Pramāṇa, Vol. 17, No. 3, September 1981, pp. 229–244. © Printed in India.

Scattering on energy-dependent separable potentials

K NIYOGI,* R N PANIGRAHI, D ROY and B TALUKDAR
*Department of Physics, Bolpur College, Bolpur, 731 204, India
Department of Physics, Visva-Bharati University, Santiniketan 731 235, India

MS received 21 November 1980; revised 18 July 1981.

Abstract. We derive a coordinate space approach to energy-dependent separable potentials and clearly demonstrate its calculational advantage. The results presented include expressions for (i) low-energy scattering parameters and (ii) off-energy-shell T and K matrices. We study energy dependence of the effective potential with particular emphasis on cross-channel suppression effects.

Keywords. Nuclear reaction; scattering theory; energy-dependant separable potentials; low-energy scattering parameters; off-shell transition; reactance amplitudes.

1. Introduction

In a scattering experiment when inelastic channels open up, elastic and inelastic scatterings compete for their share of incoming flux. The effective potential in the elastic channel is thus made energy dependent. Such problems are treated by the multichannel formulation of the scattering theory. Energy dependence of the elastic channel potential results from formal elimination of coupled channels (Feshbach 1958). Expressions for low-energy scattering parameters (scattering length a and effective range r) and off-shell scattering amplitude for the multichannel scattering process are needed in many physical applications (Satoh and Nogami 1970; Alberg *et al* 1973; Myhrer 1973). The multichannel problem becomes amenable to analytical treatment if the form factors of the channel interaction are chosen separable. Exact expressions for a and r are obtained from the behaviour of the momentum-space physical wave function (outgoing wave solution) of the Schrödinger equation (Nogami and Satoh 1970). The off-shell amplitudes are constructed by solving Lippmann-Schwinger equations (Ernst *et al* 1974) for the T and K matrices.

The momentum-space approach indicated above is a straightforward application of the Tabakin's method (Tabakin 1964) for the single channel problem. It involves evaluation of certain tedious contour integrals (Bagchi and Mulligan 1974). Also we note that the off-shell result does not appear to be a simple extension of the corresponding on-shell one.

Coz *et al* (1970) made use of a co-ordinate space (r-space) method to treat the on-shell single channel scattering by separable potentials. Recently one of us (Talukdar *et al* 1979c) derived off-energy-shell results for such potentials by a wave function approach which may be regarded as a natural extension of the on-shell work. As opposed to the method of Tabakin (1964) and its subsequent extension, here all results can be derived by the use of tabulated integrals of elementary transcendental

functions. In this paper we implement the r-space method of Coz *et al* and of Talukdar *et al* to treat energy-dependent separable potentials. In the course of our study we shall see that the merit of the present approach is its simplicity and also that a number of different methods are available to deal with both on- and off-shell scattering situations. The present analysis of the multi-channel scattering will proceed as follows:

For the on-shell case the set of coupled Schrödinger equation will be converted to integral equations with appropriate boundary conditions and the effective potential in the elastic channel determined by eliminating the inelastic channel wave functions. The method is thus equivalent to constructing the wave function in a subspace $\mathcal{H}' \in \mathcal{H}$ of the complete Hilbert space $\mathcal{H}$. Given the effective potential, the low-energy scattering parameters can be calculated from the asymptotic values of the physical as well as standing wave solutions. The variable phase approach (Calogero 1967) may be used to construct expressions for a and r by working with the regular solution. We note that the integral equation formulation of the problem involves evaluation of multiple Volterra-type integrals, the number of which tends to increase geometrically with the number of participating inelastic channels. The situation can however be made more tractable by working directly with the integro-differential equation which enters into the theory (Gordon 1970; Talukdar *et al* 1979a; Talukdar and Das 1979).

The off-shell case may be treated in a similar manner. An inhomogenous form of the Schrödinger equation (van Leeuwen and Reiner 1961) is used for the purpose. As in the on-shell case, both the integral equation and integro-differential equation approaches are available to construct expressions for off-shell T and K matrices.

In § 2 we treat the on-shell scattering by extending the method of Coz *et al* (1970). For simplicity of presentation we deal with the two-channel problem for s-wave scattering only. Generalization of our method to larger number of channels and higher partial waves is really trivial. We develop integro-differential equation method in § 3.

In principle both methods of §§ 2 and 3 permit an easy off-energy shell extension. But because of the obvious virtue of simplicity, we shall use in § 4 the development of § 3 to derive off-shell results for T and K matrices. As already noted we have used non-local potentials to describe the channel interaction. Certain characteristic features of these potentials are that, for a given choice of the parameters, they can support spurious states and bound states embedded in the continuum (Mulligen *et al* 1976). In § 5 we examine in some detail how the energy dependence of the elastic channel coupling constant affects these states.

2. Integral equation method

Consider the two-channel s-wave radial Schrödinger equation

$$\left(\frac{d^2}{dr^2} + k^2\right) \Psi(r) = \frac{2\mu}{\hbar^2} \int_0^\infty V(r, r') \Psi(r')\, dr', \tag{1}$$

where $V(r, r')$ is a real symmetric 2×2 matrix which depends on r and r' only and, k and μ are the diagonal wave numbers and mass matrices. Obviously, the matrices

k and μ are characterized by the channel wave numbers k_1 and k_2 and reduced channel masses μ_1 and μ_2 respectively. In general, the wave function Ψ is a 2×2 matrix each column of which is itself a solution, the columns differing by their boundary conditions. We consider

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{2}$$

and introduce a separable form of $V(r, r')$, namely

$$V(r, r') = \begin{pmatrix} \lambda_{11} v_1(r) v_1(r') & \lambda_{12} v_1(r) v_2(r') \\ \lambda_{12} v_2(r) v_1(r') & \lambda_{22} v_2(r) v_2(r') \end{pmatrix}. \tag{3}$$

