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Dead time corrections to photon counting statistics I: Classical theory

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Abstract. A complete solution is given to the problem of calculating the dead time corrections to the counting statistics of an arbitrary doubly stochastic Poisson process with a non-negative random intensity function. It is shown that for the particular case of an optical field with constant intensity, the general dead time modified counting formula leads to a corrected version of results earlier derived by Bedard.

Keywords. Ideal detector; dead time corrections; doubly stochastic Poisson process; optical field with constant intensity; dead time modified photon counting formula.

1. Introduction

The original investigations of Mandel (Mandel 1958; Mandel 1963) and others (Mandel *et al* 1964; Kelley and Kleiner 1964; Glauber 1965) on the theory (both classical and quantum) of photon counting experiments, were concerned only with the case of the so-called *ideal detector*—a detector which does not become inoperative (or get *blocked*) for a short period (dead time) after registering a count. But it is well known that for any counting experiment, a realistic theory has to take into account the effect of dead time on the counting statistics in order that one can make better comparisons between the experimental results and the theoretical predictions. In the framework of the classical theory of photodetection, the dead time corrections were first considered by Johnson *et al* (1966) who made use of some earlier results on the dead time problem due to Feller (1948) and De Lotto *et al* (1964). A general analysis of the dead time effects for the case of an optical field of constant intensity was given by Bedard (1967) whose results have been widely discussed and applied in the literature (Mehta 1970; Cantor and Teich 1975; Teich and Mcgill 1976; Saleh 1978, etc. For a bibliography on dead time effects, see Müller 1975). There has not so far been any investigation of the dead time effects starting from a fundamental quantum theoretic framework. Even the classical theory as developed by Bedard is of limited applicability as it is restricted to the case of a field of constant intensity. Moreover, the results obtained by Bedard are not entirely correct as they are based on the incorrect assumption that if the detector has a constant dead time τ, then in an interval of time $[0, T]$ a maximum of m counts can be realised only if $T \geqslant m\tau$, whereas it is always possible to realise m counts provided only that $T \geqslant (m-1)\tau$.

In this paper and its sequel we shall present a complete solution to the problem of incorporating dead time corrections to the photon counting statistics, both in classical and quantum theory. The present paper is devoted to the study of dead time effects in classical theory, and is based on the general coincidence approach to classical counting processes developed by Macchi (1975), which is briefly reviewed in § 2. In § 3 we shall show that from very general physical arguments one is led to a suitable ansatz for the so called exclusion probability densities even in cases when dead time effects are present. Based on this ansatz we work out the dead time modifications to the counting statistics which are infact valid for any doubly stochastic Poisson process with a non-negative random intensity function. In § 4, we apply the results of § 3 to the particular case of an optical field with constant intensity and show that our approach leads to a corrected version of Bedard's results.

2. Preliminaries on the counting statistics of an ideal detector

The photon counting formula for the case of a fluctuating classical optical field was first worked out by Mandel (1958) on the basis of the assumption that the counting probabilities in different intervals are statistically independent. His result has been shown to be applicable to a very general class of situations by means of the methods of classical probability theory—in particular, the so called coincidence approach to classical counting processes (see for example Macchi 1975; Srinivas 1977, 1978; Saleh 1978). In this approach one assumes firstly that the counting process is 'semi regular' which means that the so called exclusion probability densities (EPD) $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ exist for all $m \geqslant 1$ and $0 \leqslant t_i \leqslant T$ $(i = 1, 2, \ldots, m)$. The EPD $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ is the probability density that, in a counting experiment performed over the interval $[0, T]$, *one* count is registered around each of the times t_i $(i = 1, 2, \ldots, m)$ and *none* in the rest of interval $[0, T]$. Then, if we rule out the possibility of occurrence of multiple counts, the counting probability $\Pr(m; [0, T])$ that m counts are registered in the interval $[0, T]$ is given by

$$\Pr(m; [0, T]) = \int_0^T dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1\, \tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m), \tag{1}$$

for all $m \geqslant 1$.

