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Confinement of massless particles: Photon

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Abstract. Confinement of massless particles in a suitably chosen dielectric medium is considered. Light waves of selected frequencies are shown to be confined in a medium with dielectric constant $\epsilon(r) = a/r - b^2$. A wave theoretical analysis gives equispaced frequency spectrum for the confined light, the radial dependence of its electric wave vector resembling that of hydrogen atom wave functions. In the large frequency limit an eikonal approximation of the problem gives elliptic orbits for the confined rays. Higher frequency orbits are shown to be closer to the centre of the medium than the lower frequency ones.

Keywords. Confinement; hydrogen-like wave function; equispaced frequency spectrum; eikonal approximation; elliptic orbits; massless particles; photon.

1. Introduction

The introduction of the concept that hadrons contain ' quarks ' as the more fundamental constituents of matter has been an important achievement in theoretical physics. Quantum Chromodynamics, a field theory based on the SU(3) colour symmetry of quarks has emerged as the most successful candidate theory for describing strong interactions of hadrons. Initially it was believed that quarks were too heavy to be observed at the available accelerator energies; but now it is conjectured that they are very light point particles confined permanently to the interior of hadrons by some mechanism. There have been several attempts to develop a theory of quark confinement at phenomenological level. The MIT-SLAC bag model, the string model and the non-relativistic potential model are the major advances in this direction. Each of the above models, in addition to providing a mechanism for quark confinement, describes some important properties of hadrons.

Recently the idea of 'colour dielectric constant' has been introduced by Lee (1979) to explain the phenomenon of quark confinement. The vacuum is considered as a perfect colour diaelectric medium and all hadrons are viewed as domain structure in the physical vacuum.

The use of the concept of colour dielectric constant to describe confinement of coloured quarks (assumed to be massless) and gluons inside hadrons seems to be promising because of the observation of analogous confinement of photon in media having suitable dielectric constants. Some examples are: confinement of the path of a light beam to a single plane (Born and Wolf 1970), in a medium having spherically symmetric dielectric constant, helical path of light rays in Selfoc fibres (Ghatak and Thyagarajan 1978) and formation of perfect image in Maxwell's fish eye (Born and Wolf 1970). In the present paper we show that the solution of Maxwell's electromagnetic equations in a medium having suitably chosen dielectric constant yields permanent confinement of light of selected frequencies in closed orbits in the medium. These frequencies are equally spaced forming a frequency spectrum analogous to the equispaced energy spectrum of a simple harmonic oscillator. Solutions with higher frequencies are shown to be closer to the centre than those with lower ones. In the large frequency region ray analysis using the eikonal approximation is shown to confirm these results.

Since the equations of motion of the coloured gluon field are written in a form identical to those of the Maxwell's electromagnetic equations and the Dirac equations for massless, coloured, spin $\frac{1}{2}$, quarks can also be cast into Maxwell-like form; the model of a hadron consisting of massless quarks moving in a colour dielectric medium would lead to confinement in a similar manner. This will be dealt with in a subsequent paper. Here we consider the confiement of photon. In § 2 we make a wave theoretical analysis to show the confinement and in § 3 the same results are obtained by ray analysis using eikonal approximation in the large frequency limit. The paths of the confined rays are seen to be ellipses with frequency independent semimajor axis and the semiminor axis decreasing with increase in frequency. The physical consequences of these results are discussed.

2. Wave theoretical analysis

The confinement of light is discussed wave theoretically by solving Maxwell's equations,

$$\nabla \cdot \mathbf{B} = 0, \tag{1a}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{1b}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1c}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = 0, \tag{1d}$$

in a source free medium. Here **E** is the electric field strength, **D** is the electric displacement, **B** is the magnetic flux density and **H** is the magnetic field strength. In an isotropic medium, we have

$$\mathbf{D} = \epsilon\, \mathbf{E}, \tag{2a}$$

and

$$\mathbf{B} = \mu \mathbf{H}, \tag{2b}$$

where ϵ is the dielectric constant and μ is the magnetic permeability of the medium.

