Renormalisation group approach to the ideal Bose gas in d dimensions

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Abstract. Critical behaviour of a d-dimensional ideal Bose gas is investigated from the point of view of the renormalisation-group approach. Rescaling of quantum-field amplitudes is avoided by introducing a scaling variable inversely proportional to the thermal momentum of the particles. The scaling properties of various thermodynamic quantities are seen to emerge as a consequence of the irrelevant nature of this variable. Critical behaviour is discussed at fixed particle density as well as at fixed pressure. Connection between susceptibility and correlation function of the order-parameter for a quantum system is elucidated.

1. Introduction

Theoretical understanding of the behaviour of many-body systems near critical points has advanced significantly in the last few years as a result of the development of the renormalisation-group (RG) theory by Wilson and coworkers (Wilson 1971; Wilson and Kogut 1974). The new theory identifies the critical point of a system with the fixed point of a series of transformations characterised by a change in the length scale. It provides a satisfactory explanation of important qualitative features of critical phenomena such as universality and also enables one to calculate in suitable approximation schemes values of critical exponents, scaling equations of state, etc.

Although the basic ideas of the RG approach are quite general, the mathematical formalism for their implementation has been developed almost exclusively in the context of classical spin models of the Landau-Ginzburg type (see, *e.g.* Wilson and Kogut 1974; Ma 1976). This formalism is not immediately applicable to phase transitions which are basically of a quantum mechanical origin, *e.g.* the λ-transition in liquid ⁴He. As a matter of fact, an essential ingredient of the formalism, namely rescaling of the field amplitudes, must be dispensed with in the case of a system described by a quantum field since rescaling would not preserve commutation rules obeyed by the field amplitudes. Attempts have been made (Tanaka 1975; Baldo *et al* 1976) to overcome this problem by writing the partition function of a Bose system as a functional integral. This approach, however, is not satisfactory since it requires one to introduce without justification (De Dominicis 1975) the classical limit $(e^{\beta E} - 1) \rightarrow \beta E$.

The primary aim of this paper is to show that ideas of the RG theory can be applied

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to quantum systems without requiring rescaling of field amplitudes. To this end we consider the simplest example of a phase transition of a quantum mechanical origin, *viz*. the Bose-Einstein transition in a d-dimensional gas of ideal bosons, and work out its critical behaviour. The method can be applied to an assembly of interacting bosons also (Singh 1975). The case of the ideal gas, however, is of special interest on account of the fact that calculations can be carried through without making assumptions or approximations which are unavoidable but difficult to justify when one considers the case of interacting particles. It may be pointed out that rescaling of amplitudes is not essential even for a classical field, the method of this paper being equally applicable to that case also.

In the following sections the construction of the RG transformation for an ideal Bose gas and its consequences as regards critical behaviour are presented in detail. It is hoped that the relatively simple nature of the calculations will enable readers not directly interested in quantum many-body theory to get a feeling of the essential ideas and methodology of the RG theory when applied to a system described in terms of a quantum-mechanical hamiltonian. The contents of the various sections are as follows: § 2 explains how RG transformations may be performed on the hamiltonian of an ideal Bose gas. A symmetry-breaking term is included in the hamiltonian to bring the theory at par with that of a ferromagnet. Scaling of the thermodynamic potential is discussed in § 3. In § 4 critical behaviour at fixed density is worked out while critical behaviour at fixed pressure is dealt with in § 5. Scaling of the correlation function of the order parameter and its connection with susceptibility forms the content of § 6.

2. RG transformation for non-interacting bosons

We consider a system of non-interacting, spinless bosons, each of mass $m/2$, contained in a d-dimensional box of volume $V=L^d$. Using periodic boundary conditions, the hamiltonian of the system (in units such that $\hbar=1$) may be written as

$$H = \sum_k \frac{k^2}{m} a_k^\dagger a_k, \tag{1}$$

where a_k denotes the annihilation operator for the single-particle state of momentum k. The summation in (1) extends over the momentum space defined by

$$k = \frac{2\pi}{L}(n_1, n_2, ..., n_d), \tag{2}$$

where the n's are integers.

Bogolubov (1960) has pointed out that a convenient way to discuss the statistical mechanics of systems capable of possessing a condensed mode, is to supplement the hamiltonian by a suitable symmetry-breaking term. For the Bose system the symmetry to be broken is the gauge symmetry and the term to be added to H is

$$h_s = -\frac{\nu}{2} \int dV \left[\psi(r) + \psi^\dagger(r) \right], \tag{3}$$

where ν denotes a positive c-number and

$$\psi(r) = \sum_k \frac{a_k}{\sqrt{V}} \exp(ik \cdot r),$$ (4)

denotes the bosons field operator. In view of the periodic boundary conditions, (3) can be rewritten as

$$h_s = -\frac{\nu}{2}\left(\frac{a_0}{\sqrt{V}} + \frac{a_0^\dagger}{\sqrt{V}}\right) V.$$ (5)

In analogy with the case of a ferromagnet, the real part of $\psi(r)$ is called the order-parameter for the Bose system and ν is referred to as the field conjugate to the order-parameter. Although the physically interesting case corresponds to letting $\nu \to 0$ at the end of the calculation, we shall allow ν to stay finite since this offers the advantage of discussing the critical behaviour of the system in close analogy with that of a ferromagnet in an external magnetic field.

The grand partition function for the system can be written as

$$Z = \text{Tr} \exp(-H_0),$$ (6)

where the dimensionless hamiltonian H_0 is given by

$$H_0 = \sum_k s(k^2 p_c^{-2} + r)a_k^\dagger a_k + H_s,$$ (7)

$$H_s = -\frac{h}{2}\left(\frac{a_0}{\sqrt{V}} + \frac{a_0^\dagger}{\sqrt{V}}\right) V,$$ (8)

$$s = \beta p_c^2/m, \quad r = -m\mu p_c^{-2}, \quad h = \beta \nu.$$ (9)

In the above equations, μ denotes the chemical potential and β the inverse of the product of the absolute temperature T and the Boltzmann constant k_B. We have introduced an arbitrary momentum p_c which, in what follows, will play the role of an upper cut-off in defining the RG transformation. Note that the reduced hamiltonian (7) is independent of p_c.

The thermodynamic potential per unit volume (Ω) is given by

$$\Omega = (\beta V)^{-1} \ln Z.$$ (10)

The thermodynamic average M of the real part of the order-parameter is

$$M = \tfrac{1}{2}\left\langle \frac{a_0}{\sqrt{V}} + \frac{a_0^\dagger}{\sqrt{V}} \right\rangle = -\frac{\partial}{\partial h}(\beta \Omega).$$ (11)

The basic aim of the RG method is to eliminate short wavelength modes by performing a partial trace on the density matrix. For a Bose system, this purpose can be achieved by dividing the set of occupation number operators $\{a_k^\dagger a_k\}$ into two subsets. Let S_1 denote the subset of occupation number operators having $|k| > p_c$ and S_2 the subset of occupation number operators having $|k| < p_c$ where p_c denotes an arbitrary momentum. Let h_1 denote the space spanned by simultaneous eigenvectors of the subset S_1 and h_2 the space spanned by simultaneous eigenvectors of the subset S_2. The Hilbert space of the system is then given by the product $h_1 \otimes h_2$ and the grand partition function can be written in the form

$$Z = \mathop{\mathrm{Tr}}_{(h_2)} \mathop{\mathrm{Tr}}_{(h_1)} \exp(-H_0),$$

$$= C(r, s, V; p_c, \infty) \mathop{\mathrm{Tr}}_{(h_2)} \exp(-H_0(r, s, h; |k| < p_c), \tag{12}$$

where
$$C(r, s, V; p_c, \infty) = \mathop{\mathrm{Tr}}_{(h_1)} \exp\left[-s \sum_{|k|>p_c} (k^2 p_c^{-2} + r) a_k^\dagger a_k\right], \tag{13}$$

$$H_0(r, s, h; |k| < p_c) = \sum_{|k|<p_c} s(k^2 p_c^{-2} + r) a_k^\dagger a_k + H_s, \tag{14}$$

The introduction of a cut-off momentum p_c, which may be assumed as large as we like, is an essential step in defining the RG transformation. Note that unlike the spins on a lattice where the wave-vectors are confined within a Brillouin zone, there is no natural momentum cut-off for a system of bosons.