In (3) λ_{ij}'s are constant strength parameters and v_i's, the form factors of the potential. Combining (1), (2) and (3) we get

$$\left(\frac{d^2}{dr^2} + k_1^2\right) \Psi_1(k_1, k_2, r) = \frac{2\mu_1}{\hbar^2} v_1(r) \left[\lambda_{11} \int_0^\infty v_1(s)\, \Psi_1(k_1, k_2, s)\, ds + \lambda_{12} \int_0^\infty v_2(s)\, \Psi_2(k_1, k_2, s)\, ds\right], \tag{4a}$$

and

$$\left(\frac{d^2}{dr^2} + k_2^2\right) \Psi_2(k_1, k_2, r) = \frac{2\mu_2}{\hbar^2} v_2(r) \left[\lambda_{12} \int_0^\infty v_1(s)\, \Psi_1(k_1, k_2, s)\, ds + \lambda_{22} \int_0^\infty v_2(s)\, \Psi_2(k_1, k_2, s)\, ds\right]. \tag{4b}$$

2.1 *Physical wavefunction*

Considering the 11 entrance channel, the integral forms of (4a) and (4b) are

$$\Psi_1^+(k_1, k_2, r) = \sin k_1 r + \frac{2\mu_1}{\hbar^2} \int_0^\infty G_{11}^{(+)}(k_1, r, r')\, v_1(r')\, dr' \times \left[\lambda_{11} \int_0^\infty v_1(s)\, \Psi_1^{(+)}(k_1, k_2, s)\, ds + \lambda_{12} \int_0^\infty v_2(s)\, \Psi_2^{(+)}(k_1, k_2, s)\, ds\right], \tag{5a}$$

and

$$\Psi_2^{(+)}(k_1, k_2, r) = \frac{2\mu_2}{\hbar^2} \int_0^\infty G_{22}^{(+)}(k_2, r, r')\, v_2(r')\, dr' \times \left[\lambda_{12} \int_0^\infty v_1(s)\, \Psi_1^{(+)}(k_1, k_2, s)\, ds + \lambda_{22} \int_0^\infty v_2(s)\, \Psi_2^{(+)}(k_1, k_2, s)\, ds\right]. \tag{5b}$$

Here $\Psi_1^{(+)}$ and $\Psi_2^{(+)}$ refer to the physical wave function in the entrance and inelastic channels respectively and, $G_{11}^{(+)}$ and $G_{22}^{(+)}$ are the appropriate Green's functions. Note that like (5a), (5b) is also inhomogeneous in $\Psi_2^{(+)}$, the first term in the right side being the inhomogeneity term. Since separable kernels are involved, we can solve (5b) to get

$$\Psi_2^{(+)}(k_1, k_2, r) = \frac{2\mu_2}{\hbar^2}\Big[\lambda_{12}\int_0^\infty G_{22}^{(+)}(k_2, r, r')\, v_2(r')\, dr'$$

$$\times \int_0^\infty v_1(s)\, \Psi_1^{(+)}(k_1, k_2, s)\, ds + \left(\frac{2\mu_2}{\hbar^2}\right)\frac{\lambda_{12}\lambda_{22}}{\hbar^2 D_1^+(k_2)}$$

$$\times \int_0^\infty G_{22}^{(+)}(k_2, r, r')\, v_2(r')\, dr' \int_0^\infty v_2(r)\, dr \int_0^\infty G_{22}^{(+)}(k_2, r, r')$$

$$\times v_2(r')\, dr' \int_0^\infty v_1(s)\, \Psi_1^{(+)}(k_1, k_2, s)\, ds\Big], \tag{6}$$

where the Fredholm determinant

$$D_1^+(k_2) = 1 - \frac{2\mu_2\lambda_{22}}{\hbar^2}\int_0^\infty v_2(r)\, dr \int_0^\infty G_{22}^{(+)}(k_2, r, r')\, v_2(r')\, dr'. \tag{7}$$

Substituting (6) in (5a) we get the integral equation for $\Psi_1^{(+)}(k_1, k_2, r)$ in the form

$$\Psi_1^{(+)}(k_1, k_2, r) = \sin k_1 r + \frac{2\mu_1}{\hbar^2}\gamma(E)\int_0^\infty\int_0^\infty G_{11}^{(+)}(k_1, r, r') \times$$

$$\times v_1(r')\, dr'\, v_1(s)\, \Psi_1^{(+)}(k_1, k_2, s)\, ds, \tag{8}$$

where $$\gamma(E = k_2^2) = \lambda_{11} + \frac{2\mu_2\lambda_{12}^2}{\hbar^2 D_1^+(k_2)}\int_0^\infty v_2(r)\, dr \int_0^\infty G_{22}^{(+)}(k_2, r, r')\, v_2(r')\, dr'. \tag{9}$$

Equation (8) exhibits that the effective potential in the elastic channel possesses an energy-dependent coupling constant, which arises from formal elimination of the coupled inelastic channels. In particular, we have

$$V_{\text{eff}}(E) = \gamma(E)\,|\, v_1 > < v_1\,|\,. \tag{10}$$

Making use of explicit forms of the s-wave Green's functions (Newton 1966) it is easy to see from (10) that as $k_2 \to \infty$, $\gamma(E)/\lambda_{11}$ goes to unity. This corresponds to the familiar fact that when the threshold for the inelastic channel is very high, we have only elastic scattering. This observation has formed the basis for writing a dispersion relation for $[1/\gamma(E)] - 1$ and a possible analytic solution of the related inverse scattering problem (Londergan and Moniz 1973).

