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Nontrivial quadratic gauge-fixing in Yang-Mills theories†

ASHOK DAS
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742, USA

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Abstract. We study renormalizability of a quadratic gauge-fixing choice involving gauge fields. We show that this can be renormalized simultaneously maintaining the BRS invariance since this respects the underlying global SU(n) invariance. However, this choice, too, induces quartic ghost terms in conformity with our earlier results.

Keywords. Yang-Mills theory; gauge-fixing; quadratic gauge-fixing; BRS symmetry.

1. Introduction

In two recent papers (Das and Namazie 1980; Das 1981) we studied quadratic gauge-fixing conditions in nonabelian gauge theories. Such gauge-fixing conditions have been extensively studied in abelian theories (Joglekar 1974; Shizuya 1976; Midorikawa 1979) and the need to study their behaviour in nonabelian theories arises if one follows the ideas of unification. The unified gauge theories contain, in addition to gluons, weak bosons and photon, the heavy X, Y bosons which mediate processes like the proton decay (Chang *et al* 1981). On the other hand, since these particles are extremely heavy, at low energies one would like to define an effective field theory as an expansion in the inverse mass parameter (Hagiwara and Vakazawa 1980; Ovrut and Schnitzer 1980). In doing so one would also like to maintain the low energy symmetries manifest. It is natural, then, to choose the gauge-fixing conditions for the off-diagonal gauge bosons "covariantly" with respect to the unbroken gauge groups (Weinberg 1980). This is essentially how nonlinear gauge-fixing conditions may enter nonabelian gauge theories.

We have already shown (Das and Namazie 1980; Das 1981) that quadratic gauge-fixing conditions in nonabelian gauge theories in principle induce quartic ghost terms (Curci and Ferrari 1976a, 1976b, 1978) through renormalisation. Since the Faddeev-Popov (1967) procedure only gives quadratic terms in ghost fields and since BRS invariance (Becchi *et al* 1975, 1976) is essential to prove unitarity in such theories, one should appeal to the BRS symmetry in selecting the quartic ghost interactions. The BRS symmetry does not select out a unique quartic ghost Lagrangian. However, one can utilise the fact that the BRS symmetry itself possesses a global invariance corresponding to the original gauge group. Therefore, the Lagrangian that can be renormalised preserving the BRS symmetry is the one that respects the above global

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invariance or at the most breaks it softly (through operators of dim < 4) which then selects out a unique Lagrangian. In the simple theory of SO(3) gauge fields which we used before, the most general Lagrangian which maintains the above global invariance does not involve terms quadratic in the gauge fields in the gauge fixing condition. The reason is not hard to understand. SO(3) is a very simple group in the sense that it only has antisymmetric structure constants. Therefore, one cannot construct a nonvanishing term quadratic in gauge fields which also respects the global invariance. On the other hand, in larger nonabelian groups, which possess both symmetric and antisymmetric structure constants, it may be possible to choose nontrivial gauge-fixing conditions involving terms quadratic in the gauge fields. We study the renormalisability of such a gauge-fixing condition in an SU(n) theory. We describe the theory and the gauge-fixing conditions in § 2. The BRS symmetry and the Ward identities are also derived for completeness. In § 3 we point out additional modifications that may have to be incorporated and study the question of renormalisability. Our conclusions are presented in § 4.

2. Theory

Let us consider an SU(n) gauge theory where $n>2$. If T^a denote the generators of the group in the adjoint representation, then with the normalisation fixed by

$$\mathrm{Tr}\,(T^a T^b) = 2\delta^{ab}, \qquad a, b = 1, 2, \ldots, n^2 - 1$$

we have

$$T^a T^b = \frac{2}{n}\,\delta^{ab} + (d^{abc} + i f^{abc})\, T^c, \tag{1}$$

where d^{abc} and f^{abc} are respectively the totally symmetric and the totally antisymmetric structure constants of the group and are defined by

$$[T^a, T^b] = 2i f^{abc}\, T^c,$$

and

$$\{T^a, T^b\} = \frac{4}{n}\,\delta^{ab} + 2 d^{abc}\, T^c. \tag{2}$$

From the Jacobi identities one can write down the following relations involving these structure constants

$$f^{abp} f^{pcd} + f^{cap} f^{pbd} + f^{bcp} f^{bad} = 0,$$

$$d^{abp} f^{pcd} + d^{cap} f^{pbd} + d^{bcp} f^{pad} = 0,$$

$$f^{abp} f^{pcd} = \frac{2}{n}\,(\delta^{ac}\,\delta^{bd} - \delta^{ad}\,\delta^{bc}) - (d^{adp}\, d^{pbc} - d^{acp}\, d^{pdb}). \tag{3}$$