The space h_2 on which $H_0(|k| < p_c)$ operates may again be factorised by dividing the occupation numbers belonging to S_2 into two subsets in the following manner: Let ζ denote a number large compared to unity. The two subsets of S_2 are then: (i) the set of occupation numbers corresponding to k's lying in the interval $p_c \zeta^{-1} < |k| < p_c$, and (ii) the set of occupation numbers corresponding to k's lying in the interval $0 \leqslant |k| < p_c \zeta^{-1}$. It is convenient to denote by q's vectors having magnitudes in the interval $(0, p_c \zeta^{-1})$ and by p's vectors having magnitudes in the interval $(p_c \zeta^{-1}, p_c)$. The space h_2 can be factorised as

$$h_2 = h(q) \otimes h(p), \tag{15}$$

where $h(q)$ denotes the subspace spanned by simultaneous eigenvectors of all $\{a_q^\dagger a_q\}$ and $h(p)$ the space spanned by simultaneous eigenvectors of all $\{a_p^\dagger a_p\}$. We may now write (12) in the form

$$Z = C Z_0(r, s, V; p_c, \zeta^{-1} p_c) Z_1, \tag{16}$$

$$Z_0 = \mathop{\mathrm{Tr}}_{\{h(p)\}} \exp\left[-\sum_p s(p^2 p_c^{-2} + r) a_p^\dagger a_p\right], \tag{17}$$

$$Z_1 = \mathop{\mathrm{Tr}}_{\{h(q)\}} \exp(-H_1), \tag{18}$$

$$H_1 = \sum_q s(q^2 p_c^{-2} + r) a_q^\dagger a_q + H_s. \tag{19}$$

We now try to bring the 'low momentum' hamiltonian H_1 into the same form as H_0 given by (14) by restoring the range of momenta in (19) to the interval $(0, p_c)$. For this purpose we rescale the wave-vectors q according to

$$k_1 = \zeta q, \tag{20}$$

and the volume V according to

$$V_1 = \zeta^{-d} V. \tag{21}$$

The hamiltonian H_1 can then be written as

$$H_1 = s \sum_{|k_1| < p_c} (k_1^2 p_c^{-2} + r_1) \, \zeta^{-2} \, b_{k_1}^\dagger \, b_{k_1} - \frac{h_1}{2} \, V_1^{1/2} (b_0 + b_0^\dagger), \tag{22}$$

where

$$r_1 = \zeta^2 \, r, \tag{23}$$

$$h_1 = \zeta^{d/2} \, h, \tag{24}$$

and the operators b_k are defined by

$$b_k = a_{k \zeta^{-1}} \tag{25}$$

They satisfy the same commutation rules as the a_k's. It may be pointed out that although the k_1 vectors in (22) lie in the interval $0 \leqslant |k_1| < p_c$, in view of (2) and (20), they have components given by

$$k_1 = \frac{2\pi}{(L/\zeta)} (n_1, n_2, \ldots, n_d). \tag{26}$$

The k_1's in other words, are wave-vectors corresponding to periodic boundary conditions on a box of volume V_1 given by (21). The density of states in the k_1-space accordingly is

$$V_1 \, d^d \, k_1/(2\pi)^d. \tag{27}$$

In the theory of classical spin fields, the factor ζ^{-2} occurring in the first term in (22) is absorbed trivially by rescaling the field amplitudes which play the role of random variables in the theory. In the case of a quantum system this cannot be done without altering the commutation rules. For example, if we try to absorb the factor ζ^{-2} by defining new operators

$$A_k = \zeta^{-1} \, b_k, \tag{28}$$

the commutation rules satisfied by the A_k's are

$$[A_k, A_{k'}^\dagger] = \zeta^{-2} \, \delta_{k, k'}. \tag{29}$$

The simplest way to avoid this difficulty is not to rescale the quantum amplitudes b_k. One may rather rescale the parameter s (or equivalently the mass m of the particles) according to

$$s_1 = \zeta^{-2} s. \tag{30}$$

The hamiltonian H_1 then takes the form

$$H_1(r_1, s_1, h_1; V_1) = s_1 \sum_{|k_1| < p_c} (k_1^2 p_c^{-2} + r_1)\, b_{k_1}^\dagger b_{k_1} - \tfrac{1}{2} h_1 V_1^{1/2}(b_0 + b_0^\dagger) \tag{31}$$

while Z_1 can be written as

$$Z_1 = \mathop{\mathrm{Tr}}_{h(k_1)} \exp\left[-H_1(r_1, s_1, h_1; V_1)\right], \tag{32}$$

where $h(k_1)$ denotes the space spanned by simultaneous eigenvectors of all the operators $\{b_{k1}^\dagger b_{k1}\}$. It is identical with the space $h(q)$.

The process of obtaining $H_1(r_1, s_1, h_1; V_1)$ from $H_0(r, s, h; V)$ is an example of the RG transformation. Equations (23), (24) and (30) connecting the parameters (r_1, s_1, h_1) with (r, s, h) are called recursion relations.

The hamiltonian $H_1(r_1, s_1, h_1; V_1)$ being of the same form as $H_0(r, s, h; V)$, we can repeat the process by which H_1 was obtained from H_0. After l renormalisation group transformations, we get

$$Z = C A_l \exp\left[-H_l(r_l, s_l, h_l; V_l)\right], \tag{33}$$

$$A_l = Z_0\left(s, r, V; \frac{p_c}{\zeta}, p_c\right) \prod_{m=1}^{l-1} Z_0\left(r_m, s_m, V_m; \frac{p_c}{\zeta}, p_c\right), \tag{34}$$

$$H_l = s_l \sum_{|k| < p_c} (k^2 p_c^{-2} + r_l)\, b_k^\dagger b_k - \tfrac{1}{2} h_l V_l^{1/2}(b_0 + b_0^\dagger). \tag{35}$$

The parameters (r_l, s_l, h_l, V_l) are related to $(r_{l-1}, s_{l-1}, h_{l-1}, V_{l-1})$ by

$$r_l = \zeta^2 r_{l-1}, \tag{36}$$

$$h_l = \zeta^{d/2} h_{l-1}, \tag{37}$$

$$s_l = \zeta^{-2} s_{l-1}, \tag{38}$$

$$V_l = \zeta^{-d} V_{l-1}. \tag{39}$$

Each factor $Z_0(r_m, s_m, V_m)$ in (34) is defined according to (17) with the provision that the p vectors entering into the right-hand side of (17) be those appropriate to a box of volume $(L/\zeta^m)^d$.

In the terminology of relevant and irrelevant variables (Wegner 1972), (r, h) are relevant variables while s is an irrelevant variable. Equation (39) shows that V, if finite, is also an irrelevant variable. However, as true critical behaviour is exhibited by a system in the thermodynamic limit $V \to \infty$ only, we can take V_l itself as approaching infinity.