For Yamaguchi form factors (Yamaguchi 1954) $v(r)\, v(r') = \exp[-\alpha(r + r')]$; (9) reads

$$\gamma(E) = \lambda_{11} \frac{1 + \dfrac{2\mu_2 \lambda'}{\hbar^2 (a^2 + k_2^2)} \left(\dfrac{a + i k_2}{a^2 + k_2^2} - \dfrac{1}{2a} \right)}{1 + \dfrac{2\mu_2 \lambda_{22}}{\hbar^2 (a^2 + k_2^2)} \left(\dfrac{a + i k_2}{a^2 + k_2^2} - \dfrac{1}{2a} \right)}, \tag{11}$$

where $$\lambda' = \lambda_{22} - \lambda_{12}^2/\lambda_{11}. \tag{12}$$

We solve (8) to get

$$\Psi_1^{(+)}(k_1, k_2, r) = \sin k_1 r + \frac{2\mu_1 k_1 \lambda_{11} \left[1 + \dfrac{2\mu_2 \lambda'}{\hbar^2 (a^2 + k_2^2)} \left(\dfrac{a + i k_2}{a^2 + k_2^2} - \dfrac{1}{2a} \right) \right]}{\hbar^2 (a^2 + k_1^2)^2 D^+(k_1, k_2)} \times [\exp(-ar) - \exp(i k_1 r)]. \tag{13}$$

In writing (11) and (13) we have used the well-known expression for outgoing wave Green's function given by Newton (1966).

Here

$$D^+(k_1, k_2) = 1 + \frac{2\mu_1 \lambda_{11}}{\hbar^2 (a^2 + k_1^2)} \left[\frac{a + i k_1}{(a^2 + k_1^2)} - \frac{1}{2a} \right] + \frac{2\mu_2 \lambda_{22}}{\hbar^2 (a^2 + k_2^2)} \left(\frac{a + i k_2}{a^2 + k_2^2} - \frac{1}{2a} \right) + \frac{4\mu_1 \mu_2 \lambda_{11} \lambda'}{\hbar^4 (a^2 + k_1^2)(a^2 + k_2^2)} \times \left(\frac{a + i k_1}{a^2 + k_1^2} - \frac{1}{2a} \right) \left(\frac{a + i k_2}{a^2 + k_2^2} - \frac{1}{2a} \right). \tag{14}$$

Clearly $D^+(k_1, k_2)$ refers to the Fredholm determinant associated with the physical solution in the elastic channel, the effect of the inelastic channels being taken care of. Comparing $\lim_{r \to \infty} \Psi^{(+)}(k_1, k_2, r)$ with the prescribed asymptotic form of the physical solution written in terms of on-shell T matrix we get

$$k_1 \cot \delta = -a + \frac{k_1^2 + a^2}{2a} - \frac{(a^2 + k_1^2) \left\{ a(a + l)^2 + \dfrac{\mu_2}{\hbar^2} \lambda_{22} \right\}}{\dfrac{2\mu_1 \lambda_{11}}{\hbar^2} \left\{ a(a + l)^2 + \dfrac{\mu_2}{\hbar^2} \lambda' \right\}}, \tag{15}$$

where $$l = -i k_2, \tag{16}$$

and δ, the s-wave phase shift. The scattering length a and effective range r obtained from (15) in the limit $k_1 \to 0$, are given by

$$\frac{1}{a} = \frac{\alpha}{2}\left[1 + \frac{\alpha^3\{\alpha(\alpha+k)^2 + \gamma_{22}\}}{\gamma_{11}\,\alpha(\alpha+k)^2 + d(\gamma)}\right], \tag{17}$$

and

$$r = \frac{3}{\alpha} - \frac{4}{a\alpha^2} - \frac{\mu_2\,\gamma_{12}^2\,\alpha^5(\alpha+k)}{\mu_1\,k\{\gamma_{11}\,\alpha(\alpha+k)^2 + d(\gamma)\}^2}. \tag{18}$$

In writing (17) and (18) we have used

$$k = (2\mu_2\,\Delta)^{1/2}, \tag{19a}$$

$$\Delta = l^2(2\mu_2)^{-1} + k^2(2\mu_1)^{-1}, \tag{19b}$$

$$\gamma_{11} = \frac{\lambda_{11}}{\hbar^2}\mu_1, \tag{20a}$$

$$\gamma_{22} = \frac{\lambda_{22}}{\hbar^2}\mu_2, \tag{20b}$$

$$\gamma_{12} = \frac{\lambda_{12}}{\hbar^2}(\mu_1\,\mu_2)^{1/2}, \tag{20c}$$

and

$$d(\gamma) = \gamma_{11}\,\gamma_{22} - \gamma_{12}^2. \tag{21}$$

2.2 *Standing wave solution*

Here we shall work with the standing wave solution $\Psi_1^p(k_1, k_2, r)$ in the entrance channel and show how construction of $\Psi_1^p(k_1, k_2, r)$ provides a useful method to calculate low energy scattering parameters. The Green's function appropriate to the situation is given by

$$G_{11}^p(k_1, r, r') = G_{11}^R(k_1, r, r') - \frac{1}{k_1}\sin k_1 r \cos k_1 r'. \tag{22}$$

In terms of (22) the principal value wavefunction can be written in the form

$$\Psi_1^p(k_1, k_2, r) = \sin k_1 r + \frac{2\mu_1\,\gamma(E)}{\hbar^2 k_1}\Big[\int_0^r \sin k_1(r-r')\,v_1(r')\,dr' \times \int_0^\infty v_1(s)\,\Psi_1^p(k_1, k_2, s)\,ds - \int_0^\infty \sin k_1 r \cos k_1 r'\,v_1(r')\,dr' \times \int_0^\infty v_1(s)\,\Psi_1^p(k_1, k_2, s)\,ds\Big]. \tag{23}$$

