Pramāṇa, Vol. 17, No. 3, September 1981, pp. 217–227. © Printed in India.

Dead time corrections to photon counting statistics II: Quantum theory

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MS received 2 May 1980

Abstract. The recent formulation of the quantum theory of photodetection, based on the quantum theory of continuous measurements, is extended to the case of a (nonideal) detector which has non-zero dead time. A general result is proven which expresses the dead time modified counting statistics in terms of the counting statistics of an *associated ideal detector*. As an illustration, the dead time corrections to the counting statistics of a single-mode free field are worked out, and these corrections are shown to be identical in form to the dead time corrections for a classical optical field of constant intensity.

Keywords. Quantum theory of continuous measurements; ideal detector; dead time effects; dead time modified photon counting formula; single-mode free field.

1. Introduction

In our previous paper (Srinivas 1981) on dead time corrections to photon counting statistics, we discussed the modifications introduced by dead time effects in the counting statistics of a classical counting process. In the present investigation we shall consider the problem of working out the dead time corrections to the counting statistics of a quantum counting process. It so happens that this problem has not been discussed earlier in the literature.

The main feature of a counting experiment as performed by an ideal detector (*i.e.* a detector which does not become inoperative after registering a count) is that the detector performs continuous measurements over an extended period of time, say the interval $[0, T]$, and records when the counts occur. Most of the conventional approaches to the quantum theory of photodetection (see for example Mandel *et al* 1964; Kelley and Kleiner 1964; Glauber 1965; Lax and Zwanziger 1973) do not take into account correctly the modification of the field distribution as produced by the action of the detector. It is for this reason that the well-known quantum Mandel formula leads to unphysical results (see Srinivas and Davies 1981) such as the occurrence of negative counting probabilities, etc. Recently it has been shown (Srinivas and Davies 1981) that the quantum theory of continuous measurements can be employed to formulate a completely satisfactory theory of photon counting experiments. It has also been shown that for the case of a single-mode free field, the results that are obtained on the basis of the quantum theory of continuous measurements are identical with those obtained earlier by Mollow (1968), Scully and Lamb (1969) and Selloni *et al* (1977).

A quantum theoretic analysis of the dead time effects on the photon counting statistics is also possible only in a framework based on the theory of continuous measurements, and we shall review this theory briefly in § 2. We shall then take up in § 3 the case of a detector with a constant dead time τ. Now the measurement performed by the detector can no longer be taken to be continuous over the entire interval $[0, T]$, but is interrupted by dead time periods of duration τ, each time a count is registered. The main point is that during such a dead time period the detector performs no measurements, and thus the evolution of the system will be completely governed by its own Hamiltonian exactly as in a situation where the detector is absent. Once we take into account the fact that during the dead time periods the system evolves as per its own Hamiltonian, and also that these periods are of equal duration τ and occur after each count is registered*, then we can employ the basic prescriptions of quantum theory for successive measurements and work out the dead time modified counting statistics. We shall show that in general the dead time corrected counting probabilities $\Pr^{\rho}(m; [0, T]; \tau)$ can be expressed in terms of certain dead time free counting probabilities $\widetilde{\Pr}^{\rho}(m; [0, T]; 0)$ as measured by an 'associated ideal detector'. As an illustration of the general theory we shall in § 4 evaluate explicitly the dead time corrections to the photon counting statistics of a single-mode free field. We find that in this case $\widetilde{\Pr}^{\rho}(m; [0, T]; 0)$ become identical to $\Pr^{\rho}(m; [0, T]; 0)$ the counting probabilities as measured by the original detector itself when the dead time effects were absent. We thus obtain the rather interesting result that the dead time corrections to the counting statistics of a single-mode free field in quantum theory are expressible in exactly the same form as the corrections that we derived in classical theory (Srinivas 1981) for the case of an optical field of constant intensity.