The EPD of a semi-regular counting process can be shown (Macchi 1975) to be symmetric non-negative functions satisfying the relation

$$\sum_{m=1}^{\infty} \frac{1}{m!} \int_0^T dt_m \int_0^T dt_{m-1} \cdots \int_0^T dt_1\, \tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m) = 1 - \Pr(0; [0, T]) \leqslant 1. \tag{2}$$

Also conversely, every set of symmetric non-negative functions $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ satisfying (2), characterize a unique semiregular counting process.

For a fluctuating classical optical field, the basic random quantity is the associated random analytic signal $V(r, t)$ (where we have suppressed the polarisation indices) and the related random intensity function

$$I(t) = \iiint_D V^* (\mathbf{r}, t)\, V(\mathbf{r}, t)\, d^3\mathbf{r}, \tag{3}$$

where D is the volume of the detector. Now the basic physical assumption made in the classical theory of photo-detection is that the EPD of the counting process can be expressed in terms of the random intensity function $I(t)$ as follows:

$$\tilde{p}_{[0,T]}\,(t_1, t_2, \ldots, t_m) = \lambda^m \left\langle I(t_1)\, I(t_2) \ldots I(t_m) \exp\left[-\lambda \int_0^T I(t')\, dt'\right]\right\rangle, \tag{4}$$

where $\lambda > 0$ is the *quantum efficiency* of the detector and the angular brackets $\langle\ \rangle$ denote the operation of taking the expectation value. Implicit in the above ansatz (4) is the assumption that the random function $I(t)$ is such that the right hand side of (4) is always a well defined non-negative function* of $t_1, t_2, \ldots, t_m$ so that the EPD as defined by (4) do indeed characterize a semiregular counting process. Now, by employing (2)-(4), we obtain the well-known Mandel formula,

$$\Pr(m;\, [0, T]) = \left\langle \frac{W^m}{m!} \exp(-W) \right\rangle, \tag{5}$$

for all $m \geqslant 0$, where

$$W = \lambda \int_0^T I(t')\, dt'. \tag{6}$$

The photon counting statistics as given by (5) and (6) is characteristic of the so-called doubly stochastic Poisson process with intensity function $I(t)$.

The basic physical assumption as expressed in (3) can often be reformulated in terms of the so called coincidence probability densities (CPD). The CPD $h(t_1, t_2, \ldots, t_m)$ is the probability density that one count is registered around each of the times t_i $(i = 1, 2, \ldots, m)$. If the moments of the counting process are finite, then the CPD exist, and are symmetric non-negative functions related to the CPED as follows:

$$h(t_1, t_2, \ldots, t_m) = \sum_{\gamma=0}^{\infty} \frac{1}{\gamma!} \int_0^T d\theta_1 \int_0^T d\theta_2 \ldots \int_0^T d\theta_\gamma$$

$$\times\ \tilde{p}_{[0,T]}\,(t_1, t_2, \ldots, t_m, \theta_1, \theta_2, \ldots, \theta_\gamma). \tag{7}$$

*We may note that for this it is not necessary that $I(t)$ itself be a non-negative random function though, in classical photodetection theory, $I(t)$ is always non-negative by virtue of (3).

The factorial moments of the counting process can be expressed in terms of the CPD as

$$\langle N^{(k)} \rangle = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1) \Pr(n; [0, T])$$

$$= \int_0^T dt_1 \int_0^T dt_2 \ldots \int_0^T dt_k\, h(t_1, t_2, \ldots, t_k), \tag{8}$$

for all $k \geqslant 1$. Under suitable conditions one can invert (7) to express the EPD in terms of the CPD and obtain

$$\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m) = \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{\gamma!} \int_0^T d\theta_1 \int_0^T d\theta_2 \ldots \int_0^T d\theta_\gamma$$

$$\times h(t_1, t_2, \ldots, t_m, \theta_1, \theta_2, \ldots, \theta_\gamma). \tag{9}$$

For the doubly stochastic Poisson process characterized by (4) the CPD have a particularly simple expression in terms of the random intensity function $I(t)$, *viz*,

$$h(t_1, t_2, \ldots, t_k) = \lambda^k \langle I(t_1)\, I(t_2) \ldots I(t_k) \rangle, \tag{10}$$

as can be easily seen from (4) and (7). We may thus note that the basic physical assumption in the classical theory of photodetection can be formulated either as in (4) or as in (10), and leads to a doubly stochastic Poisson process as given by the Mandel formula (5).