Solving (23) with the Yamaguchi form factor we get

$$\Psi_1^p(k_1, k_2, r) = \sin k_1 r - \frac{2\mu_1 \gamma(E)}{\hbar^2 (a^2 + k_1^2)^2} \times \frac{k_1}{1 + \frac{2\mu_1 \gamma(E)}{\hbar^2}\left(\frac{1}{2a (a^2 + k_1^2)} - \frac{a}{(a^2 + k_1^2)^2}\right)} \times (\exp [- a r] - \cos k_1 r). \tag{24}$$

Comparing the asymptotic form of (24) with the prescribed asymptotic value of the standing wave solution, namely,

$$\Psi_1^p (k_1, k_2, r) \underset{r \to \infty}{\sim} \sin k_1 r + \tan \delta \cos k_1 r, \tag{25}$$

we obtain the expression for $k_1 \cot \delta$ in agreement with the result obtained in the earlier section.

2.3 *Regular wavefunction and phase-amplitude method*

With the effective potential in (10), the regular wavefunction $\phi_1 (k_1, k_2, r)$ in the entrance channel can be written as

$$\phi_1 (k_1, k_2, r) = \sin k_1 r + \frac{2\mu_1 \gamma(E)}{\hbar^2} \int_0^\infty G_{11}^R (k_1, r, r') v_1 (r') dr' \times \int v_1 (s) \phi_1 (k_1, k_2, s) ds, \tag{26}$$

where $$G_{11}^R (k_1, r, r') = k_1^{-1} \sin k_1 (r - r'), r' < r, \tag{27}$$

$$= 0, r' > r.$$

We shall use the basic philosophy of the phase-amplitude method (Calogero 1967) to solve (26) for $k_1 \cot \delta$. To that end we combine (26) and (27) to write

$$\phi_1 (k_1, k_2, r) = \sin k_1 r \left[1 + \frac{2\mu_1 \gamma (E)}{\hbar^2 k_1} \int_0^r \cos k_1 r' v_1 (r') dr' \times \int_0^\infty v_1 (s) \phi_1 (k_1, k_2, s) ds\right] - \frac{2\mu' \gamma (E)}{\hbar^2 k_1} \cos k_1 r \times \int_0^r \sin k_1 r' v_1(r') dr' \int_0^\infty v_1 (s) \phi_1 (k_1, k_2, s) ds, \tag{28}$$

and introduce the following ansats

$$a(r)\cos\delta(r) = 1 + \frac{2\mu_1\gamma(E)}{\hbar^2 k_1}\int_0^\infty \cos k_1 r' v_1(r')\,dr' \times \int_0^\infty v_1(s)\phi_1(k_1, k_2, s)\,ds, \tag{29}$$

and

$$a(r)\sin\delta(r) = \frac{-2\mu_1\gamma(E)}{\hbar^2 k_1}\int_0^{r\infty} \sin k_1 r' v_1(r')\,dr' \times \int_0^\infty v_1(s)\phi_1(k_1, k_2, s)\,ds. \tag{30}$$

In (29) and (30), $a(r)$ and $\delta(r)$ stand for the so-called amplitude and phase functions. We are, however, interested in the phase function which asymptotically gives the phase shift and satisfies the initial condition $\delta(0) = 0$.

Combining (27) and (30) and taking the limit $r \to \infty$ we get

$$k_1\cot\delta = \frac{k_1 + \frac{2\mu_1\gamma(E)}{\hbar^2}\int_0^\infty \cos k_1 r' v_1(r')\,dr' \int_0^\infty v_2(s)\phi_1(k_1, k_2, s)\,ds}{-\frac{2\mu_1\gamma(E)}{\hbar^2}\int_0^\infty \sin k_1 r' v_1(r')\,dr' \int_0^\infty v_1(s)\phi_1(k_1, k_2, s)\,ds}. \tag{31}$$

If we now solve (28) for

$$\int_0^\infty v_1(s)\phi_1(k_1, k_2, s)\,ds$$

for the Yamaguchi potential by the usual technique derived in §2.1 and substitute the value in (31) we get $k_1\cot\delta$ in complete agreement with equation (15). Thus the regular wavefunction can also be used to construct expressions for a and r.

We end this section by noting that the physical wavefunction in the momentum space was used by earlier workers to construct expressions for low energy scattering parameters. Here we observe that the coordinate space approach can be used with the three boundary conditions to obtain these results in a relatively noncomplicated manner. One does not require to evaluate any contour integral.

3. Integro-differential equation method

Introducing

$$\gamma_{11} = \frac{2\mu_1}{\hbar^2}\lambda_{11}, \quad \gamma_{12} = \frac{2\mu_1}{\hbar^2}\lambda_{12}, \quad \gamma_{22} = \frac{2\mu_2}{\hbar^2}\lambda_{22}, \tag{32}$$

$$\left.\begin{aligned} C_1 &= \int_0^\infty \exp(-as)\,\Psi_1(k_1, k_2, s)\,ds \\ \text{and} \qquad C_2 &= \int_0^\infty \exp(-as)\,\Psi_2(k_1, k_2, s)\,ds \end{aligned}\right\}, \tag{33}$$

(4a) and (4b) become

$$\left(\frac{d^2}{dr^2} + k_1^2\right)\Psi_1(k_1, k_2, r) = \exp(-ar)\,[\gamma_{11}C_1 + \gamma_{12}\,C_2], \tag{34}$$

and

$$\left(\frac{d^2}{dr^2} + k_2^2\right)\Psi_2(k_1, k_2, r) = \exp(-ar)\,[\gamma_{12}\,C_1 + \gamma_{12}\,C_2]. \tag{35}$$