In the present case we take $\mu = 1$ and the dielectric constant to be spherically symmetric.

$$\epsilon(\mathbf{r}) = \epsilon(r) \equiv \eta^2(r), \tag{3}$$

where η is the refractive index of the medium.

In that case (1b) becomes

$$\nabla\cdot\mathbf{D} = \epsilon\nabla\cdot\mathbf{E} + \frac{1}{r}(\mathbf{r}\cdot\mathbf{E})\frac{\partial\epsilon(r)}{\partial r} = 0\,.$$

Now we impose the condition $\mathbf{r}\cdot\mathbf{E} = 0$, which satisfies the transversality of light waves and simplifies (1b). Therefore (2) take the form

$$\nabla\cdot\mathbf{B} = 0, \tag{4a}$$

$$\nabla\cdot\mathbf{E} = 0, \tag{4b}$$

$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0, \tag{4c}$$

$$\nabla\times\mathbf{B} - \frac{1}{c}\,\epsilon(r)\frac{\partial\mathbf{E}}{\partial t} = 0. \tag{4d}$$

Assuming the time dependence of $\mathbf{E}$ and $\mathbf{B}$ to be

$$\mathbf{E}\,(\mathbf{r}, t) = \mathbf{e}\,(\mathbf{r}) \exp\,(-i\,\omega\, t), \tag{5a}$$

and

$$\mathbf{B}\,(\mathbf{r}, t) = \mathbf{b}\,(\mathbf{r}) \exp\,(-i\,\omega\, t). \tag{5b}$$

Equations (4c) and (4d) become

$$\nabla\times\mathbf{e} - i\frac{\omega}{c}\mathbf{b} = 0, \tag{6a}$$

$$\nabla\times\mathbf{b} + i\frac{\omega}{c}\epsilon\,(r)\,\mathbf{e} = 0, \tag{6b}$$

which give $\nabla^2\,\mathbf{e}\,(\mathbf{r}) + \frac{\omega^2}{c^2}\,\eta^2\,(r)\,\mathbf{e}\,(\mathbf{r}) = 0.$ (7)

The vector wave solutions of this equation can be written in terms of the solutions

$$u\,(r,\ \theta, \phi) = R_l\left(\eta\frac{\omega}{c}r\right) Y_{lm}\,(\theta, \phi), \tag{8}$$

of the scalar wave equation

$$\nabla^2\,u\,(r,\ \theta,\ \phi) + \frac{\omega^2}{c^2}\,\eta^2\,(r)\,u\,(r,\ \theta,\ \phi) = 0. \tag{9}$$

In (8) the radial functions

$$R_l\left(\eta\frac{\omega}{c}r\right)$$

satisfy the differential equation

$$r^2 R_l'' + 2r R_l' + \left\{\eta^2 \frac{\omega^2}{c^2} r^2 - l(l+1)\right\} R_l = 0 . \tag{10}$$

The solutions of (7) are easily shown (Stratton 1941) to be

$$\mathbf{e} = \nabla \times (\mathbf{r}\, u), \quad \text{and} \quad \mathbf{b} = -i\frac{c}{\omega}\nabla \times \mathbf{e},$$

which component-wise read as:

$$\mathbf{e} = \begin{pmatrix} 0 \\ \dfrac{1}{\sin\theta}\dfrac{\partial Y_{lm}}{\partial \phi} \\ -\dfrac{\partial Y_{lm}}{\partial \theta} \end{pmatrix} \mathrm{R}_l, \tag{11a}$$

and $$\mathbf{b} = -i\frac{c}{\omega}\begin{pmatrix} \dfrac{R_l}{r} l(l+1)\, Y_{lm} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rR_l)\dfrac{\partial Y_{lm}}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(rR_l)\dfrac{1}{\sin\theta}\dfrac{\partial Y_{lm}}{\partial \phi} \end{pmatrix}. \tag{11b}$$

Solutions with $l = 0$ are excluded as they would lead to vanishing of the electromagnetic field in toto.