3. Dead time corrections to the counting statistics of a doubly stochastic Poisson process

In the last section we briefly reviewed the classical theory of photodetection for an ideal detector. We shall now consider how this theory is to be modified when dead time effects come in. Our analysis of these dead time effects is very general and is applicable for any doubly stochastic Poisson process with a non-negative random intensity function.

We consider a detector which has a constant dead time τ and also assume that the detector starts performing measurements at time $t = 0$. Our objective is to obtain an expression for the dead time corrected counting probability $\Pr(m; [0, T]; \tau)$ that m counts are registered in the interval $[0, T]$, such that in the limiting case of an ideal detector ($\tau = 0$) we recover the Mandel formula —*i.e.*

$$\lim_{\tau \to 0} \Pr(m; [0, T]; \tau) = \left\langle \frac{W^m \exp(-W)}{m!} \right\rangle. \tag{11}$$

We shall achieve this by appropriately modifying the basic ansatz(4) that was employed in arriving at the Mandel formula (5) for the case of an ideal detector.

When the counter starts performing measurements at time $t = 0$, the first count can be registered at any time $t_1 \geqslant 0$, but then the counter will be inoperative during the period $(t_1, t_1 + \tau)$ and the time t_2 at which the second count is registered is always such that $t_2 \geqslant t_1 + \tau$ and so on. Thus it is easy to see that m counts can be registered by the counter in the interval $[0, T]$ only if $T \geqslant (m-1)\,\tau$*. In other words,

$$\Pr(m;\ [0, T];\ \tau) = 0, \tag{12}$$

if $T < (m-1)\,\tau$. Hence the normalization condition on the probabilities $\Pr(m;\ [0, T];\ \tau)$ is of the form

$$\sum_{m=0}^{\infty} \Pr(m;\ [0, T];\ \tau) = \sum_{m=0}^{M} \Pr(m;\ [0, T];\ \tau) = 1, \tag{13}$$

where M is an integer such that $(M-1)\,\tau \leqslant T < M\,\tau$. Finally, another important restriction on the probabilities $\Pr(m;\ [0, T];\ \tau)$ is that for the case when no counts are registered ($m = 0$), the dead time effects should be absent —*i.e.*

$$\Pr(0;\ [0, T];\ \tau) = \Pr(0;\ [0, T];\ 0), \tag{14}$$

where the right side is as given by the Mandel formula, (11).

Having taken note of the basic requirements that have to be satisfied by the probabilites $\Pr(m;\ [0, T];\ \tau)$, we shall now discuss how the basic ansatz (4) is to be modified in order to arrive at these dead time corrected probabilities. It is obvious that when dead time effects are present, the EPD $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ should vanish whenever $|t_i - t_j| < \tau$ for any $i \neq j$. On the other hand when the condition $|t_i - t_j| \geqslant \tau$ is satisfied for all $i \neq j$, it is natural to assume that $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ is related to the random intensity function $I(t)$ in the same manner as in (4), except that the exponential $\exp[-\lambda \int_0^T I(t')\,dt']$ occuring in the right hand side of (4), (which signifies that no counts are detected in $[0, T]$ except around times $t_1, t_2, \ldots, t_m$), should be replaced by a suitable factor which takes into account the fact that the counter is inoperative during the periods $(t_i, t_i + \tau)$; $i = 1, 2, \ldots, m$. We are therefore led to the following ansatz for the dead time modified EPD:

$$\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m) = 0, \tag{15a}$$

whenever $|t_i - t_j| < \tau$ for some $i \neq j$;

$$\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m) = \lambda^m \langle I(t_1)\, I(t_2) \ldots I(t_m)$$
$$\times \exp\Big[-\lambda \int_0^{t_a} I(t')\,dt'\Big] \prod_{\gamma \neq a} \exp\Big[\lambda \int_{t_\gamma}^{t_\gamma + \tau} I(t')\,dt'\Big] \rangle, \tag{15b}$$

