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Two-potential formula continued off-the-energy-shell

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Abstract. We derive an off-energy-shell generalization of the two-potential formula by using a coordinate-space approach and apply the formalism to construct algorithms for studying spatial behaviour of the fully off-shell T matrix. We also suggest some future applications of the proposed theory.

Keywords. Nuclear reactions; scattering theory; off-shell two-potential formula; interpolating equations for T matrix.

1. Introduction

There exist experimental situations which involve scattering by additive interactions. The calculation of state vectors or scattering amplitudes for such problems need not start with the kinetic energy as a zero order Hamiltonian. Instead, one begins by calculating the states, Green's function, etc for a model Hamiltonian and arrives at the formula in the title.

Applications of the two-potential formula (TPF) (Yost *et al* 1936; Gell-Mann and Goldberger 1953) immediately followed its discovery. With the availability of high-speed digital computers, there has been a surge of interest in this line of investigation. The TPF is effective in treating a wide variety of phenomena ranging from the atomic domain to GeV region (Hinckelmann and Spruch 1971; Crowley 1979; Kunz *et al* 1979; Gustafsson and Lambert 1978).

Harrington (1965) suggested that the TPF should be continued off-the-energy-shell for its possible usage in nuclear three-body problems involving charged particles. This was exploited by Alessandrini *et al* (1968), Tandy and McCarthy (1972), Zachary (1971), Bajzer (1974) and van Haeringen and van Wageningen (1975) to calculate the off-shell proton-proton T matrix. However, much advance has not been made because of the rather involved algebra.

Attempts were made to construct analytic T matrices for Coulomb (distorting potential) plus rational separable potentials. For a separable kernel the Fredholm series becomes a polynomial (Mikhlin 1964). Thus it was possible to derive results in terms of a momentum-space approach. But the method involves typical contour integrals which are difficult to perform (Bagchi and Mulligan 1974).

In this work we implement a coordinate-space (r-space) approach (Fuda and Whiting 1973; Talukdar *et al* 1977, 1979) to derive an off-shell TPF and use it to construct algorithms for calculating off-shell T matrix within the framework of the

variable-phase-approach (VPA) to potential scattering (Calogero 1967). Our construction procedure is similar to that of Tikochinsky (1970, 1977) who obtained an interpolating equation for the on-shell T matrix as an immediate consequence of the usual two-potential formula. We include the derivation of the off-shell TPF in § 2 and apply it in § 3 to develop an equation for the interpolating off-shell T matrix with the following motivations.

Potential models are often used to represent the basic interaction between elementary particles. Typically, the nucleon-nucleon potentials contain a hard core, a phenomenological intermediate region, and a one-pion exchange tail. Different potentials are known to have the same value for r greater than certain r', and differ only for $r < r'$, although they fit the scattering data (Feshbach and Kerman 1967). Thus scattering experiments can probe the potential down to r', but not closer. It is, therefore, of some general interest to express the transition matrix (T) in a form which exhibits the effect of the action of different regions of the potential by using the basic philosophy of the VPA. The on- and half-off-shell cases have been treated by Calogero (1967) and by Sobel (1968) respectively. Understandably, these studies bear relevance only to the discussion of elastic scattering and half-shell reactions (Picker *et al* 1971). But the three-body equation of Faddeev (1961) has as input the two-body T matrices off-the-energy-shell.

In § 2 we also examine the on- and half-off-shell limits of our fully off-shell equation. We present a few concluding remarks in § 4.

2. Off-shell two-potential formula

Consider the two-body Hamiltonian

$$H = H_D + V_1, \tag{1}$$

with $$H_D = H_0 + V_D. \tag{2}$$

Here H_0 is the kinetic energy operator, and V_D and V_1, the distorting and residual-potential operators. We write $V = V_D + V_1$ for the total potential operator. In the representation space all V's refer to spherically symmetric local potentials. Throughout this paper unsubscripted variables refer to scattering by V. We use subscripts D and 1 for variables relating to scattering on V_D and V_1 respectively. The angular momentum l, however, will occur as subscript in all appropriate places. The T matrices T and T_D satisfy the Lippmann-Schwinger equations