The gauge invariant Lagrangian is given by

$$L_{\text{inv}} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu},$$

where $\quad D_\mu A^a_\nu = \partial_\mu A^a_\nu - gf^{abc} A^b_\mu A^c_\nu.$

and $$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu. \tag{4}$$

The theory is invariant under the local gauge transformation

$$\delta A^a_\mu (x) = D_\mu \alpha^a (x), \tag{5}$$

where $\alpha^a (x)$ is the gauge parameter. The structure of the theory is singular because of the underlying local gauge invariance and, therefore, to quantise the theory covariantly in the path integral formulation we must add a gauge-fixing Lagrangian. We choose our gauge-fixing Lagrangian to be

$$L_{GF} = - F^a \left(\partial{\cdot}A^a + \frac{\lambda}{2} d^{abc} A^b_\mu A^c_\mu\right) + \frac{\beta}{2} F^{a2}. \tag{6}$$

Here we have introduced an auxiliary field F^a to write our gauge-fixing condition. When this field is integrated out in the path integral formulation, one recovers the familiar structure of the gauge-fixing Lagrangian

$$L_{GF} = -\frac{1}{2\beta}\left(\partial{\cdot}A^a + \frac{\lambda}{2} d^{abc} A^b_\mu A^c_\mu\right)^2, \tag{7}$$

except for the fact that our gauge-fixing condition is now nonlinear. We would, however, choose to work with the form of the Lagrangian (6) for various reasons—the most relevant reason for the present discussion being that the auxiliar field formulation helps understand the divergent structure much better.

The modification of the theory by the gauge-fixing Lagrangian would require a compensating ghost Lagrangian (Faddeev and Popov 1967) given by

$$L_{\text{ghost}} = \partial_\mu \bar{C}^a D_\mu C^a - \lambda d^{abc} A^b_\mu \bar{C}^a D_\mu C^c. \tag{8}$$

The theory is now described by the effective Lagrangian

$$L_{\text{eff}} = L_{\text{inv}} + L_{GF} + L_{\text{ghost}}, \tag{9}$$

and the generating functional for the Green's functions are given by

$$\begin{aligned} Z(J_\mu, J, \eta, \bar{\eta}) &= \exp\, [iW(J_\mu, J, \eta, \bar{\eta})] \\ &= \int [d\phi] \exp\left\{i \int d^4x\, [L_{\text{eff}} + J^a_\mu A^a_\mu + J^a F^a + i(\bar{\eta}^a C^a - \bar{C}^a \eta^a)]\right\}. \end{aligned} \tag{10}$$

Here $[d\phi]$ generically stands for integration over all field variables. We would like to point out here that the local gauge invariance of the theory given by (5) is manifestly broken by the addition of the gauge-fixing and the ghost Lagrangian. However, the effective Lagrangian still possesses a global invariance originally discovered by Becchi *et al* (1975, 1976) and has the form

$$\begin{aligned} \delta A^a_\mu &= \omega D_\mu C^a, \\ \delta C^a &= \frac{\omega g}{2} f^{abc} C^b C^c, \\ \delta \bar{C}^a &= -\omega F^a, \\ \delta F^a &= 0. \end{aligned} \tag{11}$$

Here ω is an anticommuting global parameter and it is not difficult to see that the effective Lagrangian does not change under the above set of BRS transformations. We would like to point out here that the BRS symmetry itself remains invariant under global SU(n) rotation of all fields including the ghosts and the auxiliary fields. The BRS symmetry is essential in proving unitarity in gauge theories and it also helps understand the renormalisability of the theory through Ward identities which we derive next.

