Pramāṇa, Vol. 16, No. 5, May 1981, pp. 375–383. © Printed in India.

A NUT-like electrovac spacetime

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MS received 17 November 1980; revised 27 March 1981

Abstract. A solution of the Einstein-Maxwell equations corresponding to source-free electromagnetic field is obtained. The solution is algebraically special. A particular case of the solution is considered which includes Brill's solution. The details regarding the solution are also discussed.

Keywords. General relativity; Algebraically special solution; Einstein-Maxwell equations; electromagnetic field.

1. Introduction

There are two types of solutions of Einstein-Maxwell equations in general relativity, namely algebraically general solution and algebraically special solution. Inspite the fact that an exact gravitational solution radiating from a finite source must be algebraically general (Sachs 1961), many investigators have shown keen interest in obtaining algebraically special solutions. One of the reasons behind this is that the Schwarzschild exterior solution, the Kerr solution (Kerr 1963) and the NUT solution (Newman *et al* 1963) are familiar members of this class. The aim of the present paper is to derive a NUT-like algebraically special solution of Einstein-Maxwell equations with the help of the complex vectorial formalism formulated by Cahen *et al* (1967). A lucid account of this formalism is also given by Israel (1970). It will not be out of place to give a very brief summary of this formalism.

2. Complex vectorial formalism

Consider a four-dimensional pseudo Riemannian space-time manifold V_4. Let l_α and n_α be two future pointing real null vector fields and m_α be a complex null vector field on V_4. They are such that the metric on V_4 has the form

$$g_{\alpha\beta} = 2\, l_{(\alpha}\, n_{\beta)} - 2m_{(\alpha}\, \overline{m}_{\beta)}, \tag{1}$$

with bar denoting the complex conjugation. Here and in what follows the Greek and the first half of Latin indices will range from 1 to 4 and the second half of the Latin indices will range from 1 to 3.

Introducing the basic 1-forms

$$\theta^1 = l_a dx^a, \quad \theta^4 = n_a\, dx^a, \quad \theta^2 = m_a\, dx^a, \quad \theta^3 = \overline{\theta^2}. \tag{2}$$

One can write (1) as

$$(ds)^2 = 2\,(\theta^1\, \theta^4 - \theta^2\, \theta^3) = g_{ab}\, \theta^a\, \theta^b. \tag{3}$$

Here x^a are the local co-ordinates in V_4. Let Z^p be a basis for the complex 3-space ξ^3 of self-dual 2 forms, given as

$$z^1 = \theta^3 \wedge \theta^4, \quad \theta^2 = \theta^1 \wedge \theta^2, \quad z^3 = \tfrac{1}{2}\,(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3). \tag{4}$$

The metric γ_{pq} for the space ξ^3 is given as

$$\gamma^{pq} = 2\,\delta^{(p}_{1}\,\delta^{q)}_{2} - \tfrac{1}{2}\,\delta^p_3\,\delta^q_3. \tag{5}$$

In the absence of torsion in the Riemannian space, the affine connection 1 forms ω_b^a and the curvature 2-forms Ω^a_b are determined by the following equations known as Cartan's structural equations:

$$d\theta^a = -\,\omega^a_b \wedge \theta^b, \tag{6}$$

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \tag{7}$$

where d and $\wedge$ denote respectively the exterior differentiation and the exterior product. The connection 1-forms ω_{ab} and the curvature 2-forms Ω_{ab} are related to Ricci rotation coefficients Γ_{abc} and curvature R_{abcd} as follows:

$$\omega_{ab} = \Gamma_{abc}\, \theta^c,$$

$$\Omega_{ab} = \tfrac{1}{2}\, R_{abcd}\, \theta^c \wedge \theta^d. \tag{8}$$

In complex 3 space ξ^3, (6) is replaced by

$$dz^p = \tfrac{1}{2}\, \epsilon^{pmn}\, \sigma_m \wedge z_n \tag{9}$$

where σ_m are three valued 1-forms which serve as six connection 1 forms ω_{ab}. σ_m are related to ω_{ab} as follows.