*It should be noted that Bedard (1967) and Teich and coworkers wrongly assert that m counts can occur in the interval $[0, T]$ *only if* $T \geqslant m\,\tau$.

whenever $|t_i - t_j| \geqslant \tau$ for all $i \neq j$ and $t_a \leqslant T < t_a + \tau$ where $t_a = \max\{t_1, t_2, \ldots, t_m\}$;

$$\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m) = \lambda^m \langle I(t_1)\, I(t_2) \ldots I(t_m) \times \exp\left(-\lambda \int_0^T I(t')\, dt'\right) \prod_\gamma \exp\left(\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\, dt'\right) \rangle, \tag{15c}$$

whenever $|t_i - t_j| \geqslant \tau$ for all $i \neq j$ and $T \geqslant t_a + \tau$ where, as before, $t_a = \max\{t_1, t_2, \ldots, t_m\}$.

From (15a)–(15c) it is clear that $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ is a symmetric function which is non-negative whenever $I(t)$ is a non-negative random function. We shall later show that (2) is also satisfied so that the EPD (15a)–(15c) do indeed characterize a semiregular counting process. However, we shall also see that the dead time modified counting statistics is no longer characteristic of a doubly stochastic Poisson process.

In order to obtain the counting formula for $\Pr(m; [0,T]; \tau)$ all that needs to be done is to substitute (15a)–(15c) in (1). Here we should distinguish between the three cases $T < (m-1)\tau$, $(m-1)\tau \leqslant T < m\tau$, and $T \geqslant m\tau$ where $m \geqslant 1$.

Case I : $T < (m-1)\tau$.

It is easy to see from (15a) that the EPD $\tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m)$ vanishes if $T < (m-1)\tau$. Hence we have

$$\Pr(m; [0,T]; \tau) = 0, \tag{16}$$

if $T < (m-1)\tau$, which is in accordance with what we stated earlier.

Case II: $(m-1)\tau \leqslant T < m\tau$.

It follows from (15a) and (1) that

$$\Pr(m; [0,T]; \tau) = \int_{(m-1)\tau}^{T} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \cdots \int_0^{t_2-\tau} dt_1\, \tilde{p}_{[0,T]}(t_1, t_2, \ldots, t_m). \tag{17}$$

Now substituting (15b) in (17), we get

$$\Pr(m; [0,T]; \tau) = \int_{(m-1)\tau}^{T} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \cdots \int_0^{t_2-\tau} dt_1 \times \lambda^m \langle I(t_1)\, I(t_2) \ldots I(t_m) \exp\left(-\lambda \int_0^{t_m} I(t')\, dt'\right) \times \prod_{\gamma=1}^{m-1} \exp\left(\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\, dt'\right) \rangle, \tag{18}$$

if $(m-1)\tau \leqslant T < m\tau$.

Case III: $T \geqslant m\,\tau$.

For this case, if we substitute (15b), (15c) in (17), we get

$$\Pr(m;\ [0, T];\ \tau) = \int_{(m-1)\tau}^{T-\tau} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \cdots \int_{0}^{t_2-\tau} dt_1$$

$$\times\ \lambda^m \left\langle I(t_1) \ldots I(t_m) \exp\left(-\lambda \int_0^T I(t')\,dt'\right) \prod_{\gamma=1}^{m} \exp\left(\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\,dt'\right) \right\rangle$$

$$+ \int_{T-\tau}^{T} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \cdots \int_{0}^{t_2-\tau} dt_1$$

$$\times\ \lambda^m \left\langle I(t_1) \ldots I(t_m) \exp\left(-\lambda \int_0^{t_m} I(t')\,dt'\right)\right.$$

$$\left. \times \prod_{\gamma=1}^{m-1} \exp\left(\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\,dt'\right) \right\rangle, \tag{19}$$

whenever $T \geqslant m\,\tau$.

