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Asymptotic chiral symmetry, meson masses and decay constants

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Abstract. Weinberg's spectral function sum rules are examined to study the axial vector mass spectrum at the level of SU(4). New mass relations and general mass constraints are derived to predict the masses of the charmed axial vector mesons and the $I=0$ pseudoscalar decay constants.

Keywords. Chiral symmetrs; spectral function; sum rules; pseudoscalar mesons; axial vector mesons.

1. Introduction

Over the years, Weinberg's two spectral function sum rules (Weinberg 1967) have been successfully employed to ascertain the extent of symmetry violations of hadrons. While the success of these sum rules at the level of SU(3) is well-known (Das *et al* 1967), at the level of SU(4) or even SU(5), where the breaking of symmetry is expected to be considerably larger, reasonable results have been obtained (Dicus 1975; Ueda 1976; Pham *et al* 1976; Gautam *et al* 1979) in predicting the masses of new vector mesons (belonging to the Ψ and Υ multiplet) by modifying the 2nd sum rule to take into account higher order symmetry breaking effects.

In this paper, we examine the SU(4) axial vector mass spectrum along with their vector counterparts and derive new mass constraints between $D(1285)$, $E(1420)$ and Ψ_A (the $c\bar{c}$ axial vector analog of Ψ) mesons; in addition we also obtain estimates of the decay constants of $I=0$ pseudoscalar mesons. Throughout this paper, we shall assume, owing to the fact that the SU(6) mass relation

$$m^2_{A_1} + m^2_E = 2m^2_{Q_1}$$

is fairly well satisfied by A_1, E and Q_1 (thus suggesting that these are members of 1^{++} family) (for a discussion on this point see Caffarelli and Kang 1976) that A_1, Q_1, $D(1285)$, $E(1420)$, D^{**}, F^{**} along with Ψ_A belong to the $(15)+(1)$ representation of SU(4), (D^{**} and F^{**} being the charmed and charmed-strange axial vector mesons).

In § 2, we present the formalism where we write down Weinberg's 1st and the modified second spectral function sum rules. Section 3 deals with the mass relations obtained from these sum rules while numerical results are given in § 4. Finally a brief summary of our work is given in the concluding § 5.

2. Formalism

Assuming in analogy with the SU(3) theory, the validity of asymptotic symmetry, Weinberg's first sum rule can be written as

$$\int_0^\infty dm^2 \frac{\rho_v^{i,j}(m^2)}{m^2} = \int_0^\infty dm^2 \left[\frac{\rho_v^{i,j}(m^2)}{m^2} + \rho_0^{i,j}(m^2)\right] = C\,[\delta_{ij}+(k-1)\,\delta_{i0}\,\delta_{j0}], \quad (1)$$

where $\rho_v^{i,j}$ is the spin 1 spectral function of the SU(4) vector currents V_μ^i, $\rho_A^{i,j}$ and $\rho_0^{i,j}$ are the spin 1 and spin 0 analog of SU(4) axial vector currents A_μ^i, and C and k are constants independent of the SU(4) index i. The second sum rule in the exact SU(4) limit

$$\int_0^\infty dm^2\, \rho_v^{i,j}(m^2) = \int_0^\infty dm^2\, \rho_A^{i,j}(m^2) = \text{constant} \quad (2)$$

has however to be modified to take into account the effects of symmetry-breaking and provide consistency with the first sum rule (Das *et al* 1967; Sakurai 1967). We parametrise this breaking at the SU(4) level by writing

$$\int_0^\infty dm^2\, \rho_v^{i,j}(m^2) = \int_0^\infty dm^2\, \rho_A^{i,j}(m^2) = C\,[M\delta_{ij} + l\,\delta_{i0}\,\delta_{j0} + N(d_{8ij} + a d_{15ij})], \quad (3)$$

where SU(4) invariant parameters M, N, and a are to be determined from experimental inputs.