$$-\,\omega^1_1 = \omega^4_4 = \tfrac{1}{2}\,(\sigma_3 + \bar{\sigma}_3)\,; \quad -\,\omega^2_2 = \omega^3_3 = \tfrac{1}{2}\,(\sigma_3 - \bar{\sigma}_3)$$

$$\omega^1_3 = \omega^2_4 = -\,\tfrac{1}{2}\,\sigma_1 \qquad \omega^1_2 = \omega^3_4 = -\,\tfrac{1}{2}\,\bar{\sigma}_1 \tag{10}$$

$$\omega^4_2 = \omega^3_1 = \tfrac{1}{2}\,\sigma_2 \quad ; \quad \omega^4_3 = \omega^2_1 = \tfrac{1}{2}\,\sigma_2$$

Cartan's second equation of structure (7) can be written in ξ^3 as

$$\sum_p = d\,\sigma_p - \tfrac{1}{2}\,\epsilon_{pmn}\,\sigma^m \wedge \sigma^n, \tag{11}$$

where Σ_p are three complex valued 2-forms which are related to 2-forms Ω_{ab} in exactly the same manner as σ_p are related to ω_{ab}. Σ_p being a complex 2-form, can be expressed in terms of z^p and z^{-p}:

$$\sum_p = C_{pq}\,Z^q - \frac{1}{\sigma} R\,\gamma_{pq}\,Z^q + E_{pq}\,\bar{Z}^2. \tag{12}$$

Here C_{pq} is a complex-valued trace-free symmetric tensor which corresponds to the Weyl tensor, $E_{p\bar{q}}$ is hermitian tensor corresponding to the trace-free part of the Ricci tensor, R is the scalar curvature: C_{pq} are related to the five Newman-Penrose components ψ_A in terms of which the Petrov classification can be made. In fact

$$C_{pq} = 2\begin{pmatrix} \psi_0 & -\psi_2 & \psi_1 \\ -\psi_2 & -\psi_4 & 2\psi_3 \\ \psi_1 & 2\psi_3 & -4\psi_2 \end{pmatrix}. \tag{13}$$

The Einstein-Maxwell field equations for the source-free electromagnetic fields can be expressed as

$$R_{\alpha\beta} = F_{\alpha\gamma}\,F^{\gamma}_{\beta} - (1/4)\,g_{\alpha\beta}\,F_{\gamma\delta}\,F^{\gamma\delta}, \tag{14}$$

$$F^{\alpha\beta}_{;\,\beta} = 0, \tag{15}$$

where $F_{\alpha\gamma}$ are the components of the electromagnetic field tensor.

Equation (14) can be written as

$$E_{p\bar{q}} = -\,2F_p\,\bar{F}_q,\;\; R = 0. \tag{16}$$

Here the self-dual part of the electromagnetic field tensor $F_{\alpha\beta}$ can be expressed in the form

$$F^+ = F_p\,Z^p. \tag{17}$$

Equation (15) can be expressed in the form

$$dF^+ = 0. \tag{18}$$

Thus Einstein-Maxwell equations in the language of complex formalism, are (16) and (18).

3. The metric and Maxwell equations

We consider the metric (Patel and Thaker 1980)

$$(ds)^2 = 2\,(du + g \sin \alpha d\beta)\, dr - 2L\,(du + g \sin \alpha d\beta)^2$$
$$- M^2\,(d\alpha^2 + \sin^2 \alpha d\beta^2). \tag{19}$$

Here L and M are functions of r, α and β also g is function of α and β. We use r, u, α and β as co-ordinates. Introducing the basic 1 forms

$$\theta^1 = du + g \sin \alpha d\beta,\ \sqrt{2}\,\theta^2 = M\,(d\alpha + i \sin \alpha d\beta),$$
$$\theta^4 = dr - L\theta^1,\ \theta^3 = \overline{\theta^2}. \tag{20}$$