It can easily be seen that the counting probability $\Pr(m;\ [0, T];\ \tau)$ as given by (16), (18) and (19) is a continuous function of T even when $T = (m-1)\,\tau$ or $T = m\tau$. Also, the dead time corrected counting statistics reduces to the Mandel formula in the limiting case of an ideal detector. For, when $\tau = 0$, it is only case III which is relevent and we get from (19) that

$$\Pr(m;\ [0, T];\ 0) = \int_0^T dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1$$

$$\times\ \lambda^m \left\langle I(t_1) \ldots I(t_m) \exp\left(-\lambda \int_0^T I(t')\,dt'\right) \right\rangle$$

$$= \left\langle \frac{\left(\lambda \int_0^T I(t')\,dt'\right)^m}{m!} \exp\left[-\lambda \int_0^T I(t')\,dt'\right] \right\rangle, \tag{20}$$

which is in agreement with (11).

We shall now show that the counting probabilities (16), (18), (19) do satisfy the condition

$$\sum_{m=1}^{\infty} \Pr(m;\ [0, T];\ \tau) \leqslant 1, \tag{21}$$

and that if we impose the normalization

$$\sum_{m=0}^{\infty} \Pr(m;\ [0, T];\ \tau) = 1, \tag{22}$$

then the zero count probability $\Pr(0;\ [0, T];\ \tau)$ that we obtain, does satisfy the condition (14). For this purpose, it is convenient to define the functions $\Omega(m, T, \tau)$ and $\Theta\ (m, T, \tau)$ as follows:

$$\Omega(m, T, \tau) = \int_{(m-1)\tau}^{T} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \ \ldots \int_{0}^{t_2-\tau} dt_1$$

$$\times\ \lambda^m \left\langle I(t_1) \ldots I(t_m) \exp\left(-\lambda \int_0^{t_m} I(t')\, dt'\right)\right.$$

$$\left. \times \prod_{\gamma=1}^{m-1} \exp\left[\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\, dt'\right]\right\rangle, \tag{23}$$

for all $T \geqslant (m-1)\,\tau$ and $m \geqslant 1$;

$$\Theta\ (m, T, \tau) = \int_{(m-1)\tau}^{T-\tau} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \ldots \int_{0}^{t_2-\tau} dt_1$$

$$\times\ \lambda^m \left\langle I(t_1) \ldots I(t_m) \exp\left[-\lambda \int_0^T I(t')\, dt'\right]\right.$$

$$\left. \times \prod_{\gamma=1}^{m} \exp\left[\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\, dt'\right]\right\rangle, \tag{24}$$

whenever $T \geqslant m\,\tau$ and $m \geqslant 1$, and

$$\Theta\ (0, T, \tau) = \left\langle \exp\left[-\lambda \int_0^T I(t')\, dt'\right]\right\rangle. \tag{25}$$

If $I(t)$ is a non-negative random function, both $\Omega\ (m, T, \tau)$ and $\Theta\ (m, T, \tau)$ are non-negative quantities.

The dead time modified counting statistics given by (16), (18), (19) can now be reexpressed in the following form for $m \geqslant 1$:

$$\Pr(m;\ [0, T];\ \tau) = 0,$$

$$\text{if } T < (m-1)\,\tau; \tag{26a}$$

$$\Pr(m; [0, T]; \tau) = \Omega(m, T, \tau),$$

$$\text{if } (m-1)\tau \leqslant T < m\tau; \tag{26b}$$

$$\Pr(m; [0, T]; \tau) = \Theta(m, T-\tau, \tau)$$

$$+ \Omega(m, T, \tau) - \Omega(m, T-\tau, \tau),$$

$$\text{if } T \geqslant m\tau. \tag{26c}$$

Now, if $T \geqslant (m-1)\tau$, we have from (23) that

$$\Omega(m, T, \tau) = \int_0^{T-(m-1)\tau} dt_1 \int_{t_1+\tau}^{T-(m-2)\tau} dt_2 \ldots \int_{t_{m-1}+\tau}^{T} dt_m$$