The outgoing wave solution of (34) and (35) are (Talukdar and Das 1979)

$$\Psi_1^{(+)}(k_1, k_2, r) = \sin k_1 r + \frac{\gamma_{11}\,C_1 + \gamma_{12}\,C_2}{a_2 + k_1^2}\,[\exp(-ar) - \exp(ik_1 r)], \tag{36}$$

$$\Psi_2^{(+)}(k_1, k_2, r) = \frac{\gamma_{12}\,C_1 + \gamma_{22}\,C_2}{a^2 + k_2^2}\,[\exp(-ar) - \exp(ik_2 r)]. \tag{37}$$

In writing (36) and (37) we assume that the incident plane wave is contained only in the entrance channel. To determine C_1 and C_2 we substitute (36) and (37) in (34) and (35) and arrive at the simultaneous equations

$$C_1\,d_{11} + C_2\,d_{12} = l_1,$$

and

$$C_1\,d_{21} + C_2\,d_{22} = l_2, \tag{38}$$

where

$$d_{11} = \gamma_{11} + \frac{\gamma_{11}^2}{a^2 + k_1^2}\left\{\frac{a + ik_1}{a^2 + k_1^2} - \frac{1}{2a}\right\} + \frac{\gamma_{12}^2}{a^2 + k_2^2}\left\{\frac{a + ik_2}{a^2 + k_2^2} - \frac{1}{2a}\right\}, \tag{39}$$

$$d_{12} = \gamma_{12} + \frac{\gamma_{12}\,\gamma_{22}}{a^2 + k_1^2}\left\{\frac{a + i\,k_1}{a^2 + k_1^2} - \frac{1}{2a}\right\} + \frac{\gamma_{12}\,\gamma_{22}}{a^2 + k_2^2}\left\{\frac{a + i\,k_2}{a^2 + k_2^2} - \frac{1}{2a}\right\}, \tag{39b}$$

$$d_{21} = d_{12}, \tag{39c}$$

$$d_{22} = \gamma_{22} + \frac{\gamma_{12}^2}{a^2 + k_1^2}\left\{\frac{a + i\,k_1}{a^2 + k_1^2} - \frac{1}{2a}\right\} + \frac{\gamma_{22}^2}{a^2 + k_2^2}\left\{\frac{a + i\,k_2}{a^2 + k_2^2} - \frac{1}{2a}\right\}, \tag{39d}$$

and

$$l_1 = \frac{\gamma_{11}\,k_1}{a^2 + k_1^2}, \qquad l_2 = \frac{\gamma_{12}\,k_1}{a^2 + k_1^2}. \tag{40}$$

Solving the set of simultaneous equations in (38) we can write

$\Psi_1^{(+)}(k_1, k_2, r)$ in the form

$$\Psi_1^{(+)}(k_1, k_2, r) = \sin k_1 r - \frac{\gamma_{11} N_1 + \gamma_{12} N_2}{D_t (\alpha^2 + k_1^2)} [\exp(i k_1 r) - \exp(-\alpha r)], \tag{41}$$

where

$$D_t = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}, \tag{42a}$$

$$N_1 = \begin{vmatrix} l_1 & d_{12} \\ l_2 & d_{22} \end{vmatrix}, \tag{42b}$$

and

$$N_2 = \begin{vmatrix} d_{11} & l_1 \\ d_{21} & l_2 \end{vmatrix}. \tag{42c}$$

By carrying out certain trivial algebraic manipulations one can easily show that (41) is in exact agreement with (13). An important element of this treatment is that here we had to deal only with infinite integrals, which tend to simplify the calculation considerably. Also we note that elimination of the coupled channels enters the calculation *via* the set of equations in (38). This represents a point of contrast with the treatment in § 2.

4. Off-shell scattering amplitude

In the wavefunction approach (r-space) off-shell T and K matrices are calculated by using the solutions of an inhomogeneous Schrödinger equation which, for non-local separable potential reads (van Leeuwen and Reiner 1961)

$$\left(\frac{d^2}{dr^2} + k^2\right) \Psi(k, q, r) = (k^2 - q^2) \sin qr + \lambda v(r) \times \int_0^\infty v(s) \Psi(k, q, s)\, ds. \tag{43}$$

Here q is an off-shell momentum and k is the centre of mass momentum. If $\Psi^{(+)}(k, q, r)$ and $\Psi^{(p)}(k, q, r)$ represent the outgoing and standing wave solutions of (43), the off-shell T and K matrices are given by

$$T(p, q, k^2) = \frac{2\lambda}{\pi pq} \int_0^\infty \int_0^\infty dr\, ds \sin(pr)\, v(r)\, v(s)\, \Psi^{(+)}(k, q, s), \tag{44}$$

and

$$K(p, q, k^2) = \frac{2\lambda}{\pi pq} \int_0^\infty \int_0^\infty dr\, ds \sin(pr)\, v(r)\, v(s)\, \Psi^{(p)}(k, q, s), \tag{45}$$

where p is another off-shell momentum different from q. When $p = k$, the expressions in (44) and (45) give the half-off-shell results.

The appropriate equations for studying off-shell effects for the two-channel problem under consideration are given by

$$\left(\frac{d^2}{dr^2} + k_1^2\right) \Psi_1 (k_1, k_2, q, r) = (k_1^2 - q^2) \sin qr + \exp(-\alpha r) [\gamma_{11} C_1 + \gamma_{12} C_2], \tag{46a}$$

and

$$\left(\frac{d^2}{dr^2} + k_2^2\right) \Psi_2 (k_1, k_2, q, r) = \exp(-\alpha r) [\gamma_{12} C_1 + \gamma_{22} C_2], \tag{46b}$$

where C_1 and C_2 of § 3 are redefined as

$$C_1 = \int_0^\infty \exp(-\alpha s) \Psi_1 (k_1, k_2, q, s)\, ds, \tag{47a}$$

and

$$C_2 = \int_0^\infty \exp(-\alpha s) \Psi_2 (k_1, k_2, q, s)\, ds. \tag{47b}$$

In writing the above equations we have explicitly introduced the Yamaguchi form factors.