We can express (19) as

$$(ds)^2 = 2\,(\theta^1\,\theta^4 - \theta^2\,\theta^3). \tag{21}$$

Using (20) we can obtain $d\theta^a$, which by using the defining expressions (4) for z^p, will give us dz^p. Using these expressions for dz^p, Cartan's first equation of structure given by (9) will then determine the connection 1 forms σ_p. The calculations of σ_p are given in Patel and Thaker (1980), we shall reproduce the expressions for σ_p for ready reference

$$\sigma_1 = -2\,[(M_r/M) - i\,(f/m^2)]\,\theta^2,$$
$$\sigma_2 = -\sqrt{2}\,[(L_\alpha/M) - i\,(L_\beta/M)\,\text{cosec}\,\alpha]\,\theta^1 + 2L\,[(M_r/M) - i\,(f/M^2)]\,\theta^3,$$
$$\sigma_3 = -2\,[L_r + i\,(Lf/M^2)]\,\theta^1 - \sqrt{2}\,(F - iE)\,\theta^2 + \sqrt{2}\,(F + iE)\,\theta^3$$
$$+ 2i\,(f/M^2)\,\theta^4. \tag{22}$$

Here $2f = g_\alpha + g \cot \alpha$, $M^2 F = M_\alpha + M \cot \alpha$, $M^2 E = M_\beta\,\text{cosec}\,\alpha$ and suffixes denote partial derivatives *viz.* $L_r = \partial L/\partial r$, etc.

The absence of terms involving θ^3 and θ^4 in σ_1 indicates that the congruence k^α of null tangents is geodesic as well as shear-free.

Using the expressions (22) for σ_p in Cartan's second structure equation given by (11), we can compute the curvature 2 forms Σ_p.

The expressions for Σ_p are recorded in Patel and Thaker (1980) and are not repeated here. Using these expressions for Σ_p and the identity given by (12) we can compute $E_{p\bar{q}}$; R and the complex valued trace-free symmetric tensor C_{pq}. $E_{p\bar{q}}$ and R are given by

$$E_{1\bar{1}} = 2\,[(M_{rr}/M) - (f^2/M^4)],$$
$$E_{1\bar{2}} = \overline{E}_{2\bar{1}} = 0,$$

$$E_{1\bar{3}} = \bar{E}_{3\bar{1}} = (1/M)\,[(M_r/M)_\alpha + (f/M^2)_\beta \operatorname{cosec} \alpha + i\,\{(f/M^2)_\alpha - (M_r/M)_\beta \operatorname{cosec} \alpha\}],$$

$$E_{2\bar{3}} = \bar{E}_{3\bar{2}} = (\sqrt{2}/M)\,[L_{r\alpha} + L(M_r/M)_\alpha - \{L(f/M^2)_\beta + 2L_\beta\,(f/M^2)_\beta\} \operatorname{cosec} \alpha - i\,\{L(f/M^2)_\alpha + 2L_\alpha\,(f/M^2) + L_{r\beta} \operatorname{cosec} \alpha + L(M_r/M)_\beta \operatorname{cosec} \alpha\}], \tag{23}$$

$$E_{2\bar{2}} = (1/M^2)\,[L_{\alpha\alpha} + L_\alpha \cot \alpha + L_{\beta\beta} \operatorname{cosec}^2 \alpha] + L^2\, E_{1\bar{1}}$$

$$E_{3\bar{3}} = 2L_{rr} - 4L\,\{M_r^2/M^2 - 3f^2/M^4\} + (2/M^2)\,[(M_\alpha/M)_\alpha + (M_\beta/M)_\beta \operatorname{cosec}^2 \alpha + (M_\alpha/M) \cot \alpha - 1],$$

$$R = 2L_{rr} + 8L_r\,(M_r/M) + 4L\,\{M_r^2/M^2 + f^2/M^4\} - (2/M^2)\,[(M_\alpha/M)_\alpha + (M_\beta/M)_\beta \operatorname{cosec}^2 \alpha + (M_\alpha/M) \cot \alpha - 1] + 4L\, E_{1\bar{1}}$$

Since $E_{1\bar{2}} = 0$ it follows from the field equation $E_{1\bar{2}} = -2F_1\bar{F}_2$ that either (i) $F_1 = 0$ or (ii) $F_2 = 0$ or (iii) $F_1 = 0$ and $F_2 = 0$.