$$\times \lambda^{m-1} \left(-\frac{d}{dt_m}\right) \Big\langle I(t_1) \ldots I(t_{m-1})$$

$$\times \exp\left(-\lambda \int_0^{t_m} I(t_1)\, dt'\right) \prod_{\gamma=1}^{m-1} \exp\left(\lambda \int_{t_\gamma}^{t_\gamma+\tau} I(t')\, dt'\right) \Big\rangle,$$

where we have assumed that the operation of taking the derivative commutes with that of taking expectations values. Further simplification of the above equation leads to the result

$$\Omega(m, T, \tau) = \Omega(m-1, T-\tau, \tau)$$

$$- \Theta(m-1, T, \tau), \tag{27}$$

whenever $T \geqslant (m-1)\tau$. Repeated application of (27) leads to

$$\Omega(m, T, \tau) = \Omega(1, T-(m-1)\tau, \tau)$$

$$- \sum_{\gamma=1}^{m-1} \Theta(\gamma, T-(m-1-\gamma)\tau, \tau)$$

$$= 1 - \sum_{\gamma=0}^{m-1} \Theta(\gamma, T-(m-1-\gamma)\tau, \tau), \tag{28}$$

because, from (23) and (25) we have that

$$\Omega(1, T-(m-1)\tau, \tau) = \int_0^{T-(m-1)\tau} dt_1 \left(-\frac{d}{dt_1}\right) \Big\langle \exp\left[-\lambda \int_0^{t_1} I(t')\, dt'\right] \Big\rangle$$

$$= 1 - \Theta(0, T-(m-1)\tau, \tau)$$

We can now employ (28) in (26a)–(26c) to express the dead time modified counting statistics in the following final form for all $m \geqslant 1$:

$$\Pr(m;\ [0, T];\ \tau) = 0, \tag{29a}$$

$$\text{if } T < (m-1)\,\tau;$$

$$\Pr(m;\ [0, T];\ \tau) = 1 - \sum_{\gamma=0}^{m} \Theta(\gamma, T-(m-1-\gamma)\,\tau, \tau) \tag{29b}$$

$$\text{if } (m-1)\,\tau \leqslant T < m\,\tau;$$

$$\Pr(m;\ [0, T];\ \tau) = \sum_{\gamma=0}^{m} \Theta(\gamma, T-(m-\gamma)\,\tau, \tau)$$

$$-\sum_{\gamma=0}^{m-1} \Theta(\gamma, T-(m-1-\gamma)\,\tau, \tau), \tag{29c}$$

$$\text{if } T \geqslant m\,\tau.$$

Now, in order to check the normalization condition (21), let us fix T such that $(M-1)\,\tau \leqslant T < M\,\tau$ so that $\Pr(m;\ [0, T];\ \tau) = 0$ if $m > M$. Now employing (29a)–(29c) we get

$$\sum_{m=1}^{\infty} \Pr(m;\ [0, T];\ \tau) = \sum_{m=1}^{M} \Pr(m;\ [0, T];\ \tau)$$

$$= 1 - \sum_{\gamma=0}^{M-1} \Theta(\gamma, T-(M-1-\gamma)\,\tau, \tau)$$

$$+ \sum_{m=1}^{M-1} \Big\{\sum_{\gamma=0}^{m} \Theta(\gamma, T-(m-\gamma)\,\tau, \tau)$$

$$- \sum_{\gamma=0}^{m-1} \Theta(\gamma, T-(m-1-\gamma)\,\tau, \tau)\Big\}$$

$$= 1 - \Theta(0, T, \tau). \tag{30}$$

Hence we see that (21) is satisfied, and if we impose the normalization condition (22), then we obtain

$$\Pr(0;\ [0, T];\ \tau) = \Theta(0, T, \tau) = \langle \exp[-\lambda \int_0^T I(t')\,dt'] \rangle$$

$$= \Pr(0;\ [0, T];\ 0), \tag{31}$$

in accordance with our earlier requirement (14).