In the limit $V \to \infty$ the transformation (36)–(38) has a fixed point (r^*, h^*, s^*) given by

$$r^* = 0, \quad h^* = 0, \quad s^* = 0. \tag{40}$$

It is also clear that this fixed point is reached only if

$$r \to 0, \quad h \to 0. \tag{41}$$

The RG theory identifies the fixed point with a critical point of the system. Keeping in view the definition of r (equation 9), we conclude that the critical point of the ideal Bose gas corresponds to

$$\mu \to 0, \quad h \to 0. \tag{42}$$

As is well known, these conditions characterise the point of Bose-Einstein transition is an ideal Bose gas. We shall show in § 4 that at fixed particle density, and at temperatures lower than the temperature T_c corresponding to (42), the order-parameter M is non-zero.

The parameter s being an irrelevant variable, plays no role in determining the critical point of the system. Its scaling behaviour, however, is of importance in determining the critical behaviour of the system as will become evident in the following section.

3. Scaling of thermodynamic potential

On using the expression (33) for Z in (10), we obtain for the thermodynamic potential per unit volume

$$-\beta \Omega = \frac{1}{V} \ln C + \frac{1}{V} \ln A_l$$

$$+ \frac{\zeta^{-ld}}{V_l} \ln \mathrm{Tr} \exp [- H_l(r_l, h_l, s_l; V_l)] \tag{43}$$

The expression (34) for A_l can be simplified by noting that

$$Z_0 \left(r_m, \ s_m, \ V_m; \frac{p_c}{\zeta}, \ p_c \right) = Z_0 \left(r_{m-1}, \ s_{m-1}, \ V_{m-1}; \frac{p_c}{\zeta^2}, \ \frac{p_c}{\zeta} \right). \tag{44}$$

One gets

$$A_l = Z_0 \left(r, s, V; \frac{p_c}{\zeta^l}, p_c \right). \tag{45}$$

Equations (36) to (39) and (45) show that the result (43) obtained for the thermo-dynamic potential by l successive RG transformations, each with a cut-off factor ζ, is the same as would be obtained by a single RG transformation with a cut-off factor ζ^l. This is an illustration of the semi-group property of the RG transformations.

In what follows, we shall denote ζ^l by b, and (r_l, h_l, s_l, V_l) by (r_1, h_1, s_1, V_1). The recursion relations connecting (r_1, h_1, s_1, V_1) with (r, h, s, V) are then given by (23), (24), (30) and (21) with ζ replaced by b. Equation (43) for Ω takes the form

$$-\beta\Omega = \frac{1}{V} \ln C + \frac{1}{V} \ln Z_0 \left(r, s, V; \frac{p_c}{b}, p_c \right)$$

$$+ \frac{b^{-d}}{V_1} \ln \operatorname{Tr} \exp \left[- H_1 (r_1, s_1, h_1, V_1) \right], \tag{46}$$

$$H_1(r_1, s_1, h_1, V_1) = \sum_{|k| < p_c} s_1 (k^2 p_c^{-2} + r_1) b_k^\dagger b_k - \tfrac{1}{2} h_1 V_1^{1/2} \left(b_0 + b_0^\dagger \right). \tag{47}$$

The symmetry-breaking term in (47) can be got rid off by performing the following canonical transformation:

$$B_k = b_k, \; k \neq 0, \tag{48}$$

$$B_0 = b_0 + \frac{1}{2} \frac{h_1 V_1^{1/2}}{s_1 r_1}, \; r_1 \neq 0. \tag{49}$$

The result is

$$H_1 = s_1 \sum_{|k| < p_c} (k^2 p_c^{-2} + r_1) B_k^\dagger B_k - \frac{1}{4} \frac{h_1^2 V_1}{s_1 r_1}, \tag{50}$$

$$\operatorname{Tr} \exp (- H_1) = \exp \left(\frac{1}{4} \frac{h_1^2 V_1}{s_1 r_1} \right) \prod_{|k| < p_c} [1 - \exp (-s_1 k^2 p_c^{-2} - s_1 r_1)]. \tag{51}$$

Equation (51) is meaningful only if r_1 (and hence r) is non-negative; otherwise $\operatorname{Tr} \exp (-H_1)$ is divergent. The Z_0 in (46) is given by an expression similar to the second factor on the right hand side of (51). On taking the limit $V \to \infty$, we find

$$-\beta\Omega = \operatorname*{Lt}_{V \to \infty} V^{-1} \ln C -$$

$$\int_{p_c b^{-1} < |k| < p_c} \frac{d^d k}{(2\pi)^d} \ln [1 - \exp (-sk^2 p_c^{-2} - sr)]$$

$$- b^{-d} \int_{0 < |k| < p_c} \frac{d^d k}{(2\pi)^d} \ln [1 - \exp (-s_1 k^2 p_c^{-2} - s_1 r_1)]$$

$$+ \frac{b^{-d}}{4} \frac{h_1^2}{s_1 r_1}. \tag{52}$$

It may be pointed out that, unlike the ordinary theory of the Bose gas without a symmetry-breaking term (Munster 1969) replacement of summations by integrations offers no problems in the present case since r_1 cannot be zero so long as the 'external field' h_1 is non-zero (cf. equation (50)).

In d-dimensions

$$\int \frac{d^d k}{(2\pi)^d} = A_0(d) \int k^{d-1} dk,$$ (53)

$$A_0(d) = 2^{-d+1} \frac{\pi^{-d/2}}{\Gamma(d/2)}$$ (54)

Integrating the second and third terms in (52) partially, we get

$$-\beta \Omega = \mathrm{Lt} \frac{1}{V} \ln C - \frac{A(d)}{2} \ln [1 - \exp(-s-sr)]$$

$$+ sA(d) \int_{b^{-1}}^{1} \frac{q^{d+1} dq}{[\exp(sq^2+sr)-1]} + \frac{b^{-d}}{4} \frac{h_1^2}{s_1 r_1}$$

$$+ b^{-d} s_1 A(d) \int_{0}^{1} \frac{q^{d+1} dq}{[\exp(s_1 q^2+s_1 r_1)-1]}$$ (55)

where $A(d) = 2 p_c^d A_0(d)/d.$ (56)

An important simplification which has no counterpart in the theory of a classical spin field, and which, as we shall see, leads to scaling property of the free energy, now occurs on account of the irrelevant nature of the parameter s. By choosing b sufficiently large, we can make s_1 as small as we like. The last term in (55) can therefore be simplified to the form

$$b^{-d} A(d) \int_{0}^{1} \frac{q^{d+1} dq}{q^2+r_1}.$$ (57)

We may now write the thermodynamic potential as the sum of two parts:

$$-\beta\Omega = -\beta\Omega_1 - \beta\Omega_2,$$ (58)

$$-\beta\Omega_1 = \mathrm{Lt} \frac{1}{V} \ln C - \frac{A}{2} \ln (1 - \exp(-s-sr_1))$$

$$+ sA \int_{b^{-1}}^{1} \frac{q^{d+1} dq}{\exp(sq^2+sr)-1},$$ (59)

$$-\beta\Omega_2 = b^{-d} A \int_{0}^{1} \frac{q^{d+1} dq}{(q^2+b^2 r)} + \frac{h^2}{4sr}.$$ (60)

The RG treatments, in general, assume (Wegner 1972, Fisher 1974) that Ω_1—which is the contribution to Ω from short wavelength modes $|\,k\,| > p_c b^{-1}$—is a regular function of the scaling variable (*e.g.* r) which is a measure of the deviation from criticality. The scaling (or singular) behaviour is consequently associated with the long-wavelength part of the thermodynamic potential. An examination of (59) however shows that this, in general, is not true.   The reason is that, in order to obtain a scaling form for Ω_2, we must choose b^{-1} proportional to $r^{1/2}$. It is easy to see that a similar contribution arises from the last term in (59) from wave-vectors q of order b^{-1}. This point has been emphasized by Ma (1976) in the context of a classical field.