We take $F_1 = 0$ and assume the following form of self dual 2 form F^+ :

$$F^+ = \phi\, Z^2 + \psi\, Z^3, \tag{24}$$

where ϕ and ψ are complex valued functions of α, β and r. Since $F_1 = 0$, it follows from the field equations that $E_{1\bar{1}} = 0$ and $E_{1\bar{3}} = 0$. These equations involve only one unknown function M. The solution of these equations can be expressed in the form

$$M^2 = (f/Y)\,(X^2 + Y^2), \tag{25}$$

where
$$X_\alpha = -\,Y_\beta \operatorname{cosec} \alpha,\quad Y_\alpha = X_\beta \operatorname{cosec} \alpha,$$

$$X_r = -1,\quad Y_r = 0. \tag{26}$$

With M given by (25) and (26) and Σ_1 given in Patel and Thaker (1980) we have verified that $C_{11} = C_{13} = 0$. Therefore the spacetime described by the line element (19) is algebraically special.

We shall now try to solve the Maxwell equation (18). Using F^+ given by (24), M^2 given by (25) and (26) and the expression for dz^p, we have verified that the equations $dF^+ = 0$ imply the following four differential equations for ϕ and ψ:

$$\psi_r + 2\psi\,[(M_r/M) - i\,(f/M^2)] = 0, \tag{27}$$

$$\psi_\alpha + i\,\psi_\beta \operatorname{cosec} \alpha = 0, \tag{28}$$

$$2\sqrt{2}\,(\phi M)_r - (\psi_\alpha - i\,\psi_\beta \operatorname{cosec} \alpha) = 0, \tag{29}$$

$$(\sqrt{2}/M)\,(\phi_\alpha + i\,\phi_\beta \operatorname{cosec} \alpha) + \sqrt{2}\,\phi\,(F + i\,E)$$
$$- L\,\psi_r - 2\,L\,\psi\;[(M_r/M) - i\,(f/M^2)] = 0. \tag{30}$$

We can use (25) and (26) to solve (27). The function ψ is given by

$$\psi = K\,(X - i\,Y)^{-2}, \tag{31}$$

where K is a complex function of α and β. Substitution of ψ from (31) into (28) yields

$$e_\alpha = h_\beta \operatorname{cosec} \alpha, \quad h_\alpha = -\,e_\beta \operatorname{cosec} \alpha, \quad K = e + ih. \tag{32}$$

Using (31) and (32) in (29) we obtain

$$\phi = (1/\sqrt{2}\,M)\;[K\,(X - i\,Y)^{-1}]_\alpha. \tag{33}$$

Finally, using all the relevant results of this section, we have verified that (30) is satisfied identically.

4. The remaining Einstein-Maxwell equations

We set $R = 0$ and use M^2 given by (25) and (26) to determine the function $2L$. We shall find that

$$2L = 2S + (2E^*X + 2F^*Y)\,(X^2 + Y^2)^{-2}, \tag{34}$$

where $$2S = (Y/f)\;[\tfrac{1}{2}(Y/f)\;\nabla^2(f/Y) \operatorname{cosec}^2 \alpha - 1 - \tag{35}$$
$$(y/f)^2\,\{(f/Y)_z^2 + (f/Y)_\beta^2\}\,\operatorname{cosec}^2\alpha],$$

where $$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \beta^2}, \quad Z = \operatorname{logtan} \alpha/2,$$

E^* and F^* are undetermined functions of α and β.

Next we take $E_{3\bar{3}} = -\,2F_3\,\bar{F}_3$. Using (25), (26), (34) and (35) in this equation we find that

$$E^* + 2SY + \frac{K\bar{K}}{4\,Y} = 0. \tag{36}$$

Further the field equation $E_{2\bar{3}} = -2F_2\bar{F}_3$ will lead to

$$F^*_\alpha = -\left(E^* + \frac{K\bar{K}}{4Y}\right)_\beta \operatorname{cosec}\alpha, \tag{37}$$

$$F^*_\beta \operatorname{cosec}\alpha = \left(E^* + \frac{K\bar{K}}{4Y}\right)_\alpha.$$

Using all the relevant results in the field equation $E_{2\bar{2}} = -2F_2\bar{F}_2$ we find that K is a complex constant and S satisfies

$$S_\alpha Y_\alpha + S_\beta Y_\beta \operatorname{cosec}^2\alpha = 0. \tag{38}$$