2. Preliminaries on the quantum theory of counting processes

The quantum theory of counting process is based on the analysis of the continuous measurements performed by the detector over the interval $[0, T]$ to record when the counts occur. We shall here summarize only those features of the theory of counting experiments which are essential for a discussion of the dead time corrections to the counting statistics and refer the reader to Davies (1969, 1976); Srinivas (1977) and Srinivas and Davies (1981) for further details. The continuous measurements performed by the detector can, in general, be characterized in terms of two measurement transformations $\rho \to S_t \rho$, $\rho \to J \rho$ on the state ρ of the system, which are defined as follows. Let ρ be the density operator state of the system (optical field) at time t_0. If the detector is active during the entire interval $(t_0, t_0 + t)$ and registers no counts, then $S_t \rho$ is the (unnormalized) state of the system at time $t_0 + t$. If the detector is active during the (infinitesimal) period $(t_0, t_0 + \Delta t)$ and records one count during that period, then the (unnormalized) state of the system at time $t_0 + \Delta t$ is

*It is important to note that we are here dealing with a rather curious situation. Though in every run of the counting experiment we set up the detector so as to perform measurements in the interval $[0, T]$, the periods during which the detector is inoperative vary from one trial to another—depending on when the counts are realised in each case. Thus, in a given trial, whether the detector is performing measurements at a given time t or not depends on the outcomes of the measurements performed during the earlier interval $[0, t)$. Of course, the counting statistics that we calculate is the ensemble average of the outcomes of all such trials.

$J\rho\Delta t$. Here both S_t and J are positive linear operators on the space of trace class operators. $\{S_t\}$ is a strongly continuous contraction semigroup (as follows directly from its definition) and J is normally assumed to be bounded*.

Once the basic measurement transformations are specified, then the usual prescriptions of quantum theory for successive measurements can be employed to calculate the various probabilities. Now for the case of an ideal detector the exclusion probability density (EPD) $\tilde{p}^{\rho}_{[0,T]}(t_1, t_2, \ldots, t_m)$ that *one* count is registered around each of the instants t_i $(i = 1, 2, \ldots, m)$ and none in the rest of interval $[0, T]$, can be shown to be given by

$$\tilde{p}^{\rho}_{[0,T]}(t_1, t_2, \ldots, t_m) = \mathrm{Tr}\,[S_{T-t_m} J S_{t_m - t_{m-1}} J \ldots S_{t_2 - t_1} J S_{t_1} \rho], \qquad (1)$$

where ρ is the state of the system at time $t = 0$ when the detector starts performing measurements. If we exclude the possibility of occurrence of multiple counts, then the counting probability $\mathrm{Pr}^{\rho}(m; [0, T])$ that m counts are obtained in the interval $[0, T]$ will be given by

$$\mathrm{Pr}^{\rho}(m; [0, T]) = \int_0^T dt_m \int_0^{t_m} dt_{m-1} \ldots \int_0^{t_2} dt_1 \times \tilde{p}^{\rho}_{[0,T]}(t_1, t_2, \ldots, t_m) = \mathrm{Tr}\,(N_T(m)\,\rho), \qquad (2)$$

for $m \geqslant 1$, where $N_T(m)$ is a linear transformation on the space of trace class operators given by

$$N_T(m) = \int_0^T dt_m \int_0^{t_m} dt_{m-1} \ldots \int_0^{t_2} dt_1 \times S_{T-t_m} J S_{t_m - t_{m-1}} J \ldots S_{t_2 - t_1} J S_{t_1}. \qquad (3)$$

Also, from the definition of the measurement transformation $\rho \to S_t\,\rho$, it follows that the probability that no counts are registered in the interval $[0, T]$ is given by

$$\mathrm{Pr}^{\rho}(0; [0, T]) = \mathrm{Tr}\,(S_T\,\rho)\,. \qquad (4)$$

Thus the counting statistics is determined from (2)–(4) once the pair of measurement transformations (S_t, J) is specified. It can be shown that the counting probabilities (2), (4) satisfy the basic normalization condition

$$\sum_{m=0}^{\infty} \mathrm{Pr}^{\rho}(m; [0, T]) = 1, \qquad (5)$$

*For the sake of applications to quantum optics it becomes necessary to consider unbounded J's also, and this has been done for various particular cases (Davies 1971; Srinivas and Davies 1981). However, in this paper we shall mostly ignore such technical questions.

if the relation

$$\mathrm{Tr}(S_t\,\rho) + \int_0^t ds\ \mathrm{Tr}\,(J\,S_s\,\rho\,) = 1, \tag{6}$$

is satisfied for all $t \geqslant 0$ and any density operator ρ. The relation (6) is also equivalent to the equation

$$\frac{d}{dt}\mathrm{Tr}\,(S_t\,\rho)\Big|_{t=0} = -\,\mathrm{Tr}\,(J\,\rho), \tag{7}$$

for all those ρ for which the equation makes sense.

