta^{ij}}{p^2}$$



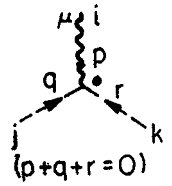
(d) Triple gluon vertex

$$gf^{ijk}\,[g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-r)_\mu + g_{\lambda\mu}(r-p)_\nu]$$



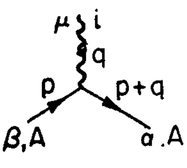
(e) Quartic gluon vertex

$$\begin{aligned}-ig^2\,[&f^{ijm}f^{klm}(g_{\mu\lambda}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\lambda}) \\ &+ f^{ikm}f^{jlm}(g_{\mu\nu}\,g_{\lambda\sigma} - g_{\mu\sigma}\,g_{\nu\lambda}) \\ &+ f^{ilm}f^{kjm}(g_{\mu\lambda}\,g_{\nu\sigma} - g_{\mu\nu}\,g_{\lambda\sigma})]\end{aligned}$$



(f) Ghost-ghost-gluon vertex

$$g\,f^{ijk}\,r_\mu$$



(g) Fermion-gluon vertex

$$ig\gamma_\mu\,T^i_{\alpha\beta}$$

Figure 2. Feynman rules used in calculations. A denotes fermion flavour, while $T^i_{\alpha\beta}$ are generators of the colour group which satisfy the commutation relation $[T^i, T^j] = if^{ijk}\,T^k$. g is the unrenormalized coupling constant and α is the unrenormalized gauge parameter.

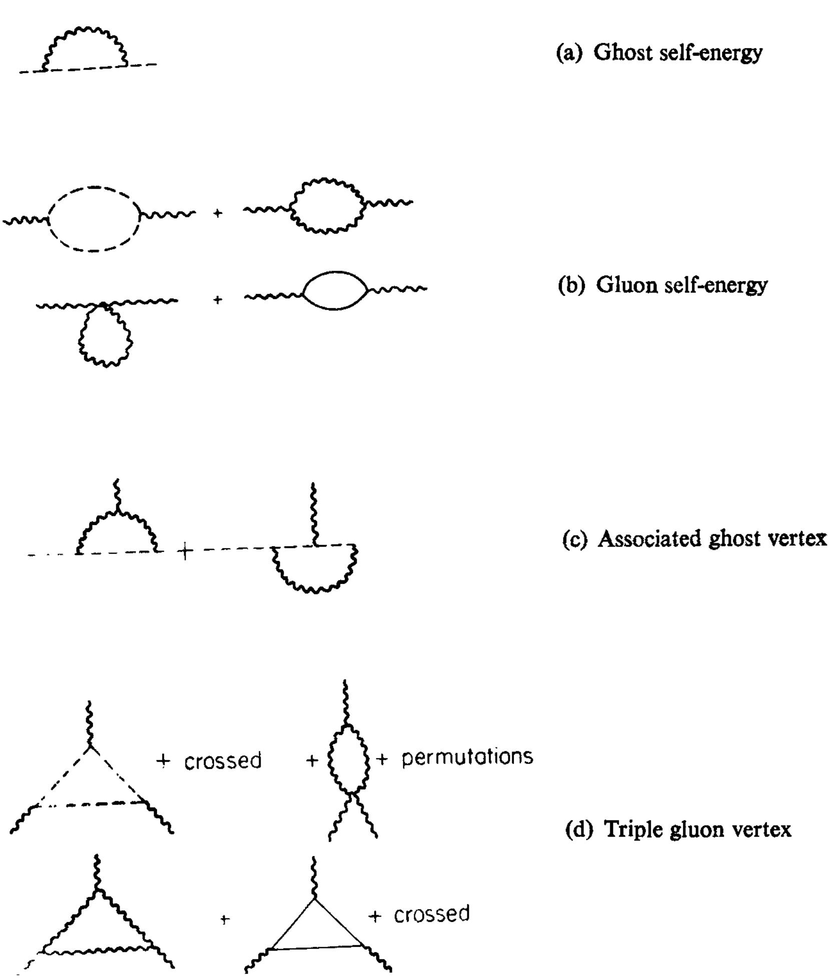


Figure 3. Diagrams which contribute to one-loop order to the renormalization constants.

$$Z_1 = 1 + \frac{g^2}{16\pi^2}\left[C_2(G)\left(\frac{17}{6} - \frac{3}{2}\alpha\right) - \frac{8}{3}T(R)\right]\left(\frac{1}{\epsilon} + \bar{\gamma}\right)$$
$$+ \frac{g^2}{16\pi^2}\left[C^2(G)\,(A_1 + B_1\,\alpha + C_1\,\alpha^2 + D_1\,\alpha^3) - \frac{8}{3}T(R)\,F_1\right]$$
$$+ O(g^4). \tag{21}$$

Here $2\bar{\gamma} \equiv (\ln 4\pi - \gamma_E)$ and γ_E is the Euler-Mascheroni constant. The gauge group (with generators T^i) factors are defined by $f^{iab}\,f^{jab} = \delta^{ij}\,C_2(G)$ and $\mathrm{Tr}(T^i\,T^j) = \delta^{ij}\,T(R)$. For N_f fermion flavours, $2T(R) = N_f$, while $C_2(G) = 3$ for QCD.