The corresponding electromagnetic field tensor $F_{\alpha\beta}$ can easily be obtained:

$$F_{12} = (X\psi_1 + Y\psi_2)_\alpha, \quad F_{13} = (X\psi_1 + Y\psi_2)_\beta \operatorname{cosec}\alpha, \tag{39}$$

$$F_{14} = \psi_1, \quad F_{23} = -g\sin\alpha\,(X\psi_1 + Y\psi_2)_\alpha - M^2\psi_2\sin\alpha,$$

$$F_{24} = 0, \quad F_{34} = (g\sin\alpha)\psi_1, \quad \psi = \psi_1 + i\psi_2.$$

Here we have named the coordinates as

$$X^1 = u, \quad X^2 = \alpha, \quad X^3 = \beta, \quad X^4 = r.$$

Now E^* is determined from (37) and S from (35). These expressions for E^* and S must satisfy (36) and (38) and we have only one unknown function $2f$ (i.e. $g_\alpha + g\cot\alpha$) at our disposal. Thus we have one additional equation.

However in the case $f = Y$ we have $2S = -1$ and (38) is satisfied identically. In the general case $f \neq Y$, we have verified that (35), (36), (37) and (38) are not consistent. Therefore we shall consider the case $f = Y$ only.

5. The case f = Y

In this case we have

$$M^2 = X^2 + Y^2, \tag{40}$$

where X and Y satisfy (26). We take the following solution of (26) as an example (Patel and Thaker 1980)

$$X = -r + A\sin\beta\operatorname{cosec}\alpha, \tag{41}$$

$$Y = a - A\cos\beta\cot\alpha,$$

where a and A are constants of integration. In this case (35) gives $2S = -1$ and therefore from (36) we have

$$E^* + \frac{K\bar{K}}{4Y} = Y.$$

Using this relation in (37) we get

$$F^* = A \sin \beta \operatorname{cosec} \alpha - m = X + r - m, \tag{42}$$

where m is a constant of integration. In this case the result (34) gives us

$$2L = 1 + \frac{2X(r-m) - K\bar{K}/2}{X^2 + Y^2} \tag{43}$$

Using Y given in (41) in $f = Y$ we obtain

$$g \sin \alpha = -2\, a \cos \alpha - 2\, A \cos \beta \sin \alpha. \tag{44}$$

One can therefore write the line element in the final form

$$\begin{aligned}(ds)^2 &= 2\, [du - 2\, (a \cos \alpha + A \cos \beta \sin \alpha)\, d\beta]\, dr \\ &\quad - (X^2 + Y^2)\,(d\,\alpha^2 + \sin \alpha^2\, d\beta^2) \\ &\quad - \left[1 + \frac{2x\,(r-m) - K\bar{K}/2}{X^2 + Y^2}\right] \\ &\qquad [\, du - 2\, (a \cos \alpha + A \cos \beta \sin \alpha)\, d\beta]^2,\end{aligned} \tag{45}$$

where X and Y are given by (41). The components of the electromagnetic field tensor $F_{\alpha\beta}$ for this particular case can be easily obtained from (39). When $K = 0$ the metric (45) reduces to the generalized NUT metric, discussed by Patel and Thaker (1980). When $A = 0$, it is easy to verify that the metric (45) reduces to the metric

$$\begin{aligned}(ds)^2 &= 2\,(du - 2\, a \cos \alpha\, d\beta)\, dr - (r^2 + a^2)\,(d\alpha^2 + \sin^2 \alpha\, d\beta^2) \\ &\quad + \left[1 - \frac{2\,(mr + a^2) - K\bar{K}/2}{r^2 + a^2}\right](du - 2\, a \cos \alpha\, d\beta)^2.\end{aligned} \tag{46}$$

The metric (46) is the metric discussed by Brill (1964) with slight change of notations. Putting $K = 0$ in (46) we obtain the well-known NUT metric. Thus our solution (45) includes Brill's solution as a particular case.

Acknowledgements

The authors are grateful to the referee for his valuable comments and suggestions. One of the authors (SCT) is highly indebted to the University Grants Commission, New Delhi for the award of teacher fellowship.

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