Finally we may note that though every pair (S_t, J) which satisfies (6) or (7) characterizes a quantum counting process, there is a natural or canonical choice for the semigroup $\{ S_t \}$ once we are given some J and also the evolution $\rho \to U_\tau\,\rho$ of the system when no measurements are performed. If H is the Hamiltonian of the system (in the absence of any measurement interaction with the detector) then

$$U_\tau\,\rho = \exp\,(-i/\hbar\ \ H\tau)\,\rho\,\exp\,(i/\hbar\ \ H\tau) \tag{8}$$

Now, let R be the unique positive operator defined by the equation

$$\mathrm{Tr}\,(J\,\rho) = \mathrm{Tr}\,(\rho\,R), \tag{9}$$

for all density operators ρ. Then the transformation $\rho \to S_t\,\rho$ given by

$$S_t\,\rho = \exp\,[(-i/\hbar\ \ H - R/2)\,t]\,\rho\,\exp\,[(i/\hbar\ H - R/2)\,t], \tag{10}$$

is easily seen to be such that the condition (6) or (7) is automatically satisfied. Also when the measurement interaction is switched off (so that $R = 0$), then S_t goes over into the Hamiltonian evolution U_t.

3. Dead time corrections to the counting statistics of a quantum counting process

In the preceding section we saw that the counting statistics of an ideal detector is completely characterized by the pair of measurement transformations (S_t, J). We shall now discuss how the counting formula (2) will have to be modified when the detector is assumed to have a constant dead time τ during which period it will be inoperative after it registers any count. In the classical theory of photodetection we saw that the dead time modified EPD had to be postulated separately [(15a)–(15c) of Srinivas 1981] though their general form could more or less be deduced from physical grounds. On the other hand, we shall now show that in quantum theory, the fundamental principles of the theory are themselves suffiicent to provide us with a prescription for calculating the dead time modified EPD, and hence also the corresponding corrections to the counting statistics. The main point here is that during the periods when the detector is not performing any measurements, the system evolves as per its own Hamiltonian. Thus during any dead time period of the detector, the evolution of the system is given by $\rho \to U_\tau \rho$, where

$$U_\tau \rho = \exp [(-i/\hbar) H \tau \rho] \exp [(i/\hbar) H \tau], \tag{11}$$

where H is the Hamiltonian of the system when not coupled to the detector. Of course, when the detector is operative the changes in the state of the system are given by the measurement transformations $\rho \to S_t \rho$ and $\rho \to J \rho$, which were defined before and are constrained to satisfy (6) or (7).

Let us assume that the detector starts performing measurements at $t = 0$. The first count can occur at any time $t_1 \geqslant 0$, but the times t_2, t_3, etc at which the later counts occur have to be such that $t_1 \leqslant t_2 - \tau \leqslant t_3 - 2\tau$, etc. Hence in an interval $[0, T]$ a maximum of m counts can occur if $(m-1)\tau \leqslant T < m\tau$. Now if we apply the basic measurement transformations $\rho \to S_t \rho$, $\rho \to J \rho$ in appropriate succession along with the transformation $\rho \to U_\tau \rho$ for the dead time periods, we obtain the following expression for the dead time modified EPD.