Writing the matrix elements of SU(4) currents V_μ^i and A_μ^i as

$$\sqrt{2q_0}\,\langle 0|V_\mu^i|\omega(q)\rangle = g_\omega^i\,\epsilon_\mu(q);\ \sqrt{2q_0}\,\langle 0|A_\mu^i|D(q)\rangle = g_D^i\,\epsilon_\mu(q)$$

$$\sqrt{2q_0}\,\langle 0|V_\mu^i|K^{*j}(q)\rangle = g_{K^*}^i\,\epsilon_\mu(q);\ \sqrt{2q_0}\,\langle 0|A_\mu^i|\pi(q)\rangle = if_\pi\, q_\mu,\ \text{etc.} \quad (4)$$

and restricting ourselves to ground state mesons only, we obtain from equation (1) the set of relations

$$C = g_\rho^2/m_\rho^2 = g_{K^*}^2/m_{K^*}^2 = g_{D^*}^2/m_{D^*}^2 = g_{F^*}^2/m_{F^*}^2 = g_{A_1}^2/m_{A_1}^2 + f_\pi^2,$$

$$= g_{Q_1}^2/m_{Q_1}^2 + f_K^2 = g_{D^{**}}^2/m_{D^{**}}^2 + f_D^2 = g_{F^{**}}^2/m_{F^{**}}^2 + f_F^2, \quad (5)$$

where Q_1, D^{**}, F^{**} are the axial-vector counterparts of the vector mesons K^*, D^* and F^* and

$$C = \sum_V \frac{(g_V^8)^2}{m_V^2} = \sum_V \frac{(g_V^{15})^2}{m_V^2} = \frac{1}{X}\sum_V \frac{(g_V^0)^2}{m_V^2},$$

$$= \sum_A \frac{(g_A^8)^2}{m_A^2} + \sum_P (f_p^8)^2 = \sum_A \frac{(g_A^{15})^2}{m_A^2} + \sum_P (f_p^{15})^2,$$

$$= \frac{1}{X}\left[\sum_A \frac{(g_A^0)^2}{m_A^2} + \sum_P (f_p^0)^2\right], \tag{6}$$

and

$$0 = \sum_V \frac{g_V^8 g_V^{15}}{m_V^2} = \sum_V \frac{g_V^{15} g_V^0}{m_V^2} = \sum_V \frac{g_V^0 g_V^0}{m_V^2},$$

$$= \sum_A \frac{g_A^8 g_A^{15}}{m_A^2} + \sum_P f_p^8 f_p^{15} = \sum_A \frac{g_A^{15} g_A^0}{m_A^2} + \sum_P f_p^{15} f_p^0,$$

$$= \sum_A \frac{g_A^0 g_A^8}{m_A^2} + \sum_P f_p^0 f_p^8, \tag{7}$$

where Σ_V denotes summation over ω, Φ and Ψ mesons, Σ_A over D, E and Ψ_A mesons and Σ_P over η, η' and η_c mesons, η_c being the $c\bar{c}$ pseudoscalar analog of Ψ. The modified second sum rule given by (3) leads to

$$g_\rho^2 = g_{A_1}^2 = C\left[M + N\left(\frac{1}{\sqrt{3}} + \frac{\alpha}{\sqrt{6}}\right)\right],$$

$$g_{K^*}^2 = g_{Q_1}^2 = C\left[M + N\left(-\frac{1}{2\sqrt{3}} + \frac{\alpha}{\sqrt{6}}\right)\right], \tag{8}$$

$$g_{D^*}^2 = g_{D^{**}}^2 = C\left[M + N\left(\frac{1}{2\sqrt{3}} - \frac{\alpha}{\sqrt{6}}\right)\right],$$

$$g_{F^*}^2 = g_{F^{**}}^2 = C\left[M + N\left(-\frac{1}{\sqrt{3}} - \frac{\alpha}{\sqrt{6}}\right)\right],$$

and

$$\sum_V (g_V^8)^2 = \sum_A (g_A^8)^2 = C\left[M + N\left(-\frac{1}{\sqrt{3}} + \frac{\alpha}{\sqrt{6}}\right)\right] = g_8^2,$$

$$\sum_V (g_V^{15})^2 = \sum_A (g_A^{15})^2 = C\left[M - \sqrt{\frac{2}{3}}\,\alpha\, N\right] = g_{15}^2,$$

$$\sum_V (g_V^0)^2 = \sum_A (g_A^0)^2 = C\,[M + l] = g_0^2,$$

and

$$\sum_V g_V^8 g_V^{15} = \sum_A g_A^8 g_A^{15} = \frac{CN}{\sqrt{6}}\ (= \beta), \tag{9}$$

$$\sum_V g^{15} g_V^0 = \sum_A g_A^{15} g_A^0 = \frac{CN}{\sqrt{2}}\,\alpha\ (= \sqrt{3}\,\alpha\beta),$$

$$\sum_V g_V^0 g_V^8 = \sum_A g_A^0 g_A^8 = \frac{CN}{\sqrt{2}} (= \sqrt{3}\,\beta).$$

In the following section, we shall use the set of relations (5) — (9) to obtain mass constraint equations.