We have thus demonstrated that for a counter with a constant dead time τ, the natural assumption that the EPD are given by (15a)-(15c) leads to the dead time corrected counting statistics as given by (29a)-(29c) and (31). Our analysis is very general and is infact applicable to any doubly stochastic Poisson process with a nonnegative intensity. Our results can also be extended in a straight-forward manner (see for example, Cantor *et al* 1975) to cases where the dead time is variable or even random. However, the actual computation of the dead time corrected counting probabilities (29a)-(29c) given the random intensity function $I(t)$, involves evaluating the complicated expectation values $\Theta\ (m, t, \tau)$ given by (24), and this is in general a highly nontrivial problem. The only general observation that can be made is that the dead time modified counting statistics is no longer that characteristic of a doubly stochastic Poisson process.

4. Dead time corrected counting statistics of an optical field with constant intensity

In order to illustrate the results obtained in the last section, we shall now calculate the dead time corrected counting statistics of an optical field whose random intensity function $I(t)$ happens to be a constant random variable I. As is well-known, in situations where the duration T of the counting experiment is very short compared to the coherence time T_c of the field, the random intensity function $I(t)$ can be treated as very nearly a constant random variable, for the duration of the experiment. As we remarked in the introduction, this happens to be the only case for which dead time effects have been investigated in the literature so far, starting from the pioneering work of Bedard (1967). We shall now show that our general counting formula (29a)-(29c) when applied to this case, leads immediately to a corrected version of the results obtained by Bedard.

When we set $I(t) = I$ (a constant random variable) in (24), we get

$$\Theta\ (m, T, \tau) = \int_{(m-1)\tau}^{T-\tau} dt_m \int_{(m-2)\tau}^{t_m-\tau} dt_{m-1} \dots \int_{0}^{t_2-\tau} dt_1$$

$$\times\ \lambda^m \langle\ I^m \exp\ [-\lambda\, I\, (T - m\,\tau)]\ \rangle$$

$$= \left\langle \frac{[\lambda\, I\, (T - m\,\tau)]^m}{m!} \exp\ [-\lambda\, I\, (T - m\,\tau)] \right\rangle. \tag{32}$$

When we compare this with the counting statistics (for the same field) for the case of an ideal detector as given by

$$\Pr\ (m;\ [0, T]; 0) = \left\langle \frac{(\lambda\, I\, T)^m}{m!} \exp\ [-\lambda\, I\, T] \right\rangle, \tag{33}$$

we have the relation

$$\Theta\ (m, T, \tau) = \Pr\ (m;\ [0, T - m\,\tau]; 0). \tag{34}$$

Substituting (34) in (29a)-(29c) we obtain the following result for the dead time corrected counting statistics of an optical field with constant intensity:

$$\Pr(m;\ [0, T];\ \tau) = 0, \tag{35a}$$

if $T < (m-1)\,\tau$ and $m \geqslant 1$;

$$\Pr(m;\ [0, T];\ \tau) = 1 - \sum_{\gamma=0}^{m-1} \Pr(\gamma;\ [0, T-(m-1)\,\tau];\ 0), \tag{35b}$$

if $(m-1)\,\tau \leqslant T < m\,\tau$ and $m \geqslant 1$;

$$\Pr(m;\ [0, T];\ \tau) = \sum_{\gamma=0}^{m} \Pr(\gamma;\ [0, T-m\,\tau];\ 0) - \sum_{\gamma=0}^{m-1} \Pr(\gamma;\ [0, T-(m-1)\,\tau];\ 0), \tag{35c}$$

if $T \geqslant m\,\tau$ and $m \geqslant 1$;

$$\Pr(0;\ [0, T];\ \tau) = \Pr(0;\ [0, T];\ 0). \tag{35d}$$

In the above equations the counting probabilities $\Pr(m;\ [0, T];\ 0)$ on the right hand side and those associated with an ideal detector are given by the Mandel formula (33).