We introduce into the generating functional (10) sources corresponding to the composite BRS variations

$$\begin{aligned} Z = \exp(iW) &= \int [d\phi] \exp\Big\{ i \int d^4x \, [L_{\text{eff}} + J^a_\mu A^a_\mu + J^a F^a \\ &\quad + i(\bar{\eta}^a C^a - \bar{C}^a \eta^a) + K^a_\mu (\omega^{-1} \delta A^a_\mu) + L^a (\omega^{-1} \delta C^a)] \Big\} \\ &= \int [d\phi] \exp(iS_{\text{eff}}). \end{aligned} \tag{12}$$

The generating functional does not change under a redefinition of the fields since all fields are integrated over. In particular if we make an infinitesimal BRS transformation on the fields it should be stationary.

$$\delta Z = i\delta W \exp(iW) = 0 = i \int [d\phi] \, \delta S_{\text{eff}} \exp(iS_{\text{eff}}). \tag{13}$$

This can also be expressed in terms of the sources as

$$J^a_\mu \frac{\delta W}{\delta K^a_\mu} + i\eta^a \frac{\delta W}{\delta L^a} - \frac{\delta W}{\delta J^a} \eta^a = 0. \tag{14}$$

We can define the generating functional for the $|P|$ vertex functions by a Legendre transformation

$$\begin{aligned} \Gamma(\phi, K_\mu, L) &= W(J_\mu, J, \eta, \bar{\eta}, K_\mu, L) \\ &\quad - \int d^4x \, (J^a_\mu A^a_\mu + J^a F^a + i(\bar{\eta}^a C^a - \bar{C}^a \eta^a). \end{aligned} \tag{15}$$

Equation (15) then becomes

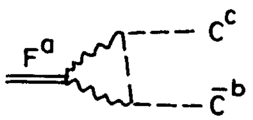
$$\frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta K^a_\mu}+\frac{\delta\Gamma}{\delta C^a}\frac{\delta\Gamma}{\delta L^a}-F^a\frac{\delta\Gamma}{\delta\bar{C}^a}=0. \tag{16}$$

This contains the Ward identities for the theory and by successively differentiating (16) with respect to different fields one can derive relations between various amplitudes that renormalisation must respect.

3. Renormalisability

Even before venturing on the study of renormalisability of the theory, it is worth asking if the theory can generate any obvious new divergent structures at higher orders. Looking at the effective Lagrangian one notices immediately that there are vertices in the theory of the form FAA, $A\bar{C}C$ and $AA\bar{C}C$. These can in principle induce divergent terms of the form $F\bar{C}C$. In fact

$$\propto \frac{(n^2-4)}{n}\cdot\frac{8}{\epsilon}f^{abc}$$



$$\propto \frac{(n^2-4)}{n}\cdot\frac{2}{\epsilon}f^{abc} \text{ where } \epsilon = 4-d.$$

It is, therefore, clear that unless one introduces into the Lagrangian terms of the from $F\bar{C}C$ to begin with, one would run into problems in multiplicative renormalisation. However, addition of just a term of the form $F\bar{C}C$ would spoil the BRS invariance of the theory which is so crucial to proving unitarity in gauge theories. Therefore, one looks for a BRS-invariant completion of terms of the form $F\bar{C}C$. In fact the new Lagrangian,

$$L_{\text{new}} = -\lambda_2 f^{abc}F^a\bar{C}^bC^c+\frac{\lambda_2 g}{4}f^{abc}f^{cpq}\bar{C}^a\bar{C}^bC^pC^q, \tag{17}$$

is invariant under the BRS transformations and one can define an effective Lagrangian

$$\tilde{L}_{\text{eff}} = L_{\text{eff}}+L_{\text{new}}, \tag{18}$$

as the starting theory. We would like to point out here that the new Lagrangian can be viewed as a gauge-fixing condition involving ghosts and its BRS completion. We have

$$\tilde{L}_{\text{eff}} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}-F^a\left(\partial\cdot A^a+\frac{\lambda_1}{2}d^{abc}A^b_\mu A^c_\mu+\lambda_2 f^{abc}\bar{C}^bC^c\right)+\frac{\beta}{2}F^{a^2}$$
$$+\partial_\mu\bar{C}^aD_\mu C^a-\lambda_1 d^{abc}A^b_\mu\bar{C}^aD_\mu C^c+\frac{\lambda_2 g}{4}f^{abc}f^{cpq}\bar{C}^a\bar{C}^bC^pC^q. \tag{19}$$