3. Mass rules

Before deriving the mass rules, let us note that the parameter C appearing in (1) and (5) can be expressed as

$$C = \frac{g_\rho^2}{m_\rho^2} = 2 f_\pi^2, \tag{10}$$

using the KSRF (Kawarabayashi and Suzuki 1966; Riazuddin and Fayyazuddin 1966) relation $g_\rho^2 = 2\, m_\rho^2\, f_\pi^2$. Thus equations (8) which take the form

$$\begin{aligned}
m_{A_1}^2 &= \frac{C}{C - f_\pi^2}\left[M + N\left(\frac{1}{\sqrt{3}} + \frac{a}{\sqrt{6}}\right)\right];\\
m_{Q_1}^2 &= \frac{C}{C - f_K^2}\left[M + N\left(-\frac{1}{2\sqrt{3}} + \frac{a}{\sqrt{6}}\right)\right];\\
m_{D^{**}} &= \frac{C}{C - f_D^2}\left[M + N\left(\frac{1}{2\sqrt{3}} - \frac{a}{\sqrt{6}}\right)\right];\\
m_{F^{**}}^2 &= \frac{C}{C - f_F^2}\left[M + N\left(-\frac{1}{\sqrt{3}} - \frac{a}{\sqrt{6}}\right)\right];
\end{aligned} \tag{11}$$

by using (5), become

$$\begin{aligned}
m_{A_1}^2 &= 2\left[M + N\left(\frac{1}{\sqrt{3}} + \frac{a}{\sqrt{6}}\right)\right],\\
(2 - f_K^2/f_\pi^2)\, m_{Q_1}^2 &= 2\left[M + N\left(-\frac{1}{2\sqrt{3}} + \frac{a}{\sqrt{6}}\right)\right],\\
(2 - f_D^2/f_\pi^2)\, m_{D^{**}}^2 &= 2\left[M + N\left(\frac{1}{2\sqrt{3}} - \frac{a}{\sqrt{6}}\right)\right],\\
(2 - f_F^2/f_\pi^2)\, m_{F^{**}}^2 &= 2\left[M + N\left(-\frac{1}{\sqrt{3}} - \frac{a}{\sqrt{6}}\right)\right].
\end{aligned} \tag{12}$$

Equations (6) and (12) then imply

$$m_{A_1}^2 = 2\, m_\rho^2, \tag{13a}$$

$$(2 - f_K^2/f_\pi^2)\, m_{Q_1}^2 = 2\, m_{K^*}^2, \tag{13b}$$

$$(2 - f_D^2/f_\pi^2)\, m_{D^{**}}^2 = 2\, m_{D^*}^2 \tag{13c}$$

$$(2 - f_F^2/f_\pi^2)\, m_{F^{**}}^2 = 2\, m_{F^*}^2, \tag{13d}$$

where (13a) is the well-known Weinberg (1967) mass relation.

It is interesting to note that (13b) to (13d) simplify to the following relations

$$m_{Q_1}^2 = 2\, m_{K^*}^2; \quad m_{D^{**}}^2 = 2\, m_{D^*}^2; \quad m_{F^{**}}^2 = 2\, m_{F^*}^2, \tag{14}$$

if we assume that all the decay constants f_π, f_K, f_D, f_F are equal. Equation (14) may be looked upon as a generalized version of the Weinberg relation (13a). Equation (14) also leads to an equal spacing sum rule

$$m_{A_1} - m_{Q_1}^2 = m_{D^{**}}^2 - m_{F^{**}}^2, \tag{15}$$

if we make use of the analogous mass rule for the vector mesons

$$m_\rho^2 - m_{K^*}^2 = m_{D^*}^2 - m_{F^*}^2, \tag{16}$$

which follows from (3).