$$\tilde{p}^{\rho}_{[0,T]} (t_1, t_2, \ldots, t_m) = 0, \tag{12a}$$

if $| t_i - t_j | < \tau$ for any $i \neq j$;

$$\begin{aligned}\tilde{p}^{\rho}_{[0,T]} (t_1, t_2, \ldots, t_m) &= \mathrm{Tr}\, (U_{T-t_m} J S_{t_m - t_{m-1} - \tau} U_\tau J \\ &\quad \ldots S_{t_2 - t_1 - \tau} U_\tau J S_{t_1} \rho) \\ &= \mathrm{Tr}\, (J S_{t_m - t_{m-1} - \tau} U_\tau J \ldots S_{t_2 - t_1 - \tau} U_\tau J S_{t_1} \rho),\end{aligned} \tag{12b}$$

if $0 \leqslant t_1 \leqslant t_2 - \tau \leqslant t_3 - 2\tau \ldots \leqslant t_m - (m-1)\tau$,

and $t_m \leqslant T < t_m + \tau$;

$$\tilde{p}^{\rho}_{[0,T]} (t_1, t_2, \ldots, t_m) = \mathrm{Tr}\, [S_{T-t_m-\tau} U_\tau J \ldots S_{t_2-t_1-\tau} U_\tau J S_{t_1} \rho], \tag{12c}$$

if $0 \leqslant t_1 \leqslant t_2 - \tau \leqslant \ldots \leqslant t_m - (m-1)\tau \leqslant T - m\tau$.

If we substitute (12a)–(12c) in (2), we obtain the following result for the dead time corrected counting probability

$$\mathrm{Pr}^{\rho} (m; [0, T]; \tau) = 0, \tag{13a}$$

if $T < (m-1)\tau$ and $m \geqslant 1$;

$$\begin{aligned}\mathrm{Pr}^{\rho} (m; [0, T]; \tau) &= \int_{(m-1)\tau}^{T} dt_m \int_{(m-2)\tau}^{t_m - \tau} dt_{m-1} \cdots \int_{0}^{t_2 - \tau} dt_1 \\ &\quad \times \mathrm{Tr}\, (J S_{t_m - t_{m-1} - \tau} U_\tau J \ldots S_{t_2 - t_1 - \tau} U_\tau J S_{t_1} \rho),\end{aligned} \tag{13b}$$

if $(m-1)\tau \leqslant T < m\tau$ and $m \geqslant 1$;

$$\Pr^{\rho}(m;\ [0, T];\ \tau) = \int_{(m-1)\tau}^{T-\tau} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \ldots \int_{0}^{t_2-\tau} dt_1$$

$$\times \operatorname{Tr}(S_{T-t_m-\tau}\, U_\tau\, J \ldots S_{t_2-t_1-\tau}\, U_\tau\, J\, S_{t_1}\, \rho)$$

$$+ \int_{T-\tau}^{T} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \ldots \int_{0}^{t_2-\tau} dt_1 \operatorname{Tr}(J\, S_{t_m-t_{m-1}-\tau}\, U_\tau\, J$$

$$\ldots S_{t_2-t_1-\tau}\, U_\tau\, J\, S_{t_1}\, \rho), \tag{13c}$$

if $T \geqslant m\tau$ and $m \geqslant 1$. Finally, from the basic interpretation of the measurement transformation $\rho \to S_t\, \rho$ and the fact that no dead time interruptions occur unless a count is detected, we obtain

$$\Pr^{\rho}(0;\ [0, T];\ \tau) = \operatorname{Tr}(S_T\, \rho)$$

$$= \Pr^{\rho}(0;\ [0, T];\ 0). \tag{13d}$$

Equations (13a)–(13d) provide the dead time modified counting statistics once S_t, J and U_τ are specified. From these equations it is obvious that $\Pr^{\rho}(m;\ [0, T];\ \tau)$ are non-negative and also continuous as functions of T even at the points $T = (m-1)\tau$, $T = m\tau$. It is also easy to see that when we take the limit $\tau \to 0$, only (13c) and (13d) are relevant and we get

$$\lim_{\tau \to 0} \Pr^{\rho}(m, [0, T], \tau) = \Pr^{\rho}(m, [0, T]), \tag{14}$$

where the right hand side is the ideal detector counting probability as given by (2)–(4). It is not equally obvious whether the counting probabilities $\Pr^{\rho}(m;\ [0, T];\ \tau)$ satisfy the appropriate normalization condition

$$\sum_{m=0}^{\infty} \Pr^{\rho}(m;\ [0, T];\ \tau) = \sum_{m=0}^{M} \Pr^{\rho}(m;\ [0, T];\ \tau) = 1, \tag{15}$$

where M is the integer such that $(M-1)\tau \leqslant T < M\tau$. We shall now show that (15) also follows as a consequence of the condition (6) [or (7)] that is imposed on the measurement transformations S_t and J in any counting process.