The solution $R_l\,(\eta\omega/c\, r)$ of (10) depends on the dielectric constant of the medium. If this is chosen to be

$$\epsilon(r) = \eta^2(r) = \frac{a}{r} - b^2, \tag{12}$$

with a and b as constants, it is found that light waves represented by (11) are permanently confined to the neighbourhood of $r = 0$. With this choice (10) now reads

$$R_l'' + \frac{2}{r} R_l' + \left\{\frac{a\omega^2}{c^2 r} - \frac{b^2\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right\} \mathrm{R}_l = 0, \tag{13}$$

whose form is same as that of radial equation for hydrogen-like atoms.

Assuming the solution $R_l \sim r^l f_l(r) \exp(-b\,\omega/c\, r)$, and changing to the dimensionless variable $\rho \equiv 2\,b\,\omega/c\, r$, one gets the equation for f_l:

$$\rho f_l'' + (2l + 2 - \rho) f_l' - \left(l + 1 - \frac{a\,\omega}{2bc}\right) f_l = 0. \tag{14}$$

This is a confluent hypergeometric equation whose solution is

$$f_l = F\left(l + 1 - \frac{a\,\omega}{2bc}, 2l + 2; \rho\right). \tag{15}$$

Now, since we seek solutions that would give confinement, the R_l must be asymptotically zero. Hence, the above confluent hypergeometric series must terminate after a certain number of terms and reduce to a polynomial the condition for which is

$$l + 1 - \frac{a\omega}{2bc} = -n_r, \tag{16}$$

with $n_r = 0, 1, 2, \ldots$ etc. This relation imposes a condition on the frequencies of the light wave in order that it gets confined. It can be written as

$$\omega_n = 2\frac{bc}{a}n, \tag{17}$$

where $$n = n_r + l + 1. \tag{18}$$

Since, as noted earlier $l = 0$ solutions are excluded, n would take up values 2, 3, 4 etc. It will be seen that the allowed frequencies are equally spaced like those of the allowed energy values of the simple harmonic oscillator in quantum mechanics.

The solution of the radial equation (13) for the confined light wave is

$$R_{nl}(r) \sim r^l \exp\left(-2\frac{b^2}{a}nr\right) F\left(l + 1 - n,\ 2l+2;\ 4\frac{b^2}{a}nr\right). \tag{19}$$

This has a resemblance with the radial wave functions of the hydrogen atom. However, unlike the hydrogen atom case the exponential factor decreases instead of increasing with frequency on account of relation (17) between ω_n and n. As a result, the higher frequency states are closer to the centre than the ones with lower frequencies. The medium thus provides a trap for confinement of light waves of selected frequencies.

3. Ray orbits

In order to obtain the path of the confined beam, we make a ray analysis using the eikonal approximation (*i.e.* wave length $\lambda \to 0$ or $\omega \to \infty$). Consider the scalar wave equation (9)

$$\nabla^2 u + \frac{\omega^2}{c^2}\eta^2 u = 0.$$

Now, putting $u = u_0 \exp\left(i\,S\,\frac{\omega}{c}\right)$ with u_0 and S independent of ω, in (9) we get in the limit $\omega \to \infty$,

$$(\nabla S)^2 = \eta^2, \tag{20}$$

which is the eikonal equation for ray optics. Here, $S = S(\mathbf{r})$ is the eikonal and its gradient ∇S gives the direction of the ray.