In order to separate out the relevant contribution from the integral in (59), we write

$$b = b_1\,b_2\,,\tag{61}$$

i.e., we regard the RG transformation with cut-off factor b as the product of two RG transformations, one corresponding to a cut-off factor b_1 and the other to a factor b_2. Equations (59) and (60) then become

$$-\beta\,\Omega_1 = -\beta\Omega_1' + b_1^{-d}\,A\,Y,\tag{62}$$

$$-\beta\,\Omega_2 = (b_1\,b_2)^{-d}\,A\,\int_0^1 \frac{q^{d+1}\,dq}{(q^2 + b_2^2\,r_1)} + \frac{h^2}{4\,sr}\,,\tag{63}$$

$$-\beta\,\Omega_1' = \mathrm{Lt}\,\frac{1}{V}\,\ln\,C - \frac{A(d)}{2}\,\ln\,[1 - \exp\,(-s - sr)]$$

$$+\,s\,A\,(d)\,\int_{b_1^{-1}}^1 \frac{q^{d+1}\,dq}{[\exp\,(sq^2 + sr) - 1]}\,,\tag{64}$$

$$Y = s_1\,\int_{b_2^{-1}}^1 \frac{q^{d+1}\,dq}{[\exp\,(s_1 q^2 + s_1\,r_1) - 1]} \simeq \int_{b_2^{-1}}^1 \frac{q^{d+1}\,dq}{q^2 + r_1},\tag{65}$$

$$r_1 = b_1^2\,r,\qquad s_1 = b_1^{-2}\,s.\tag{66}$$

The second equation in (65) holds on account of the fact that s_1 can be made arbitrarily small by choosing b_1 appropriately.

Very close to the fixed point, r_1 is a small quantity.   We choose

$$b_2^{-1} = pr_1^{1/2},\tag{67}$$

where p is a positive number. This gives

$$- \beta \Omega_2 = A(d) \, r^{d/2} \int\limits_0^p \frac{q^{d+1} \, dq}{q^2 + 1} + \frac{h^2}{4sr} .$$

(68)

The first term on the right hand side of (64) represents the contribution to the thermo-dynamic potential of momenta larger than p_c. It can be written in the form (cf. equation (52)).

$$\mathrm{Lt} \frac{1}{V} \ln C = - A_0 \, (d) \, p_c^d \int\limits_1^\infty dq \, q^{d-1} \ln [1 - \exp (- sq^2 - sr)],$$

(69)

and admits of an expansion in powers of r around $r = 0$. The introduction of an arbitrary cut-off p_c required to define the RG transformations for a Bose system thus has no effect on the singular behaviour of Ω. The second and third terms on the right hand side of (64) are also regular at $r = 0$. It follows that the first term in (62) is a regular function of r. As regards Y we note that by allowing p in (67) to be large compared to 1 we can expand the integrand in (65) in powers of r_1. We then find

$$Y = \left(\frac{1}{d} - \frac{r_1}{d-2} + \frac{r_1^2}{d-4} - \ldots \right)$$

$$- r_1^{d/2} \left(\frac{p^d}{d} - \frac{p^{d-2}}{d-2} + \frac{p^{d-4}}{d-4} - \ldots \right)$$

(70)

if d does not equal an even integer. It is thus clear that, besides a regular part, Ω_1 contains a term similar to the first term in (68).

We denote the regular and singular parts of Ω by Ω_r and Ω_s respectively. After some algebraic manipulation we find

$$- \beta \, \Omega = - \beta \, \Omega_r - \beta \, \Omega_s$$

(71)

$$- \beta \, \Omega_r = - \beta \, \Omega_0 - \frac{dA \, (d)}{2} \left[s \int\limits_{b_1^{-1}}^\infty \frac{q^{d-1} \, dq}{\exp (sq^2) - 1} + \frac{b_1^{-d+2}}{d-2} \right] r + cr^2$$

(72)

$$- \beta \, \Omega_0 = A \, (d) \left[s \int\limits_{b_1^{-1}}^\infty \frac{q^{d+1} \, dq}{\exp (sq^2) - 1} + \frac{b_1^{-d}}{d} \right]$$

(73)

$$- \beta \, \Omega_s = K \, (d) \, A \, (d) \, r^{d/2} + \frac{h^2}{4sr}$$

(74)

$$K \, (d) = \int\limits_0^p \frac{q^{d+1} \, dq}{q^2 + 1} - \left(\frac{p^d}{d} - \frac{p^{d-2}}{d-2} + \frac{p^{d-4}}{d-4} - \ldots \right)$$

(75)

The coefficient c of the r^2 term in (72) is a positive number. The quantity K defined by (75) has been written as a function of d only because p being an arbitrarily introduced number, the thermodynamic potential must be independent of p. For the same reason Ω_r must be independent of b_1. It is easy to check that the coefficient of r in (72) is simply

$$-\frac{sAd}{2}\int_0^\infty \frac{q^{d-1}\,dq}{(\exp sq^2-1)}. \tag{76}$$

For $2 < d < 4$, we can write

$$K(d) = \int_0^p \frac{q^{d-3}}{q^2+1}\,dq + \left(-\frac{p^{d-4}}{d-4}+\frac{p^{d-6}}{d-6}-\dots\right), \tag{77}$$

and, for $4 < d < 6$

$$K(d) = -\int_0^p \frac{q^{d-5}}{q^2+1}\,dq + \left(\frac{p^{d-6}}{d-6}-\frac{p^{d-8}}{d-8}+\dots\right). \tag{78}$$

In particular, for $d=3$ the series within the parenthesis in (77) is simply the expansion of arc tan (p^{-1}) for $p > 1$. Thus

$$K(d=3) = \text{arc tan } p + \text{arc tan } p^{-1} = \frac{\pi}{2}. \tag{79}$$

4. Critical behaviour at fixed density

The variables which have appeared in the theory so far are h and r. It is conventional as well as convenient from the experimental point of view, to present results concerning critical behaviour in terms of the temperature deviation

$$t = (T-T_c)/T_c. \tag{80}$$

Since $(-\Omega)$ equals the pressure P exerted by the system, we can use (71) to (74) to determine r as a function of h, t and P. Alternatively, we can express r as a function of h, t and the particle density n by using the relation

$$n = -\frac{\partial \Omega}{\partial \mu} = \frac{\partial (\beta\Omega)}{\partial (sr)}. \tag{81}$$

In this section we discuss critical behaviour of the ideal Bose gas at constant n. The case of constant P will be considered in § 5.

On differentiating (71) with respect to r, we get

$$n = f_0(\beta) - f_1(\beta) \, r^{(d/2-1)} + f_2(\beta) \frac{h^2}{r} - \frac{2c}{s} r, \tag{82}$$

$$f_0(\beta) = \frac{Ad}{2} \int\limits_0^\infty \frac{q^{d-1} \, dq}{\exp(sq^2) - 1}, \tag{83}$$

$$f_1(\beta) = KAd/2s, \tag{84}$$

$$f_2(\beta) = 1/4s. \tag{85}$$

For a given density n, the critical temperature $T_c(n)$ is determined by the fixed point condition (41). Although (82) holds only when h and r are non-zero, the result corresponding to the fixed point can be obtained by first letting $h \to 0$ and then allowing $r \to 0$. Thus

$$n = f_0(\beta_c), \tag{86}$$

$$\beta_c = (K_B T_c)^{-1}. \tag{87}$$

It may be pointed out that (86) holds for $d > 2$ only; for $d \leqslant 2$, (82) gives the unphysical result $n = \infty$ at the fixed point for any non-zero temperature.