4.1 *T-matrix*

In this case Ψ_1 and Ψ_2 are the outgoing wave solutions of (46), which we denote by the superscript $(+)$. The off-shell physical wavefunction in the 11 entrance channel can be obtained by following the procedure outlined in § 3. We have

$$\Psi_1^{(+)} (k_1, k_2, q, r) = \sin qr - \frac{\gamma_{11} N_1 + \gamma_{12} N_2}{D_t (\alpha^2 + k_1^2)} \left[\exp(i k_1 r) - \exp(-\alpha r)\right]. \tag{48}$$

Here N_1 and N_2 are given by (42), but l_1 and l_2 replaced by

$$l_1 = \frac{\gamma_{11} q}{\alpha^2 + q^2}, \qquad l_2 = \frac{\gamma_{12} q}{\alpha^2 + q^2}. \tag{49}$$

To calculate the off-shell T matrix we note that the effective potential in the present case is $\gamma (E) |v\rangle \langle v|$. Thus combining (44) and (48) we have

$$T (p, q, k_1, k_2, k^2) = \frac{2\gamma (E)}{\pi} \frac{1}{\alpha^2 + p^2} \frac{1}{D (k_1, k_2)} \frac{1}{\alpha^2 + q^2}. \tag{50}$$

As in the single channel scattering (Talukdar *et al* 1979c), the off-shell T matrix given in equation (50) for the two-channel case is separable in the off-shell momenta p and q. Looking at (14) we see that $T (p, q, k_1, k_2 k^2)$ is also separable in the channel momenta k_1 and k_2 although the dependence is somewhat complicated.

4.2 *K matrix*

Application of the standing wave boundary condition within the framework of the integro-differential equation method for the single channel scattering by separable potential has been discussed in some detail by Talukdar and Das (1979). This analysis can be applied to the multichannel case in a straightforward manner. For example the principal value off-shell wave function $\Psi_1^{(p)}$ in the two-channel case is obtained in the form

$$\Psi_1^{(p)}(q, q, k_1\, k_2\, k_1^2) = \sin qr + \frac{q\,A}{(a^2 + k_1^2)(a^2 + q^2)} (\exp(-ar) - \cos kr), \quad (51)$$

where
$$A = \frac{\gamma_{11} N_{1p} + \gamma_{12} N_{2p}}{D_s}. \quad (52)$$

Here N_{1p}, N_{2p} and D_s are given by equations like these in (42). More specifically, the results for these quantities are given by

$$N_{1p} = \gamma' \left[1 + \frac{\gamma_{22}}{2a(a^2 + k_2^2)^2}(a^2 - k_2^2)\right], \quad (53a)$$

$$N_{2p} = -\gamma' \gamma_{12} \frac{1}{2a(a^2 + k_2^2)^2}(a^2 - k_2^2), \quad (53b)$$

and
$$D_s = \gamma' \left[1 + \frac{\gamma_{11}}{2a(a^2 + k_1^2)^2}(a^2 - k_1^2) + \frac{\gamma_{22}}{2a(a^2 + k_2^2)^2}(a^2 - k_2^2)\right], \quad (53c)$$

with
$$\gamma' = \gamma_{11}\gamma_{22} - \gamma_{12}^2. \quad (54)$$

The off-shell K matrix is

$$K(p, q, k_1, k_2\, k_1^2) = \frac{2\gamma(E)}{\pi} \frac{1}{(a^2 + p^2)} \times \left(1 - \frac{A}{2a} \frac{a^2 - k_1^2}{a^2 + k_1^2}\right) \frac{1}{a^2 + q^2}. \quad (55)$$

As in the case of T matrix, $K(p, q, k_1\, k_2\, k_1^2)$ is also separable both in off-shell and channel momenta. Thus we conclude that the separability of T and K matrices are not spoiled by inter-channel interaction. However, we note that as opposed to the single-channel scattering, the K matrix here has become complex. This refers to the fact that when inelastic channels are open, the elastic phase shift becomes complex because of absorption and ultimately leads to a complex off-shell K matrix.

5. Energy-dependent potentials and specific non-local effects

In the preceding sections we have dealt with energy-dependent potentials which result from analysis of multichannel scattering. Specific results have been obtained

for non-local channel interactions. Phenomena characteristic of non-local potentials for the single channel scattering have been studied in the recent past by Mulligan *et al* (1976) and by Bagchi *et al* (1977) using Fredholm determinants $D(k)$ and $D^+(k)$ associated with regular and physical solutions of the Schrödinger equation. A zero of $D^+(k)$ for real $k \neq 0$ has been called a continuum bound state (Gourdin and Martin 1957; Martin 1958). Mulligan *et al* (1976) also demonstrated that $D(k)$ must also be zero when $D^+(k) = 0$. A zero of $D(k)$ for real $k \neq 0$ when $D^+(k) = 0$ is called a spurious state (Coz *et al* 1970). For a single channel scattering a continuum bound state cannot be associated with the Yamaguchi form factor (Mulligan *et al* 1976). However, a two-channel potential with a rank one non-local separable form factor in each of the four potential matrix elements can support a continuum bound state. We investigate below how such a state is related with the cross channel suppression effects which play a role in the binding energy calculation of hypernuclei (Hooyman and Schick 1976).