This effective Lagrangian still possesses the BRS invariance of (11) and therefore, the Ward identities of the theory are still given by (16). To study the renormalisability of the theory we now parametrise various amplitudes of the theory in the following way.

2 pt. functions (All momenta flowing in)

$$\frac{\delta^2 \Gamma}{\delta A^a_\mu(p)\, \delta A^b_\nu(-p)} = -(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\delta^{ab} I_A,$$

$$\frac{\delta^2 \Gamma}{\delta F^a(p)\, \delta A^b_\mu(-p)} = ip_\mu \delta^{ab} I_M,$$

$$\frac{\delta^2 \Gamma}{\delta F^a(p)\, \delta F^b(-p)} = \beta\delta^{ab} I_F,$$

$$\frac{\delta^2 \Gamma}{\delta \bar{C}^a(p)\, \delta C^b(-p)} = p^2 \delta^{ab} I_C,$$

$$\frac{\delta^2 \Gamma}{\delta K^a_\mu(p)\, \delta C^b(-p)} = -ip_\mu \delta^{ab} I_K. \tag{20}$$

3 pt. functions: $p + q + r = 0$

$$\frac{\delta^3 \Gamma}{\delta A^a_\mu(p)\, \delta A^b_\nu(q)\, \delta A^c_\lambda(r)} = igf^{abc}[(p-q)_\lambda \delta_{\mu\nu} + (q-r)_\mu \delta_{\nu\lambda} + (r-p)_\nu \delta_{\mu\lambda}] T_A,$$

$$\frac{\delta^3 \Gamma}{\delta F^a(p)\, \delta A^b_\mu(q)\, \delta A^c_\nu(r)} = -\lambda_1 \delta\mu\nu\, d^{abc} T_{1F},$$

$$\frac{\delta^3 \Gamma}{\delta F^a(p)\, \delta \bar{C}^b(q)\, \delta C^c(r)} = -\lambda_2 f^{abc}\, T_{2F},$$

$$\frac{\delta^3 \Gamma}{\delta \bar{C}^a(p)\, \delta A^b_\mu(q)\, \delta C^c(r)} = -igp_\mu f^{abc}\, \tilde{T}_{1A} - i\lambda_1 d^{abc} r_\mu\, \tilde{T}_{2A},$$

$$\frac{\delta^3 \Gamma}{\delta K^a_\mu(p)\, \delta A^b(q)\, \delta C^c(r)} = -gf^{abc}\, T_K,$$

$$\frac{\delta^3 \Gamma}{\delta L^a(p)\, \delta C^b(q)\, \delta C^c(r)} = gf^{abc}\, T_L. \tag{21}$$

4 *pt. functions*: $p + q + r + s = 0$

$$\frac{\delta^4 \Gamma}{\delta A^a_\mu(p)\, \delta A^b_\nu(q)\, \delta A^c_\lambda(r)\, \delta A^d_\rho(s)} = -\, g^2 \{f^{eab} f^{ecd} (\delta_{\mu\lambda}\, \delta_{\nu\rho} - \delta_{\mu\rho}\, \delta_{\nu\lambda})$$

$$+ f^{eac} f^{edb} (\delta_{\mu\rho}\, \delta_{\nu\lambda} - \delta_{\mu\nu}\, \delta_{\lambda\rho}) + f^{ead} f^{ebc} (\delta_{\mu\nu}\, \delta_{\lambda\rho} - \delta_{\mu\lambda}\, \delta_{\nu\rho})\} F_A$$

$$\frac{\delta^4 \Gamma}{\delta A^a_\mu(p)\, \delta A^b_\nu(q)\, \delta \bar{C}^c(r)\, \delta C^d(s)} = \lambda_1 g\, \delta\mu\nu\, (d^{cap} f^{pbd} + d^{cbp} f^{pad})\, \tilde{F}_A$$

$$\frac{\delta^4 \Gamma}{\delta \bar{C}^a(p)\, \delta \bar{C}^b(q)\, \delta C^c(r)\, \delta C^d(s)} = \lambda_2 g f^{abp} f^{pcd}\, F_C \tag{22}$$