The functions f_0, f_1, f_2 can be expanded in powers of $(T - T_c)$. Close to the critical point, one gets the following equation for determining r in terms of t and h:

$$t = c_1 r^{(d/2)-1} - c_2 \frac{h^2}{r^2} + c_3 r, \tag{88}$$

$$c_1 = \frac{KAd}{2sw}, \tag{89}$$

$$c_2 = 1/4s^2 w, \quad c_3 = \frac{2c}{sw}, \tag{90}$$

$$w = -\beta_c f_0'(\beta_c). \tag{91}$$

Note that the coefficients c_1, c_2, c_3 are all positive. Equation (88) cannot be solved explicitly for r. We can, however, obtain formal solutions by rewriting it in a scaled form. For $2 < d < 4$, the last term in (88) can be ignored. Introducing the variable

$$x = r \, |t|^{-2/(d-2)} \tag{92}$$

the equation takes the form

$$\epsilon \, x^2 - c_1 x^{d+2/2} = -c_2 \, (h/ \, |t|^\Delta)^2, \tag{93}$$

where ϵ denotes the signature of t and

$$\Delta = \frac{d+2}{2(d-2)}. \tag{94}$$

Let $f_\epsilon(h/|t|^\Delta)$ denote the positive root of (93). Then

$$r = |t|^{2/(d-2)} f_\epsilon(h/|t|^\Delta). \tag{95}$$

The asymptotic forms of f_ϵ can be obtained from (93). We find

$$f_\epsilon(y) \xrightarrow[y \to 0]{} \begin{cases} (\sqrt{c_2}\,y - \frac{1}{2}c_1 c_2^{d/4}\, y^{d/2} + \dots), & \epsilon = -1, \tag{96} \\[2mm] \left(c_1^{-2/(d-2)} + \dfrac{2c_2 c_1^{2/(d-2)}}{d-2}\, y^2 + \dots\right), & \epsilon = 1. \tag{97} \end{cases}$$

$$f_\epsilon(y) \xrightarrow[y \to \infty]{} (a(d)y^{4/(d+2)}\,[1 + \epsilon b(d)y^{-(2d-4)/d+2} + \dots]), \tag{98}$$

$$a(d) = (c_2/c_1)^{2/(d+2)}, \tag{99}$$

$$b(d) = \frac{2}{d+2}\left(\frac{c_2}{c_1}\right)^{(2-d)/(2+d)} \tag{100}$$

For $d>4$, the first term on the right hand side of (88) may be ignored. The equation can then be written in the form

$$\epsilon x^2 - c_3 x^3 = -c_2\,(h/|t|^{3/2})^2, \tag{101}$$

$$x = r/|t|. \tag{102}$$

Denoting by $g_\epsilon(h/|t|^{3/2})$ the positive root of (101),

$$r = |t|\,g_\epsilon(h/|t|^{3/2}). \tag{103}$$

The asymptotic forms of g are

$$g_\epsilon(y) \xrightarrow[y \to 0]{} \begin{cases} (\sqrt{c_2}\,y - \frac{1}{2}c_2 c_3 y^2 + \dots), & \epsilon = -1, \tag{104} \\[2mm] (c_3^{-1} + c_2 c_3 y^2 + \dots), & \epsilon = +1, \tag{105} \end{cases}$$

$$g_\epsilon(y) \xrightarrow[y \to \infty]{} \frac{c_2}{c_3}\, y^{3/2} + \dots \tag{106}$$

The free energy f per unit volume is given by

$$-\beta f = -\beta \Omega + nsr = -r^2 \frac{\partial}{\partial r}(\beta \Omega / r). \tag{107}$$

On substituting for Ω from (71), we find

$$\beta f = \beta \Omega_0 + KA \frac{d-2}{2} r^{d/2} - \frac{h^2}{2sr} + cr^2. \tag{108}$$

The variable r can be eliminated in favour of t and h by using (95) and (103). The result is

$$\beta f = \beta \Omega_0 + |t|^{\frac{d}{d-2}} F_\epsilon (h/|t|^\Delta), \quad 2 < d < 4, \tag{109}$$

$$F_\epsilon (y) = KA \frac{d-2}{2} f_\epsilon^{d/2} (y) - \frac{y^2}{2s f_\epsilon (y)}, \tag{110}$$

$$\beta f = \beta \Omega_0 + |t|^2 G_\epsilon (h/|t|^{3/2}), \quad d > 4, \tag{111}$$

$$G_\epsilon (y) = c g_\epsilon^2 (y) - \frac{y^2}{2s g_\epsilon (y)}. \tag{112}$$

The forms (109) and (111) for the free energy conform to the homogeneity hypothesis proposed by Widom (1965). In particular for $d > 4$, the exponents of t in (111) agree with the values given by Landau's theory of continuous phase transitions (Landau 1937). To see this in a more transparent fashion, we write the free energy in terms of the order parameter M rather than the field h.

Equations (11) and (71) imply

$$M = h/2sr. \tag{113}$$

Substitution for h in (88) and (108) yields

$$t + 4c_2 s^2 M^2 = c_1 r^{d-2/2} + c_3 r, \tag{114}$$

$$\beta g (M, T) = \beta \Omega_0 + KA \left(\frac{d-2}{2}\right) r^{d/2} + cr^2, \tag{115}$$

where the Gibbs function $g(M, T)$ is defined by

$$\beta g (M, T) = \beta f + hM. \tag{116}$$

For $d > 4$, (114) and (115) give

$$\beta g = \beta \Omega_0 + a_0 t^2 + a_2 t M^2 + a_4 M^4 \tag{117}$$

$$a_0 = \frac{s^2 w^2}{4c}, \quad a_2 = \frac{s^2 w}{2c}, \quad a_4 = \frac{s^2}{4c}. \tag{118}$$

The form (117) for the free energy is identical with that postulated in Landau's theory. The critical behaviour of the Bose gas with dimensionality larger than 4 is consequently identical with that predicted by the Landau theory. We shall not discuss this case any further.

For $2 < d < 4$, the Gibbs potential (115) takes the form

$$\beta g = \beta \Omega_0 + KA \frac{d-2}{2} \left(\frac{t + 4c_2 s^2 M^2}{c_1} \right)^{2/(d-2)}. \tag{119}$$

The equation of state is given by

$$h = \frac{\partial}{\partial M} (\beta g). \tag{120}$$

On differentiating (119) with respect to M and substituting for c_1 and c_2 from (89) and (90), the equation of state can be written in the Griffiths form (Griffiths 1967)

$$(h/M^\delta) = \psi (t/M^{1/\beta}). \tag{121}$$

where

$$\psi (x) = (2s)^{d/(d-2)} \left(\frac{w}{KAd} \right)^{2/(d-2)} \left(x + \frac{1}{w} \right)^{2/(d-2)}, \tag{122}$$

and the critical exponents δ and β are given by

$$\delta = \frac{d+2}{d-2}, \quad \beta = \tfrac{1}{2}. \tag{123}$$

Since w is a positive number, we see that M can be finite for $h \to 0$ provided $t < 0$; for $t > 0$, $M \to 0$ as $h \to 0$. The order parameter in zero external field thus exists only for $T < T_c$, and is proportional to $(T_c - T)^\beta$.