Of the above equations, (35c) and (35d) are the same as the ones originally derived by Bedard (1967) and recast in the above form by Cantor and Teich (1975). The relation (35b) was completely missed by these investigators because they wrongly assumed that $\Pr(m;\ [0, T];\ \tau)$ vanishes when $(m-1)\,\tau \leqslant T < m\,\tau$. In fact it can easily be seen that (35c) and (35d) by themselves do not lead to the correct normalization (22) for the probabilities unless (35a) and (35b) also hold. Hence the results obtained by Bedard (1967) and Teich and collaborators (Cantor and Teich 1975; Teich and Mcgill 1976) for various optical fields with different statistics for the constant intensity I, will have to be corrected as per our equation (35b) for the case $(m-1)\,\tau \leqslant T < m\,\tau$. Of course, many of the results derived under the assumption $\tau/T \ll 1$, will not be affected.

We should emphasize that apart from the above case where the calculations were rather straight-forward, the dead time corrected counting formula (29) should prove to be of considerable use in calculating the dead time effects in cases where the random intensity function $I(t)$ is no longer a constant as in, say, some of the laser models. Finally we would like to note that exactly the same relations (35a)—(35d) hold between the dead time corrected and ideal counter counting statistics in quantum theory also, for the case of a single-mode free field, as we shall show in the sequel to this paper (Srinivas 1981). However, there it will also be shown that a more general result is valid in quantum theory *viz.* that the dead time corrected counting probabilities $\Pr(m;\ [0, T];\ \tau)$ can always be expressed in the form

$$\Pr(m;\ [0, T];\ \tau) = 0, \tag{36a}$$

if $T < (m-1)\tau$; and

$$\Pr(m;\,[0,T];\,\tau) = 1 - \sum_{\gamma=0}^{m-1} \tilde{\Pr}(\gamma;\,[0,T-(m-1)\tau];\,0), \tag{36b}$$

if $(m-1)\tau \leqslant T < m\tau$; and

$$\Pr(m;\,[0,T];\,\tau) = \sum_{\gamma=0}^{m} \tilde{\Pr}(\gamma;\,[0,T-m\tau];\,0) - \sum_{\gamma=0}^{m-1} \tilde{\Pr}(\gamma;\,[0,T-(m-1)\tau];\,0), \tag{36c}$$

if $T \geqslant m\tau$, where $\tilde{\Pr}(\gamma;\,[0,T];\,0)$ refer to the usual counting probabilities with no dead time effects, but characteristic of measurements performed by an 'associated ideal detector'. This leads us to speculate whether a similar result is true in classical theory also. From (29a)–(29c) it is clear that this would indeed be the case if

$$\Theta(m,T,\tau) = \tilde{\Pr}(m;\,[0,T-m\tau];\,0), \tag{37}$$

where $\tilde{\Pr}(m;\,[0,T];\,0)$ are the dead time free counting probabilities characteristic of a doubly stochastic Poisson process with the random intensity function $\tilde{I}(t)$, which could however be different from the $I(t)$ we started with. In other words, the relations (36a)–(36c) are valid in classical theory also, whenever there exists another non-negative random function $\tilde{I}(t)$ such that

$$\Theta(m,T,\tau) = \left\langle \frac{\left(\lambda \int_0^{T-m\tau} \tilde{I}(t')\,dt'\right)^m}{m!} \exp\left[-\lambda \int_0^{T-m\tau} \tilde{I}(t')dt'\right] \right\rangle, \tag{38}$$

where $\Theta(m,T,\tau)$ is as given by (24), (25). In the case of an optical field with constant intensity we have the simple relation $I(t) = \tilde{I}(t) = I$. It would be interesting to investigate for what class of doubly stochastic Poisson processes, there does exist another random function $\tilde{I}(t)$ such that (38) is satisfied, so that we have the simple relations (36a)–(36c) between the dead time modified and the dead time free counting statistics.

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