We are using minimal subtraction scheme and, therefore, in dimensional regularization these unknown parameters I, T and F's would contain only poles in ϵ. One can derive various identities from (16) and substituting the above parametrization one finds the following relations between the unknown parameters.

$$I_K/T_K = I_A/T_A = I_C/\tilde{T}_{1A}, \qquad I_C/I_M = \tilde{T}_{2A}/T_{1F} = I_K$$

$$T_K = T_L \qquad F_A/T_A = T_A/I_A$$

$$\tilde{F}_A/\tilde{T}_{2A} = \tilde{T}_{1A}/I_A \qquad F_C\, T_{1F}/\tilde{T}_{2A} = T_{2F} \cdot \tilde{T}_{1A}/I_C. \tag{23}$$

It is not hard to see that these relations imply a scaling of the form

$$A^a_\mu = Z_{3A}^{1/2}\, A^{a^r}_\mu, \qquad F^a = Z_{3F}^{1/2}\, F^{a^r},$$

$$(C^a, \bar{C}^a) = Z_{3C}^{1/2}\, (C^{a^r}, \bar{C}^{a^r}), \qquad \beta = Z_\beta\, \beta^r,$$

$$g = Z_1/Z_{3A}^{3/2}\, g^r \qquad \lambda_1 = Z_{1F}/Z_{3F}^{1/2}\, Z_{3A}\, \lambda_1 r,$$

$$\lambda_2 = Z_{2F}/Z_{3F}^{1/2}\, Z_{3C}\, \lambda_2^r. \tag{24}$$

The Lagrangian can, therefore, be written in terms of the renormalized fields and parameters as

$$L_{\text{eff}} = -\frac{1}{4} Z_{3A}\, F^{a^r}_{\mu\nu}\, F^{a^r}_{\mu\nu} - Z_{3F}^{1/2}\, F^{a^r} \left(Z_{3A}^{1/2}\, \partial \cdot A^{r_a} + d^{abc} \frac{Z_1^r Z_{1F}}{Z_{3F}^{1/2}}\, A^{br}_\mu\, A^{cr}_\mu \right.$$

$$\left. + \lambda_2^r f^{abc} \frac{Z_{2F}}{Z_{3F}^{1/2}}\, C^{b^r}\, C^{c^r} \right)$$

$$+ Z_\beta Z_{3F} \frac{\beta r}{2} F^{ar^2} + Z_{3C} \partial_\mu \bar{C}^{ar} D_\mu C^{ar}$$

$$- \frac{Z_{1F} Z_{3C}}{Z_{3F}^{1/2} Z_{3A}^{1/2}} \lambda_1^r d^{abc} A_\mu^{br} \bar{C}^{ar} D_\mu C^{cr}$$

$$- \frac{Z_{2F} Z_1 Z_{3C}}{Z_{3F}^{1/2} Z_{3A}^{3/2}} \frac{\lambda_2^r g^r}{4} f^{abc} f^{cpq} \bar{C}^{ar} \bar{C}^{br} C^{pr} C^{qr} \tag{25}$$

4. Conclusions

We have shown in this paper that in nonabelian theories which possess both symmetric and antisymmetric structure constants one can have nonlinear gauge-fixing conditions involving gauge fields. However, such a choice does induce quartic ghost coupling as had been claimed earlier (Das and Namazie 1980; Das 1981). Since the particular gauge-fixing choice respects the underlying global invariance, it is possible to renormalise the theory simultaneously maintaining the BRS symmetry. We do not know whether this particular gauge choice would be useful anywhere but the S-matrix elements would be independent of the new couplings λ_1 and λ_2.

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