We shall first recast (12a)–(12d) in a more convenient form in terms of the counting probabilities as measured by an *associated ideal detector*. For this purpose, we introduce the positive linear transformation $\tilde{J}$ as per the equation

$$\tilde{J}(\rho) = U_\tau\, J(\rho). \tag{16}$$

Since U_τ as defined by (11) is a unitary transformation, we have

$$\int_0^t ds \operatorname{Tr}(\tilde{J} S_s \rho) = \int_0^t ds \operatorname{Tr}(J S_s \rho) = 1 - \operatorname{Tr}(S_t \rho), \tag{17}$$

and hence $S_t, \tilde{J}$ define a quantum counting process whenever S_t, J do. We shall now show that $\Pr(m; [0, T]; \tau)$ can be expressed in terms of the dead time free counting probabilities $\tilde{\Pr}(m; [0, T]; 0)$ as measured by the *associated ideal detector* whose action is characterized by the measurement transformations $S_t, \tilde{J}$. For this purpose let us consider the case when $(m-1)\tau \leqslant T < m\tau$. Then it follows from (13b) and (16) that, $\Pr(m; [0, T]; \tau)$

$$\Pr{}^\rho(m; [0, T]; \tau) = \int_0^{T_{m-1}} dt'_m \int_0^{t'_m} dt'_{m-1} \cdots \int_0^{t'_2} dt'_1$$

$$\times \operatorname{Tr}(\tilde{J} S_{t'_m - t'_{m-1}} \tilde{J} \ldots S_{t'_2 - t'_1} \tilde{J} S_{t'_1} \rho)$$

$$= \int_0^{T_{m-1}} d\theta_1 \int_0^{T_{m-1}-\theta_1} d\theta_2 \ldots \int_0^{T_{m-1} - \sum_{i=1}^{m-1} \theta_i} d\theta_m$$

$$\times \operatorname{Tr}(\tilde{J} S_{\theta_m} \tilde{J} S_{\theta_{m-1}} \ldots \tilde{J} S_{\theta_1} \rho). \tag{18}$$

where $T_{m-1} = T - (m-1)\tau$. Using (17) we can simplify (18) to obtain

$$\Pr{}^\rho(m; [0, T]; \tau) = 1 - \operatorname{Tr}(S_{T_{m-1}} \rho)$$

$$- \sum_{k=1}^{m-1} \int_0^{T_{m-1}} dt'_k \int_0^{t'_k} dt'_{k-1} \cdots \int_0^{t'_2} dt'_1$$

$$\times \operatorname{Tr}(S_{T_{m-1} - t'_k} J \ldots S_{t'_2 - t'_1} J S_{t'_1} \rho), \tag{19}$$

whenever $(m-1)\tau \leqslant T < m\tau$. We can transform (13c) in a similar manner and finally we can reexpress the dead time modified counting statistics in the following form:

$$\Pr{}^\rho(m; [0, T]; \tau) = 0, \tag{20a}$$

if $T < (m-1)\tau$ and $m \geqslant 1$;

$$\Pr{}^\rho(m; [0, T]; \tau) = 1 - \sum_{K=0}^{m-1} \tilde{\Pr}{}^\rho(K; [0, T - (m-1)\tau]; 0), \tag{20b}$$

if $(m-1)\tau \leqslant T < m\tau$ and $m \geqslant 1$;

$$\widetilde{\Pr}^{\rho}(m;\ [0, T];\ \tau) = \sum_{K=0}^{m} \widetilde{\Pr}^{\rho}(K;\ [0, T - m\tau];\ 0)$$

$$- \sum_{K=0}^{m-1} \widetilde{\Pr}^{\rho}(K;\ [0, T - (m-1)\tau];\ 0), \tag{20c}$$

if $T \geqslant m\tau$ and $m \geqslant 1$;

$$\Pr^{\rho}(0;\ [0, T];\ \tau) = \widetilde{\Pr}^{\rho}(0;\ [0, T];\ 0) = \mathrm{Tr}\,(S_T\, \rho). \tag{20d}$$