The susceptibility χ can be easily calculated from (121). One finds that for $h \to 0$, χ diverges proportional to $h^{-(4-d)/2}$ for $t < 0$, while for $t > 0$

$$\chi = c_+ t^{-\gamma} \tag{124}$$

$$\gamma = 2/(d-2) \tag{125}$$

$$c_+ = (2s)^{-d/(d-2)} \left(\frac{w}{KAd} \right)^{-2/(d-2)} \tag{126}$$

We next determine the behaviour of entropy and specific heat near the critical point. The entropy per unit volume S can be calculated from the formula

$$(S/k_B) = -\beta f - (1+t)\frac{\partial}{\partial t}(\beta f). \tag{127}$$

The specific heat per unit volume C_v, is given by

$$(C_v/k_B) = (1+t)\frac{\partial}{\partial t}(S/k_B). \tag{128}$$

To calculate S and C_v in the limit $h \to 0$, we need expansion of the function $F_\epsilon(y)$ defined by (110) for $y \to 0$. Use of the asymptotic forms (96) and (97) yields

$$F_+(y) \xrightarrow[y \to 0]{} A_1 + A_2 y^2 + O(y^4). \tag{129}$$

$$F_-(y) \xrightarrow[y \to 0]{} B_1 y + B_2 y^{d/2} + O(y^{d-1}), \tag{130}$$

$$\beta f_+ = \beta \Omega_0 + A_1 t^{2-\alpha} + A_2 t^{2-\alpha-2\Delta} h^2 + O(h^4), \tag{131}$$

$$\beta f_- = \beta \Omega_0 + B_1 |t|^{2-\alpha-\Delta} h + B_2 h^{d/2} |t|^{\left(2-\alpha-\frac{\Delta d}{2}\right)}, \tag{132}$$

$$A_1 = KA\frac{d-2}{2}\left(\frac{2sw}{KAd}\right)^{d/(d-2)}, \tag{133}$$

$$A_2 = -\tfrac{1}{2}(2s)^{-d/(d-2)}\left(\frac{w}{KAd}\right)^{-2/(d-2)}, \tag{134}$$

$$B_1 = -w^{1/2}, \tag{135}$$

$$B_2 = -KA/4s^2 w, \tag{136}$$

$$\alpha = \frac{d-4}{d-2}. \tag{137}$$

From (131) and (132), we find

$$\frac{S_+}{k_B} = \frac{S_r}{k_B} - (2-\alpha)A_1 t^{1-\alpha} + O(h^2), \tag{138}$$

$$\frac{S_-}{k_B} = \frac{S_r}{k_B} + B_1 h |t|^{1-\alpha-\Delta} + O(h^{d/2}) \tag{139}$$

$$\frac{S_r}{k_B} = -\beta \Omega_0 - (1+t)\frac{\partial}{\partial t}(\beta \Omega_0). \tag{140}$$

S_+ and S_- denote entropy for $t > 0$ and $t < 0$ respectively; S_r denotes the regular part of the entropy. The specific heat in zero field is given by

$$\frac{C_v}{k_B} = C_r, \quad t < 0, \tag{141}$$

$$= C_r - \frac{2\,A_1\,d}{(d-2)^2}\,t^{-\alpha}, \quad t > 0, \tag{142}$$

where C_r denotes the regular part of the specific heat. For $d=3$, $\alpha = -1$. In this case the specific heat has no singularity at T_c. For other values of d, one of the derivatives of C_v becomes singular at T_c.

In addition to the critical exponents β, δ, γ and α, two other exponents ζ and $\hat{\sigma}$ connected with the entropy are defined by the following relations (Gunton and Buckingham 1968)

$$(S_c - S) \propto M^{1+\zeta}, \quad T = T_c, \tag{143}$$

$$(T - T_c) \propto M^{\hat{\sigma}}, \quad S = S_c, \tag{144}$$

S_c denoting the value of the entropy at the critical point. These exponents cannot be calculated from the expansions (138) and (139) which are valid for $(h/|t|^{\Delta}) \to 0$. The easiest way to determine them is to use the formula

$$S = -\frac{\partial g\,(M, T)}{\partial T}, \tag{145}$$

in conjunction with the expression (119) for the Gibbs potential. One finds

$$\frac{S}{k_B} = \frac{S_r}{k_B} - \frac{KAd}{2\,c_1}\left(\frac{t + 4\,c_2\,s^2\,M^2}{c_1}\right)^{2/(d-2)}, \tag{146}$$

which implies

$$1 + \zeta = \frac{4}{d-2} = \hat{\sigma}. \tag{147}$$

5. Critical behaviour at constant pressure

According to equations (71) to (74), the pressure exerted by the system is given by

$$\beta P = -\beta\,\Omega_0 - s\,f_0\,(\beta)\,r + cr^2 + KAr^{d/2} + \frac{h^2}{4\,sr}; \tag{148}$$

with $f_0(\beta)$ defined by (83). For a given P, the critical temperature $T_c(P)$ is defined by the fixed point condition $r \to 0$, $h \to 0$. Equation (148) implies

$$P = -\Omega_0(\beta_c).\tag{149}$$

Close to the critical point, one can write (148) in the form

$$\beta u_0 t = s f_0(\beta) r - \frac{h^2}{4 sr} - KA r^{d/2} - cr^2,\tag{150}$$

$$u_0 = \beta_c \left(\frac{\partial \Omega_0}{\partial \beta}\right)_{\beta_c} > 0.\tag{151}$$

Ignoring terms of order $r^{d/2}$ and r^2 in (150) and solving for r, we find

$$r = t_1 + (t_1^2 + h^2)^{1/2},\tag{152}$$

$$t_1 = \beta u_0 t/(2s f_0(\beta)),\tag{153}$$

$$h_1 = h/(2s f_0^{1/2}(\beta)).\tag{154}$$

Equations (113) and (152) yield the equation of state

$$M = f_0^{1/2}(\beta) \frac{a}{(a^2+1)^{1/2} + \epsilon},\tag{155}$$

where

$$a = h_1/|t_1|,\tag{156}$$

and ϵ, as before, denotes the signature of t. For $t < 0$, as $h \to 0$, $M \to \infty$. Since M^2 represents the density of condensed particles, we conclude that temperatures less than $T_c(p)$ correspond to unphysical states of the system.

For t and h both approaching zero, the value of M depends upon the limiting value of the ratio a, and can have any value between zero and infinity. This result is not an artefact of having introduced a symmetry-breaking term in the hamiltonian. If one discusses the theory of the ideal Bose gas (without a symmetry breaking term) in terms of the variables P and T, one finds that in the thermodynamic limit $V \to \infty$, the density of zero-momentum particles on the critical line $\mu \to 0$ depends on the manner μ approaches zero, and can have any value.

For $t > 0$ and $h \to 0$

$$M = h f_0(\beta) / 2\beta u_0 t\tag{157}$$

The susceptibility exponent γ thus equals 1. If $h \neq 0$ but $t \to 0$, $M \to [f_0(\beta_c)]^{\frac{1}{2}}$. The critical exponent δ consequently cannot be defined in this case.

The entropy per particle S is given by

$$S = -\left(\frac{\partial \mu}{\partial T}\right)_{P,v} = \frac{p_c^2}{m} \left(\frac{\partial r}{\partial T}\right)_{P,v}\tag{158}$$

For $h \to 0$, (152) gives $r = 2t_1$. Substituting this value for r in the third and fourth terms on the right-hand side of (150), we get

$$r = 2t_1 + \frac{KA}{sf_0(\beta)}(2t_1)^{d/2} + \frac{4ct_1^2}{sf_0(\beta)}. \tag{159}$$

It follows that for $2 < d < 4$, the singular part of S is proportional to $t^{(d-2)/2}$. The specific heat at constant pressure therefore diverges proportional to $t^{-\alpha}$ with

$$\alpha = \frac{4-d}{2}. \tag{160}$$

6. Scaling of correlation function

In this section we wish to consider two points: (i) the relationship between susceptibility and the correlation function of the order-parameter, and (ii) the behaviour of the correlation function near the critical point.