In addition, we also obtain for axial vector mesons D, E and Ψ_A, the following mass constraint equation by eliminating the coupling constants occurring in (6), (7) and (9)

$$(m_Y^2 f_{0,8}^2 + \sqrt{3}\beta)^2\, (m_Y^2 f_{15}^2 - g_{15}^2) + (m_Y^2 f_{8,15}^2 + \beta)^2\, (m_Y^2 f_0^2 - g_0^2)$$

$$+ (m_Y^2 f_{15,0}^2 + \sqrt{3}\alpha\beta)^2\, (m_Y^2 f_8^2 - g_8^2) = (m_Y^2 f_{15}^2 - g_{15}^2)$$

$$(m_Y^2 f_0^2 - g_0^2)\, (m_Y^2 f_8^2 - g_8^2) - 2\, (m_Y^2 f_{0,8}^2 + \sqrt{3}\beta)\, (m_Y^2 f_{8,15}^2 + \beta)$$

$$(m_Y^2 f_{15,0}^2 + \sqrt{3}\alpha\beta) \tag{17a, b, c}$$

for $Y = D$, E and Ψ_A respectively, where

$$f_8^2 = C - \sum_p (f_5^{p8})^2; \quad f_1^2 = C - \sum_p (f_p^{15})^2; \quad f_0^2 = CX - \sum_p (f_p^0)^2,$$

$$f_{8,15}^2 = \sum_p (f_p^8 f_p^{15}); \quad f_{15,0}^2 = \sum_p (f_p^{15} f_p^0); \quad f_{0,8}^2 = \sum_p (f_p^0 f_p^8). \tag{18}$$

Numerical solutions of (14) and (17) are now taken up in the following section.

4. Numerical calculations

Using the experimental masses for the vector mesons ρ, K^*, D^* and F^* as inputs, (14) gives

$$M_{A_1}=1\cdot1 \text{ GeV};\quad m_{Q_1}=1\cdot26 \text{ GeV};\quad m_{D^{**}}=2\cdot84 \text{ GeV};\quad m_{F^{**}}=3\cdot03 \text{ GeV}. \tag{19}$$

While the experimental numbers for $m_{D^{**}}$ and $m_{F^{**}}$ are awaited, our prediction for the masses of A_1 and Q_1 mesons are in good agreement with their experimental values of 1·1 GeV* and 1·28 GeV respectively.

In order to solve the three equations given by (17), we first of all note that these relations involve five unknown parameters *viz.* k, l, f_8, f_{15} and f_0. However, our experience with the vector mesons shows (Ueda 1976) that it is reasonable to assume the validity of $U(4)$ symmetry: $k=1$, $l=0$. Also η may be taken as a pure octet state with $f_\eta = f_\pi$ (Gell-Mann *et al* 1968). This allows us to solve for f_{15} and f_0 from (17) by taking the experimental masses from any one of the ($I=0$, $J^P=1^+$) set of mesons *viz.* (D, Ψ_A) or (E, Ψ_A) as inputs.** By expressing f_8, f_{15} and f_0 in terms of the decay constants of the physical states η, η' and η_c (keeping in view that η_c is an almost pure $c\bar{c}$ pseudoscalar state) as

$$f_8 = f_\eta = f_\pi;\quad f_{15} = \tfrac{1}{2}(f_{\eta'} - \sqrt{3} f_{\eta_c});\quad f_0 = \tfrac{1}{2}(\sqrt{3} f_{\eta'} + f_{\eta_c}) \tag{20}$$

our results are

$$f_{\eta'}/f_\pi = 1\cdot6;\quad f_{\eta_c}/f_\pi = 0\cdot8, \tag{21}$$

corresponding to the set $m_D = 1\cdot285$ GeV, $m_{\Psi_A} = 3\cdot51$ GeV and

$$f_{\eta'}/f_\pi = 1\cdot1;\quad f_{\eta_c}/f_\pi = 0\cdot4, \tag{22}$$

for the other set $m_E = 1\cdot42$ GeV; $m_{\Psi_A} = 3\cdot51$ GeV.

5. Conclusions

We have recorded in this paper the results obtained for axial vector mesons from Weinberg's spectral function sum rules when extended to SU(4). In particular, we have obtained generalized mass relations of the form $(m^A)^2 = 2(m^v)^2$, for $I = 1$ and 1/2 axial vector and vector mesons respectively. The masses obtained of the charmed and charmed-strange axial vector mesons will provide a useful guide to the search for this class of mesons. For the isoscalars, we have obtained generalized mass constraint relations which relate the pseudoscalar decay constants to the masses of

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*It may be remarked that the new experimental data indicate that the A_1-resonance is wider and may be at a higher mass than previously thought (Particle Data Table 1980).

**The mass of the Ψ_A meson is around 3·51 GeV, see Particle Data Table (1980).

$I = 0$ axial vector mesons. Taking (m_D, m_{Ψ_A}) or (m_E, m_{Ψ_A}) as inputs, we have also obtained estimates of the decay constants $f_{\eta'}$ and f_{η_c}, which are in tune with other theoretical expectations.

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