Since the medium has spherically symmetric refractive index, the path of the beam will be confined to a plane (Born and Wolf 1970). Now, in plane polar co-ordinates, the eikonal equation reads

$$\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 = \frac{a}{r} - b^2, \tag{21}$$

where we have used

$$S(\mathbf{r}) = S(r, \theta) = S_r(r) + S_\theta(\theta), \tag{22}$$

It is clear from (21) that

$$\frac{dS_\theta}{d\theta} = \text{constant} = a_\theta \text{ (say)}, \tag{23a}$$

and hence $$\frac{dS_r}{dr} = \left(\frac{a}{r} - b^2 - \frac{a_\theta^2}{r^2}\right)^{1/2}. \tag{23b}$$

At this stage we introduce two quantities J_r and J_θ:

$$J_\theta = \oint d\theta \frac{dS_\theta}{d\theta} = 2\pi a_\theta, \tag{24}$$

and $$J_r = \oint dr \frac{dS_r}{dr} = \oint dr \left(\frac{a}{r} - b^2 - \frac{a_\theta^2}{r^2}\right)^{1/2}. \tag{25}$$

Integrating (25) by standard methods and using (24) we get

$$J_r = -J_\theta + \pi \frac{a}{b},$$

so that $$J_r + J_\theta = \pi \frac{a}{b}, \tag{26}$$

which looks very much like a relation that one comes across in the solution of Hamilton-Jacobi equation for the motion of massive charged particle in a Coulomb potential. It would therefore be interesting to see what one obtains by imposing the ‘ Bohr-Sommerfeld conditions ’,

(a) $$\frac{\omega}{c} J_r = 2\pi n_r, \tag{27}$$

with $n_r = 0, 1, 2, \ldots$, etc and

(b) $$\frac{\omega}{c} J_\theta = 2\pi (l + 1), \tag{28}$$

with $l = 1, 2, 3, \ldots$ etc.

Combining these two conditions we get

$$\frac{\omega}{c}(J_r + J_\theta) = 2\pi (n_r + l + 1) \equiv 2\pi n, \tag{29}$$

with $n = (n_r + l + 1) = 2, 3, 4, \ldots$ etc. which when substituted in (26) gives

$$\omega_n = 2 \frac{bc}{a} n. \tag{30}$$

This is precisely the same as (17) obtained from wave theoretical analysis. We thus see that in high frequency limit our ' Bohr-Sommerfeld condition ' imposed on ray optics yields results of wave optics in a manner analogous to results of wave mechanics being obtained from classical mechanics through imposition of Bohr-Sommerfeld condition.

Now, for obtaining the orbit of the light rays we make use of (22), (23) and (28) to get (see appendix)

$$\theta = \theta_0 + \frac{c}{\omega}(l + 1) \int \frac{dr}{r^2} \left\{ \frac{a}{r} - b^2 - \frac{c^2 (l + 1)^2}{\omega^2 r^2} \right\}^{-1/2}, \tag{31}$$

where θ_0 is a constant of integration. Equation (31) now leads to the orbit equation

$$\frac{1}{r} = \frac{a\,\omega^2}{2\,c^2 (l + 1)^2} \left[1 + \left\{ 1 - 4 \frac{b^2 c^2}{a^2 \omega^2} (l + 1)^2 \right\}^{1/2} \cos(\theta - \theta_0) \right], \tag{32}$$

which represents a conic section with one of the focii at the origin and having eccentricity

$$e = \left\{ 1 - \frac{4\,b^2 c^2}{a^2 \omega^2} (l + 1)^2 \right\}^{1/2}. \tag{33}$$

Substituting (30) in (32) and (33) the orbit equation becomes

$$\frac{1}{r} = 2 \frac{b^2}{a} \frac{n^2}{(l + 1)^2} \left[1 + \left\{ 1 - \frac{(l + 1)^2}{n^2} \right\}^{1/2} \cos(\theta - \theta_0) \right], \tag{34}$$

and the expression for eccentricity reads

$$e = \left\{ 1 - \frac{(l + 1)^2}{n^2} \right\}^{1/2}. \tag{35}$$

It is clear from (34) and (35) that the orbit of the confined ray is an ellipse with semimajor axis

$$r_{\text{semimajor}} = \frac{a}{2\,b^2}, \tag{36}$$

and semiminor axis

$$r_{\text{semiminor}} = \frac{a}{2\,b^2}\,\frac{l+1}{n}. \tag{37}$$

The orbit becomes circular with radius $(a/2b^2)$ when $n = l + 1$ *i.e.* when $n_r = 0$.