For a classical system, the susceptibility is proportional to the zero wave-vector Fourier component of the correlation function (see *e.g.*, Ma 1976). For a quantum system the relationship, as we shall see, is not that simple. Moreover, since one can define more than one susceptibilities for the Bose gas (depending upon whether pressure or density or some other variable is held fixed) it needs to be made explicit as to which susceptibility is directly connected with the correlation function.

The order-parameter M by definition is

$$M = \tfrac{1}{2} \, \mathrm{Tr} \left[\exp \left(-H_0 \right) \left(\frac{a_0}{\sqrt{V}} + \frac{a_0^\dagger}{\sqrt{V}} \right) \right] \Big/ \mathrm{Tr} \, \exp \left(-H_0 \right), \tag{161}$$

where H_0 is given by (7). On allowing the external field h to increase by an amount δh, H_0 changes by

$$\delta H_0 = -\tfrac{1}{2} V^{1/2} (a_0 + a_0^\dagger) \, \delta h. \tag{162}$$

On using the expansion (Abrikosov *et al* 1963)

$$\exp \left(-H_0 - \delta H_0 \right) = \exp \left(-H_0 \right) - \exp \left(-H_0 \right)$$

$$\int_0^1 d\tau \exp \left(\tau H_0 \right) \delta H_0 \exp \left(-\tau H_0 \right) + \ldots \tag{163}$$

One obtains

$$\left(\frac{\partial M}{\partial h} \right)_{\mu, T} = \int_0^1 \langle A_0 (\tau) \, A_0 (0) \rangle \, d\tau - M^2 V, \tag{164}$$

where $A_0 (\tau) = \exp \left(\tau H_0 \right) \tfrac{1}{2} (a_0 + a_0^\dagger) \exp \left(-\tau H_0 \right),$ \tag{165}

is the imaginary-time Heisenberg operator associated with the $k=0$ component of the order-parameter and $\langle \ \rangle$ denotes thermodynamic average. Equation (164) shows that the susceptibility at *constant chemical potential and temperature* is given by the integral of the imaginary time correlation function of the zero wave-vector components of the order-parameter fluctuations. The result, though derived in the context of the Bose gas, is of general validity for quantum systems.

The ordinary (time-independent) correlation function of the order-parameter fluctuations is defined as

$$\Gamma\,(x_1-x_2) = \left\langle \left(\frac{\psi\,(x_1) + \psi^\dagger\,(x)}{2} - M \right) \left(\frac{\psi\,(x_2) + \psi^\dagger\,(x_2)}{2} - M \right) \right\rangle \quad (166)$$

This can be written as a Fourier series, *viz.*

$$\Gamma\,(x_1-x_2) = \frac{1}{V} \sum_k \Gamma\,(k) \exp\,[\iota\, k\,(x_1-x_2)], \quad\quad (167)$$

$$\Gamma\,(k) = \langle A_k\, A_{-k} \rangle - \delta_{k,0}\, V M^2, \quad\quad (168)$$

$$A_k = \tfrac{1}{2}\,(a_k + a^\dagger_{-k}). \quad\quad (169)$$

$\Gamma(k)$ for $k=0$ is not the same as the expression (164) for the susceptibility. To bring out the connection between $\Gamma(k)$ and the right hand side of (164), we define the imaginary time Green's function

$$G(k, \tau) = - \langle T\, A_k\,(\tau)\, A_{-k}\,(0) \rangle, \quad\quad (170)$$

where $-1 < \tau < 1$, $A_k\,(\tau)$ is defined in the same was as $A_0(\tau)$, and T denotes the time-ordering operator which orders earlier times to the right. Invariance of trace under a cyclic permutation implies

$$G(k, \tau < 0) = G(k, 1 + \tau). \quad\quad (171)$$

The function $G\,(k,\ \tau)$ can be expanded as a Fourier series in the interval $-1 < \tau < 1$. Using (171), one gets

$$G(k, \tau) = \sum_n \exp\,(-\iota\omega_n \tau)\, G(k, \omega_n), \quad\quad (172)$$

$$G(k, \omega_n) = \int_0^1 d\tau \exp\,(\iota\omega_n\, \tau)\, G(k, \tau), \quad\quad (173)$$

where $\omega_n = 2\pi n$, n ranging over the set of integers. Equation (164) can now be rewritten as

$$(\partial M/\partial h)_{\mu,\,T} = - G(k = 0, \omega_n = 0) - M^2 V, \quad\quad (174)$$

while $\Gamma(k)$ takes the form

$$\Gamma(k) = - G(k, \omega_n = 0) - \delta_{k0} M^2 V - \sum_{n \neq 0} \exp(-\iota\omega_n\delta) G(k, \omega_n), \quad (175)$$

where $\delta \to +0$. It is evident that the $\omega_n = 0$ component of $\Gamma(k = 0)$ equals the susceptibility at constant μ and T.

The Green function $G(k, \tau)$ can be expressed in terms of the elementary Green's functions of a_k and $a_k^\dagger$, viz.

$$G(k, \tau) = \tfrac{1}{4} [\mathscr{G}(k, \tau) + \mathscr{G}(-k, -\tau) + \hat{\mathscr{G}}(k, \tau) + \check{\mathscr{G}}(k, \tau)], \quad (176)$$

$$\mathscr{G}(k, \tau) = - \langle Ta_k(\tau) a_k^\dagger(0) \rangle, \quad (177)$$

$$\hat{\mathscr{G}}(k, \tau) = - \langle Ta_k(\tau) a_{-k}(0) \rangle, \quad (178)$$

$$\check{\mathscr{G}}(k, \tau) = - \langle Ta_{-k}^\dagger(\tau) a_k^\dagger(0) \rangle = \hat{\mathscr{G}}^*(k, \tau). \quad (179)$$

The elementary Green functions can be expressed as Fourier series similar to (172). As a consequence, one finds

$$G(k, \omega_n) = \tfrac{1}{4} [\mathscr{G}(k, \omega_n) + \mathscr{G}(-k, -\omega_n) + \hat{\mathscr{G}}(k, \omega_n) + \hat{\mathscr{G}}^*(k, -\omega_n)] \quad (180)$$

The simplest way to calculate $\mathscr{G}, \hat{\mathscr{G}}$ is to solve the equation of motion of $a_k(\tau)$, viz.

$$da_k/d\tau = - [a_k, H_0], \quad (181)$$

One finds

$$a_k(\tau) = \exp(-\epsilon_k\tau) a_k(0) + \frac{hV^{1/2}}{2\epsilon_k} [1 - \exp(-\epsilon_k\tau)] \delta_{k,0}, \quad (182)$$

$$\epsilon_k = s (k^2 p_c^{-2} + r). \quad (183)$$

On substituting for $a_k(\tau)$ in (177), (178), and making use of the boundary condition (171) which is satisfied by all the Green functions, we obtain

$$\mathscr{G}(k, \omega_n) = \frac{1}{\iota\omega_n - \epsilon_k} - \frac{h^2 V}{4\epsilon_k^2} \delta_{k,0} \, \delta\omega_{n,0}, \quad (184)$$

$$\hat{\mathscr{G}}(k, \omega_n) = - \frac{h^2 V}{4\epsilon_k^2} \delta_{k,0} \, \delta\omega_{n,0}, \quad (185)$$

$$G(k, \omega_n) = \frac{\epsilon_k/2}{(\iota\omega_n)^2 - \epsilon_k^2} - \frac{h^2 V}{4\epsilon_0^2} \delta_{k,0} \, \delta\omega_{n,0}. \quad (186)$$

The expression (175) for $\Gamma(k)$ now takes the form

$$\Gamma(k) = \frac{1}{2\epsilon_k} - \sum_{n \neq 0} \frac{\epsilon_k/2}{(\iota\omega_n)^2 - \epsilon_k^2} \exp(-\iota\omega_n \delta). \tag{187}$$

As $|k|$ and $r \to 0$, $\epsilon_k \to 0$. Thus close to the critical point and for small $|k|$, the dominant part of the correlation function is represented by its $\omega_n = 0$ Fourier component which equals the susceptibility at constant μ. The classical relationship between susceptibility and correlation function is thus seen to be valid at long wavelengths for a Bose system close to its critical point. The result is believed to hold for quantum systems in general.