Here, the counting probabilities $\widetilde{\Pr}^{\rho}(m;\ [0, T];\ 0)$ are those characteristic of the *associated ideal detector* and are given by the following equation similar to (2) and (3).

$$\widetilde{\Pr}^{\rho}(m;\ [0, T];\ 0) = \int_0^T dt_m \int_0^{t_m} dt_{m-1} \ldots \int_0^{t_2} dt_1$$

$$\times\ \mathrm{Tr}\,(S_{T-t_m}\, \tilde{J}\, S_{t_m - t_{m-1}}\, \tilde{J} \ldots S_{t_2 - t_1}\, \tilde{J}\, S_{t_1}\, \rho). \tag{21}$$

We have thus seen that for a counter with a constant dead time τ, the dead time corrected counting probabilities can be expressed as in (20a)–(20d) in terms of the counting probabilities as measured by an *associated ideal detector*. We can now easily verify that the counting probabilities (20a)–(20d) do satisfy the normalization condition (15). For this purpose, let us fix T such that $(M-1)\tau \leqslant T < M\tau$ for some integer M. Then it follows from (20a)–(20d) that

$$\sum_{m=1}^{\infty} \Pr^{\rho}(m;\ [0, T];\ \tau) = \sum_{m=1}^{M} \Pr^{\rho}(m;\ [0, T];\ \tau)$$

$$= 1 - \sum_{K=0}^{M-1} \widetilde{\Pr}^{\rho}(K;\ [0, T-(M-1)\tau];\ 0)$$

$$+ \sum_{m=1}^{M-1} \left\{ \sum_{K=0}^{m} \widetilde{\Pr}^{\rho}(K;\ [0, T-m\tau];\ 0) - \sum_{K=0}^{m-1} \widetilde{\Pr}^{\rho}(K;[0, T-(m-1)\tau];0) \right\}$$

$$= 1 - \widetilde{\Pr}^{\rho}(0;\ [0, T];\ 0)$$

$$= 1 - \Pr^{\rho}(0;\ [0, T];\ \tau), \tag{22}$$

so that (15) is satisfied.

Finally, we shall make a few remarks on the mathematical structure of a quantum counting process when dead time effects are taken into account. Firstly we define the linear transformations $N_t(m, \tau)$ (on the space of trace class operators) by the relation

$$\Pr^{\rho}(m; [0, t]; \tau) = \mathrm{Tr}\,(N_t(m, \tau)\,\rho). \tag{23}$$

We can employ (13a)–(13d) and express $N_t(m, \tau)$ in terms of S_t, J and U_τ. Note that we always have

$$N_t(0, \tau) = S_t,$$

and S_t form a semigroup. However if we now define $T_t(\tau)$ by

$$T_t(\tau) = \sum_{m=0}^{\infty} N_t(m, \tau) = \sum_{m=0}^{M} N_t(m, \tau), \tag{24}$$

where M is an integer such that $(M-1)\,\tau \leqslant t < M\,\tau$, then unlike in the case of an ideal detector (see for example Davies 1976; Srinivas and Davies 1981), $T_t(\tau)$ *do not* form a semigroup whenever $\tau \neq 0$. This is of course a reflection of the fact that when dead time effects come into play, the counting process is no longer time-homogeneous. In fact the time at which the counter starts making measurements (which is $t = 0$ in all the foregoing analysis) plays a special role—for, the counter is always unblocked at that instant, whereas at any other time the counter could be blocked or unblocked depending on the particular times of occurrence of counts.