In §2 we constructed an expression for the Fredholm determinant $D^+(k_1, k_2)$ associated with the physical solution for scattering by the energy dependent potential (10) assuming that the channel potentials have the same range. If we relax this condition and write

$$v_i(r)\, v_j(r') = \exp(-a_i r) \exp(-a_j r),\ i, j = 1, 2, \tag{56}$$

the expression for $D^+(k_1, k_2)$ comes out to be

$$\begin{aligned} D^+(k_1, k_2) = 1 &+ \frac{\lambda_{11}}{a_1^2 + k_1^2}\left(\frac{a_1 + ik_1}{a_1^2 + k_1^2} - \frac{1}{2a_1}\right) + \\ &+ \frac{\lambda_{22}}{a_2^2 + k_2^2}\left(\frac{a^2 + ik_2}{a_2^2 + k_2^2} - \frac{1}{2a_2}\right) + \\ &+ \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 \frac{\lambda'}{(a_1^2 + k_1^2)(a_2^2 + k_2^2)}\left(\frac{a_1 + ik_1}{a_1^2 + k_1^2} - \frac{1}{2a_1}\right)\left(\frac{a_2 + ik_2}{a_2^2 + k_2^2} - \frac{1}{2a_2}\right). \end{aligned} \tag{57}$$

Equation (57) has been written for equal channel masses and in units in which $\hbar^2/2m$ is unity. The following special case of (57) is particularly interesting. When the cross channel is completely suppressed ($\lambda_{12} = 0$) and $k_2 = k_1$ the Fredholm determinant given above goes over to that for the Mongan case IV potential (Mongan 1969) quoted by Mulligan *et al* (1976). The Mongan potential supports a continuum bound state at $E_{\text{lab}} = 400$ MeV for the values of the parameters

$$\begin{aligned} \lambda_{11} &= 105{\cdot}875\ \text{fm}^{-3}, \\ \lambda_{22} &= -\,199{\cdot}752\ \text{fm}^{-3}, \\ a_1 &= 2{\cdot}0\ \text{fm}^{-1}, \end{aligned} \tag{58}$$

and $a_2 = 4{\cdot}0\ \text{fm}^{-1}$.

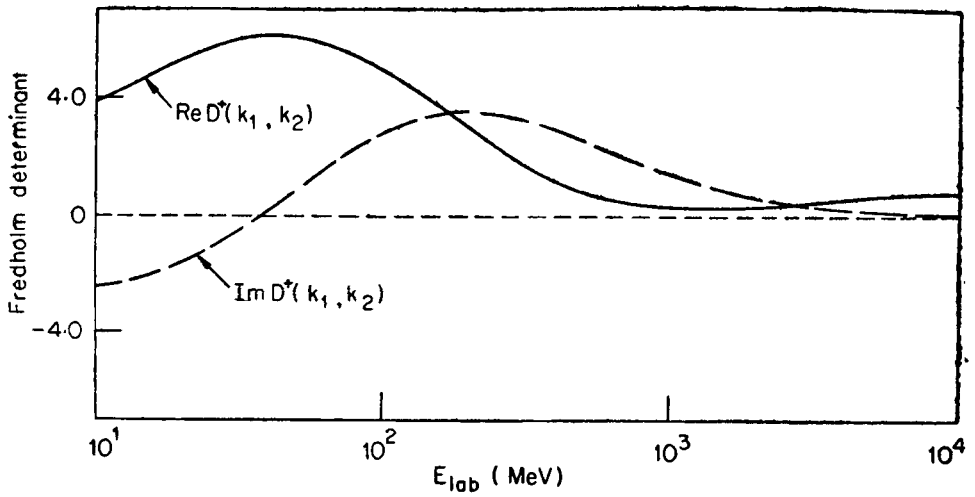

Figure 1. Fredholm determinants $D^+(k) = D^+(k, k)$ as a function of laboratory energy, E_{lab} for different choices of the cross-channel coupling.

The behaviour of the Fredholm determinant $D^+(k) = (= D^+(k, k), \lambda_{12} = 0)$ as a function of E_{lab} is illustrated in figure 1. Since the Mongan potential is used to fit the $1s_0$ nucleon-nucleon phase shifts, the energy wave number conversion factor 1/41·47 MeV^{-1} fm^{-2} is used throughout. In our plot the solid and dashed lines are used to indicate the variation of real and imaginary parts of $D^+(k)$. As with Mulligan *et al* (1976), the curves for Re $D^+(k)$ and Im $D^+(k)$ cut the energy axis at E_{lab} =400 MeV indicating that there is a continuum bound state at this energy. To examine the effect of cross channel mixing on this state we have calculated $D^+(k)$ for two other arbitrarily chosen values of λ_{12}, namely, $\lambda_{12} = \lambda_{11}/10$ and $\lambda_{22}/10$. The values of $D^+(k)$ for $\lambda_{12} = \lambda_{11}/10$ do not differ significantly from those for the completely suppressed case. In fact the two sets of numbers are not discernable on the graphical plot used by us. The continuum bound state is not removed. The results for $\lambda_{12} = \lambda_{22}/10$ are of some interest. In this case both Re $D^+(k)$ and Im $D^+(k)$ shown by solid line with cross and dashed line with cross exhibit considerable departure from the corresponding curves for $\lambda_{12} = 0$ in the low-energy region only. As the energy increases this difference diminishes and practically becomes zero after 1000 MeV. But more significantly, the bound state embedded in the continuum is removed since Re $D^+(k) \neq 0$. Because $|\lambda_{12}/10| \approx 2|\lambda_{11}/10|$ and also that square of λ_{12} occurs in the expression for $D^+(k)$, one can say that any unwanted resonance can be removed only by making allowance for large cross channel mixing.