The semimajor axis of the ellipse is thus seen to be constant for all frequencies in the $\omega \to \infty$ limit, but from (37) it transpires that for given l the semiminor axis decreases with increase in n (*i.e.* increase in frequency). Thus the higher frequency orbits are closer to the centre. This result which we have obtained here in the high frequency region was also concluded from wave theoretical analysis for all allowed frequencies in § 2.

4. Discussions

We have considered the most ideal case for the sake of simplicity. For this reason the constants a and b occurring in the expression for dielectric constant are assumed to be frequency independent; usually these are frequency dependent. Modifications of our simple theory on account of this can be done without difficulty.

Another problem is that the dielectric constant blows up at $r = 0$ similar to that of the Coulomb potential. In actual practice a medium of this nature may not exist. Modifications of our theory for realistic dielectric functions can be made in a manner analogous to the solution of Schrödinger equation for non-hydrogenic atoms.

The model medium we have considered has spherically symmetric dielectric constant which decreases with increasing radius and becomes negative after a certain radius. The space surrounding earth's surface has similar features; the atmospheric dielectric constant is spherically symmetric and decreases radially outwards. Further, it becomes negative in the ionospheric region. It is also possible that the dielectric media adopted by us may be found in some stellar bodies. In view of the recent report (Dolgov and Makshimov 1978) about the possibility of negative static dielectric constant in metals one may hopefully, be able to create such a medium in the laboratory.

The above interesting results encourage one to investigate the problem of confinement of other massless particles using the idea of dielectric constant. If the interior of the extended hadrons is assumed to be a colour dielectric medium having a suitable colour dielectric constant, the solutions of coloured gluon equations (which are identical to the Maxwell's equations) and the Dirac equation for massless, coloured quarks in that medium would lead to confinement of quarks and gluons. Details of investigations on these lines will form the subject matter of a forthcoming paper.

Appendix

Here we give a derivation of (31). From (22) and (23) we get

$$dS_\theta/d\theta = dS/d\theta = \text{Constant} = \alpha_\theta. \tag{A.1}$$

Here, $$S = \int_0^{\theta_0} \alpha_\theta \, d\theta = \alpha_\theta \, \theta_0, \tag{A.2}$$

where θ_0 is a constant of integration.

So, $$\frac{dS}{d\alpha_\theta} = \theta_0 = \frac{\partial S_\theta}{\partial \alpha_\theta} + \frac{\partial S_r}{\partial \alpha_\theta}$$

$$= \int \frac{\partial}{\partial \alpha_\theta}\left(\frac{\partial S_\theta}{\partial \theta}\right) d\theta + \int \frac{\partial}{\partial \alpha_\theta}\left(\frac{\partial S_r}{\partial r}\right) dr. \tag{A.3}$$

Now, using (23b) we get

$$\theta_0 = \theta + \int \frac{\partial}{\partial \alpha_\theta}\left(\frac{a}{r} - b^2 - \frac{\alpha_\theta^2}{r^2}\right)^{1/2} dr$$

$$= \theta - \int \frac{\alpha_\theta}{r^2}\left(\frac{a}{r} - b^2 - \frac{\alpha_\theta^2}{r^2}\right)^{-1/2} dr, \tag{A.4}$$

which by (28) and (24) leads to (31)

$$\theta_0 = \theta - \frac{c}{\omega}(l+1) \int \frac{dr}{r^2} \left\{\frac{a}{r} - b^2 - \frac{c^2 (l+1)^2}{\omega^2 r^2}\right\}^{-1/2}.$$

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