4. Dead time corrected photon counting statistics of a single-mode free field

To illustrate the general theory of dead time effects outlined in § 3 we shall evaluate the dead time corrections to the photon counting statistics of a single-mode free field which was derived by Srinivas and Davies (1981) on the basis of the quantum theory of continuous measurements. For a single-mode free field, the Hamiltonian is given by

$$H = \omega\, a^{+} a, \tag{25}$$

where a, a^{+} are respectively the creation and annihilation operators. The measurement performed by the detector is characterized by

$$J\,\rho = \lambda\, a\, \rho\, a^{+}, \tag{26}$$

where $\lambda > 0$ is a coupling parameter. Now the canonical choice of S_t as given by (9), (10) leads to

$$S_t\,\rho = \exp\,[(i\omega - \lambda/2)\, a^{+} a\, t]\;\rho\,\exp\,[(-i\omega - \lambda/2)\, a^{+} a\, t]. \tag{27}$$

In the dead time free case, the above choice of S_t and J leads to the counting probabilities

$$\Pr^{\rho}(m; [0, T]; 0) = \sum_{n=m}^{\infty} \binom{n}{m} [1 - \exp(-\lambda T)]^{m}$$

$$\times \exp(-\lambda T(n-m)) \langle n | \rho | n \rangle, \tag{28}$$

where $| n \rangle$ is the n-photon state. Equation (28), which was also earlier derived by Mollow (1968), Scully and Lamb (1969) and Selloni *et al* (1977), differs from the result which follows from the well-known quantum Mandel formula (which is obtained by replacing $\exp(-\lambda T)$ in (28) by $(1 - \lambda T)$). It is easy to see that the counting probabilities as given by (28) are always non-negative, and the mean number of photons counted is always bounded by $\mathrm{Tr}(\rho a^{+} a)$.

In order to derive the dead time corrections to the above counting formula, we need to evaluate the probabilities $\widetilde{\Pr}^{\rho}(m; [0, T]; 0)$. For this, let us consider the transformation $\tilde{J}$ given by

$$\tilde{J}\rho = U_{\tau} J(\rho)$$

$$= \lambda \exp(-i\omega a^{+} a\tau)\, a\, \rho\, a^{+} \exp(i\omega a^{+} a\tau). \tag{29}$$

If we now notice that for each n photon state we have

$$\tilde{J} | n \rangle \langle n | = \lambda n | n \rangle \langle n | = J | n \rangle \langle n |, \tag{30}$$

then it is easy to see that the probabilities $\Pr(m; [0, T]; 0)$ as defined by (21) will also be given by

$$\widetilde{\Pr}^{\rho}(m; [0, T]; 0) = \sum_{n=m}^{\infty} \binom{n}{m} [1 - \exp(-\lambda T)]^{m}$$

$$\exp[-\lambda T(n - m)] \langle n | \rho | n \rangle = \Pr^{\rho}(m; [0, T]; 0). \tag{31}$$

Hence we have the result

$$\Pr^{\rho}(m; [0, T]; \tau) = 0, \tag{32a}$$

if $T < (m - 1)\tau$ and $m \geqslant 1$;

$$\Pr^{\rho}(m; [0, T]; \tau) = 1 - \sum_{j=0}^{m-1} \Pr^{\rho}(j; [0, T - (m - 1)\tau]; 0), \tag{32b}$$

if $(m-1)\tau \leqslant T < m\tau$ and $m \geqslant 1$;

$$\Pr^{\rho}(m; [0, T]; \tau) = \sum_{j=0}^{m} \Pr^{\rho}(j; [0, T - m\tau]; 0)$$

$$-\sum_{j=0}^{m-1} \Pr^{\rho}(j; [0, T-(m-1)\tau]; 0), \tag{32c}$$

if $T \geqslant m\tau$ and $m \geqslant 1$;

$$\Pr(0; [0, T]; \tau) = \Pr(0; [0, T]; 0). \tag{32d}$$

In (32a)–(32d), $\Pr(j; [0, T]; 0)$ are the dead time free counting probabilities as given by (28).

It should be noted that (32a)–(32d) are identical to the result that we derived in the classical theory ((35a)–(35d) of Srinivas 1981) as the corrected version of the results due to Bedard (1967) for the case of an optical field with constant intensity. This is not surprising as, in the classical limit, a single mode free field will be a fluctuating field with constant (*i.e.* time independent) intensity. The important point is of course that in quantum theory the dead time corrections are always representable in a form analogous to (32a)–(32d) in terms of the counting probabilities of an *associated ideal detector*.

Acknowledgement

The author is indebted to Dr Manmohan Singh for suggesting this problem to him.

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