Denoting the dominant part of $\Gamma(k)$ by $\Gamma^{(c)}(k)$

$$\Gamma^{(c)}(k) = \frac{1}{2s(k^2 p_c^{-2} + r)}. \tag{188}$$

The scaling properties of $\Gamma^{(c)}(k)$ at fixed density are

$$\Gamma^{(c)}(k) \xrightarrow[k \to 0]{} \frac{1}{2s} |t|^{-2/(d-2)} f_\epsilon^{-1}(h/|t|^\Delta), \tag{189}$$

$$\Gamma^{(c)}(k, t = 0, h = 0) = p_c^2/2sk^2, \tag{190}$$

where f_ϵ is given by (96) to (98).

Let us see how these properties follow from the renormalisation group argument. For $k \neq 0$

$$\Gamma(k) = \tfrac{1}{4} [\langle a_k a_k^\dagger \rangle + \langle a_{-k}^\dagger a_{-k} \rangle]. \tag{191}$$

It is therefore sufficient to consider the scaling property of

$$g(k, r, s) = \langle a_k^\dagger a_k \rangle \tag{192}$$

If $|k| < p_c \, \zeta^{-1}$, we can carry out a partial trace as in § 1 over momenta p lying in the interval $p_c \, \zeta^{-1} < |p| < p_c$ and write

$$g(q, r, s) = \text{Tr} \, [e^{-H_1} a_q^\dagger a_q]/\text{Tr} \, e^{-H_1}, \tag{193}$$

where H_1 is given by (19). On rescaling momenta according to (20) and defining b_q's according to (25), we obtain

$$g(q, r, s) = g(\zeta q, r_1, s_1), \, \zeta |q| < p_c, \tag{194}$$

where r_1 and s_1 are given by the recurrence relations (23) and (30) respectively.

As the RG transformation is repeated, $s_1 \to 0$. To infer the scaling properties of g it is necessary to ascertain the dependence of the right hand side of (194) on s_1 since g

need not be a regular function of s_1 at $s_1=0$. This introduces an element of complexity in the theory of a quantum system which is absent in the classical case. The problem, however, is not insurmountable since what is required is the dependence of g on s_1 near $s_1=0$. In the present case of the ideal Bose gas, the problem is trivial since g can be easily known. One finds that for $s_1 \to 0$, g is proportional to s^{-1} (cf. (188)) so that

$$g\ (\zeta\ q,\ s_1,\ r_1) = s_1^{-1} f\ (\zeta\ q,\ r_1).\tag{195}$$

In the general case of an interacting system, the problem can be solved by examining the perturbation expansion of the quantity concerned (Singh 1976).

Combining (194) and (195) we get

$$g\ (q,\ r,\ s) = s^{-1}\ \zeta^2 f\ (\zeta q,\ \zeta^2\ r).\tag{196}$$

Choosing ζ^{-1} equal to $|q|\ p_c^{-1}$

$$g\ (q,\ r,\ s) = s^{-1}\ p_c^2\ |q|^{-2} f\ (p_c,\ p_c^2\ |q|^{-2}\ r).\tag{197}$$

A critical exponent η is associated with the correlation function by requiring that at the critical point, g be proportional to $|q|^{-2+\eta}$. It is evident that provided $f\ (p_c,\ 0)$ exists, η for the Bose gas is zero. Also, in order that the susceptibility for $r \neq 0$ may be finite

$$f\ (p_c,\ x) \xrightarrow[x \to \infty]{} \frac{A}{x},\tag{198}$$

where A is a constant. The critical behaviour of the susceptibility found in §§ 4 and 5 can be obtained from (197) and (198).

Finally, we consider the correlation length defined customarily in terms of the behaviour of the correlation function $\Gamma\ (x_1 - x_2)$ for large $|x_1 - x_2|$. If

$$\Gamma\ (x) \underset{|x| \to \infty}{\propto} \exp\ (-|x|/\xi),\tag{199}$$

then ξ is defined to be the correlation length. Wilson and Kogut (1974) pointed out that for practical purposes it is more convenient to define ξ in terms of the singularity of $\Gamma(q)$ for small q. If the leading singularity of $\Gamma(q)$ is at $q^2 = -q_0^2$, then $\xi = q_0^{-1}$. The location of this singularity is determined by comparing the derivative $\partial \Gamma/\partial q^2$ with $\Gamma(q)$ itself for $q \to 0$. Thus

$$\xi^2 \propto -\frac{\partial}{\partial q^2} \ln\ \Gamma\ (q)\ \bigg|_{q=0}\tag{200}$$

Choosing $\zeta = r^{-1/2}$ in (196) and noting that g is a function of $|q|$ only, one finds

$$g\ (q,\ r,\ s) = \frac{r^{-1}}{s}\ f(q^2\ r^{-1},\ 1).\tag{201}$$

The definition (200) gives

$$\xi \propto r^{-1/2}. \tag{202}$$

At a fixed density, r is given by (95). For $T > T_c$ and $h \to 0$

$$\xi \propto t^{-\nu}, \tag{203}$$

$$\nu = 1/(d-2). \tag{204}$$

For $T < T_c$, $\xi \to \infty$ as $h \to 0$. Using (98), we can also determine the dependence of ξ on h as $T \to T_c$. The result is

$$\xi (T = T_c) \propto h^{-2/(d+2)} \tag{205}$$

7. Concluding remarks

The work in the preceding sections demonstrates how ideas of the renormalisation group can be used to discuss critical behaviour by starting with the description of a system at a *microscopic* level in terms of a quantum mechanical hamiltonian. Rescaling of field amplitudes has been avoided by introducing a scaling variable s (Equation (30)). The scaling behaviour of various physical quantities in such a scheme emerges as a consequence of the irrelevant nature of s. Recalling the definition (9) of s, we may conclude that a Bose system near its critical point is properly described by a fictitious hamiltonian which contains only momenta vanishingly small in comparison with the thermal momentum $(m k_B T)^{1/2}$. The conclusion appears to be of general validity although it is difficult to establish it satisfactorily when interactions among the particles are present (Singh 1975).

An interesting point about the renormalisation-group transformation constructed for the Bose system is that while the parameter-transformation (36) to (38) has a fixed point, the fixed-point hamiltonian H^* vanishes on account of s^* being zero. Bell and Wilson (1974) remarked that the renormalisation-group transformation must have a non-trivial fixed-point hamiltonian in order that it may be useful for obtaining critical exponents. The transformation studied in this paper shows that the existence of a non-trivial fixed-point hamiltonian is not essential. Even when one is right at the critical point ($r=0$, $h=0$), one can calculate the quantities of interest by taking s arbitrarily close to, though not equal to, zero.

Finally, it is of interest to note that there is a correspondence between the critical exponents of the ideal Bose gas at a fixed density and the spherical model of ferromagnetism introduced by Berlin and Kac (1952). It turns out (Gunton and Buckingham 1968) that in the transition region, the logarithm of the partition function for the spherical model has exactly the same form as the logarithm of the partition function of the ideal Bose gas. As a result, the thermodynamic properties of the two models are the same in the transition region.

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