In all renormalization schemes the divergent parts (the $1/\epsilon$ terms) are the same, the differences arise in the choice of the finite terms given by the fifteen constants $\bar{\gamma}$, A_i, $\tilde{A}_i, \ldots$, etc. The constants F_1 and F_3 which depend on the quark masses m_A ($A = 1, 2, \ldots N_f$) appear only in Z_1 and Z_3 to one loop order. In the MS scheme only the $1/\epsilon$ part is kept and these satisfy (1), as is well-known. In contrast, in the momentum subtraction schemes the Z's contain a finite part. It is obvious from (18–21) that, in these schemes, unless the coefficient of α^3 (*i.e.* D_1) vanishes and the quark mass contributions (*i.e.* F_1 and F_3) are equal, there is no hope of satisfying (1) in general.

In both SMOS and AMOS Z_3 and $\tilde{Z}_3$ are the same and one finds

$$\tilde{A}_3 = 1,\ \tilde{B}_3 = 0;\ A_3 = \frac{97}{36},\ B_3 = \frac{1}{2},\ C_3 = \frac{1}{4}, \tag{22}$$

$$F_3\,(M_A) = \frac{5}{6} - \frac{1}{2N_f}\sum_{A=1}^{N_f}\left[\ln M_A + 4M_A + \left(\frac{1}{2} + M_A - 4M_A^2\right)J\,(M_A)\right], \tag{23}$$

where

$$J(M_A) = \frac{2}{(1+4M_A)^{1/2}}\ln\left\{\frac{(1+4M_A)^{1/2}+1}{(1+4M_A)^{1/2}-1}\right\},\quad M_A = \frac{m_A^2}{M^2}. \tag{24}$$

The values of the constants appearing in Z_1 and $\tilde{Z}_1$ are different in SMOS and AMOS and these are distinguished by subscripts S and A respectively. In SMOS one can easily verify that $D_{1S} = -1/24$. Moreover, one finds

$$F_{1S}\,(M_A) = \frac{3}{4} - \frac{1}{2N_f}\sum_{A=1}^{N_f}\left[\ln M_A + \left(\frac{1}{2} + 2\,M_A^2\right)J\,(M_A) + \left(\frac{2}{3} - M_A\right)K\,(M_A)\right], \tag{25}$$

where

$$K\,(M_A) = \int_0^1 y\,dy\int_0^1 dx\,[M_A + y - y^2(1 - x + x^2)]^{-1} \tag{26}$$

Since $F_{1S} \neq F_{3S}$ and $D_{1S} \neq 0$ it is not necessary to perform the extremely tedious calculation of the other constants in Z_1 and $\tilde{Z}_1$ to see, in view of the general discussion of § 3a, that in SMOS (1) is not satisfied even to one-loop order, in general (Celmaster and Gonsalves 1979).

In contrast, in AMOS

$$\tilde{A}_{1A} = 0,\quad \tilde{B}_{1A} = -\tfrac{1}{2},\quad \tilde{C}_{1A} = 0; \tag{27a}$$

$$A_{1A} = \frac{61}{36},\quad B_{1A} = 0,\quad C_{1A} = \frac{1}{4},\quad D_{1A} = 0,\quad F_{1A} = F_3. \tag{27b}$$

From (22) and (27) one can easily see that (1) is obeyed in AMOS, to one-loop order, in an arbitrary covariant gauge including quark masses.

5. Concluding remarks

The new renormalization scheme AMOS proposed here has a number of theoretically appealing features in comparison with SMOS. As shown above a simple and natural definition of the Z's satisfies (1) to all orders. This has been verified to one-loop order and explicit expressions for the Z's have been given in an arbitrary covariant gauge including quark masses. Satisfaction of (1) in AMOS guarantees equality of renormalized coupling constants defined using the triple gluon vertex and the ghost-ghost-gluon vertex. Moreover, the asymmetric subtraction point used to define the scheme makes for simpler calculation of Z_1 and $\tilde{Z}_1$. In addition, the higher order perturbative QCD corrections to a number of processes in this scheme (Dhar and Gupta 1981) are found to be small.

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