It is of considerable interest to examine the effect of inelastic channel threshold on the continuum bound state discussed above. To facilitate this we have calculated $D^+(k_1, k_2)$ for $\lambda_{12} = 0$ as a function of E_{lab} $(= k_1^2)$. For each k_1 we used $k_2 = 5k_1$ such that the inelastic channel threshold is rather high. The results are shown in figure 2. Clearly the continuum bound state is removed in this case.

6. Conclusion

We have seen that the existence of a many-channel potential leads to an energy-dependent effective elastic channel potential. As opposed to the traditional momentum

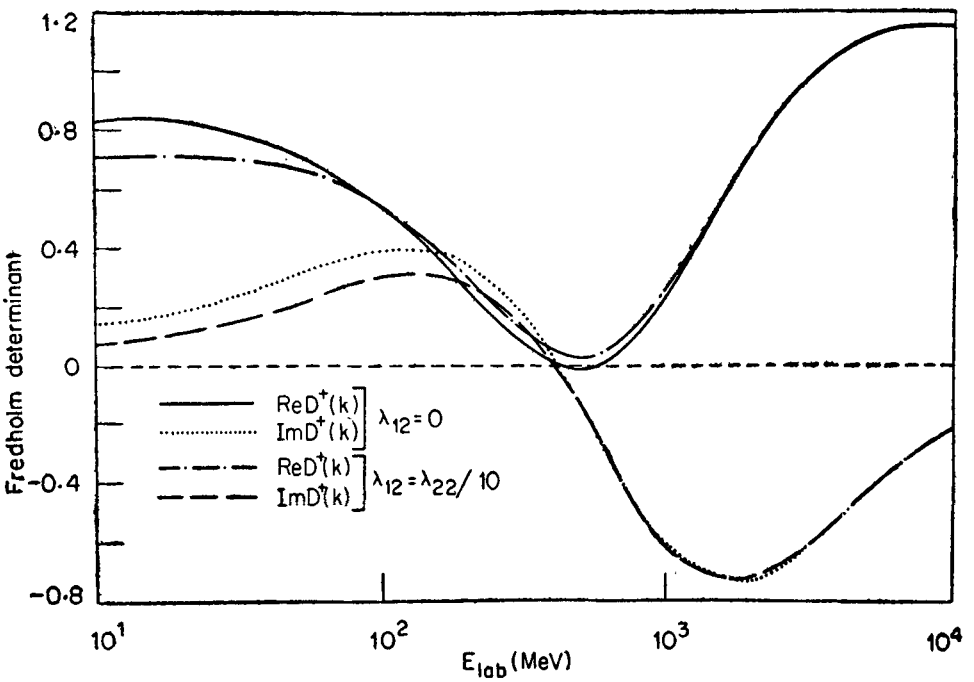

Figure 2. Fredholm determinant $D^+(k)$ for $\lambda_{12} = 0$ at a rather high inelastic channel threshold.

space approach, we have derived a coordinate space method to deal with such a potential, which treats both on- and off-shell scattering on an equal footing. The virtue of the present method is its simplicity. For example, in the present approach all results could be calculated by using tabulated integrals of elementary transcendental functions. In § 5 we have shown that for $k_2 = k_1$ the Fredholm determinant associated with the physical solution of the energy-dependent potential goes over to that for the Mongan case IV potential when the cross channel effect is completely suppressed. For some selected values of the parameters the Mongan potential supports bound states embedded in the continuum. We have examined in some detail the effect of cross channel mixing and that of inelastic channel threshold on this state. In particular it has been found that the existence of the continuum bound state depends more crucially on the inelastic channel threshold than on cross channel mixing.

References

Alberg M, Benley E M and Wilets L 1973 *Phys. Rev. Lett.* **30** 255
Bagchi B and Mulligan B 1974 *Phys. Rev.* **C10** 2197
Bagchi B, Krause T O and Mulligan B 1977 *Phys. Rev.* **C15** 1623
Calogero F 1967 *Mathematics in science and engineering XXXV* (New York: Academic Press)
Coz M, Arnold L G and Mackeller D 1970 *Ann. Phys. (N.Y.)* **59** 219
Ernst D J, Londergan J T, Moniz E J and Thaler R M 1974 *Phys. Rev.* **C10** 1708
Feshbach H 1958 *Ann. Phys. (N.Y.)* **5** 357
Gordon R G 1970 *J. Chem. Phys.* **52** 6211
Gourdin M and Martin A 1957 *Nuovo Cimento* **6** 757

Hoovman P J and Schick L H 1976 *Phys. Rev.* **C13** 2454
Londergan J T and Moniz E J 1973 *Phys. Lett.* **B45** 195
Martin A 1958 *Nuovo Cimento* **7** 607
Mongan T R 1969 *Phys. Rev.* **178** 1597
Mulligan B, Arnold L Gx, Bagchi B and Krause T O 1976 *Phys. Rev.* **C13** 2131
Myhrer F 1973 *Phys. Lett.* **B45** 96
Newton R G 1966 *Scattering theory of waves and particles* (New York: Mc Graw-Hill)
Nogami Y and Satoh E 1970 *Nucl. Phys.* **B19** 93
Satoh E and Nogami Y 1970 *Phys. Lett.* **B32** 243
Tabakin F 1964 *Ann. Phys. (N.Y.)* **30** 51
Talukdar B, Das U 1979 *Pramana* **13** 525
Talukdar B, Das U and Chakraborty S 1979a *Phys. Rev.* **C19** 322
Talukdar B, Das U, Mallick N and Sen Gupta M 1979b *J. Math. Phys.* **20** 378
Talukdar B, Das U and Mukhopadhyaya S 1979c *J. Math. Phys.* **20** 372
van Leeuwen J M J and Reiner A S 1961 *Physica* **27** 99
Yamaguchi Y 1954 *Phys. Rev.* **95** 1628