$$T(k^2) = V + V\, G_0^+(k^2)\, T(k^2), \tag{3}$$

and $$T_D(k^2) = V_D + V_D\, G_0^+(k^2)\, T_D(k^2), \tag{4}$$

respectively. Here k is the on-shell momentum related to the energy by $E = k^2 + i\epsilon$ and $G_0^+(k^2)$, the free particle Green's function. The superscript $+$ is used through-

out for the outgoing wave boundary condition. We introduce resolvents G^+ and G_D^+ and Møller wave operators Ω and Ω_D by

$$G^+(k^2) = (k^2 - H)^{-1}, \tag{5}$$

$$G_D^+(k^2) = (k^2 - H_D)^{-1}, \tag{6}$$

$$\Omega(k^2) = 1 + G_0^+(k^2)\, V\, \Omega(k^2), \tag{7}$$

and $$\Omega_D(k^2) = 1 + G_0^+(k^2)\, V_D\, \Omega_D(k^2) \tag{8}$$

From equations (3), (4), (7) and (8) we get

$$T(k^2) = V\,\Omega(k^2), \tag{9}$$

and $$T_D(k^2) = V_D\,\Omega_D(k^2). \tag{10}$$

Using the operator identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ it can easily be seen that

$$\begin{aligned} G^+(k^2) - G_D^+(k^2) &= G^+(k^2)(V - V_D)\, G_D^+(k^2) \\ &= G_D^+(k^2)(V - V_D)\, G^+(k^2). \end{aligned} \tag{11}$$

Equations (7) and (8) can also be written in alternative forms

$$\Omega(k^2) = 1 + G^+(k^2)\, V = 1 + G_0^+(k^2)\, T(k^2), \tag{12}$$

and $$\Omega_D(k^2) = 1 + G_D^+(k^2)\, V_D = 1 + G_0^+(k^2)\, T_D(k^2). \tag{13}$$

Combining equations (11), (12) and (13) we arrive at

$$\Omega(k^2) = \Omega_D(k^2) + G_D^+(k^2)(V - V_D)\,\Omega(k^2). \tag{14}$$

Introducing an off-shell momentum q we can take equations (12) and (13) in a mixed representation to read

$$\begin{aligned} \psi_l^+(k, q, r) = &-\tfrac{1}{2}\pi q\, T_l(k, q, k^2) \exp\left(-i\frac{l\pi}{2}\right) f_l(k, r) \\ &+ \frac{1}{2i}\left[\exp\left(-i\frac{l\pi}{2}\right) f_l(k, q, r) - \exp\left(i\frac{l\pi}{2}\right) f_l(k, -q, r)\right], \end{aligned} \tag{15}$$

and

$$\begin{aligned} \psi_{Dl}^+(k, q, r) = &-\tfrac{1}{2}\pi q\, T_{Dl}(k, q, k^2) \exp\left(-i\frac{l\pi}{2}\right) f_{Dl}(k, r) \\ &+ \frac{1}{2i}\left[\exp\left(-i\frac{l\pi}{2}\right) f_{Dl}(k, q, r) - \exp\left(i\frac{l\pi}{2}\right) f_{Dl}(k, -q, r)\right], \end{aligned} \tag{16}$$

where $T_l\ (k, q, k^2)$ is the lth projected half-off-shell T matrix. The objects $f_l\ (k, r)$ and $f_l\ (k, q, r)$ stand for the on- and off-shell Jost solutions. The properties and asymptotic behaviour of $f_l\ (k, q, r)$ have been discussed in some detail by Fuda and Whiting (1973). In writing equations (15) and (16) we have used the following

$$\langle \mathbf{r} \,|\, \Omega\ (k^2) \,|\, q\ lm\rangle = (2/\pi)^{1/2}\ (qr)^{-1}\ \psi_l^+\ (k, q, r)\ Y_{lm}\ (\hat{r}), \tag{17}$$

and

$$\langle \mathbf{r} \,|\, q\ lm\rangle = (2/\pi)^{1/2}\ j_l\ (qr)\ Y_{lm}\ (\hat{r}), \tag{18}$$

in addition to the well-known eigen function expansion of the free particle Green's function use was also made of

$$\langle \mathbf{r} \,|\, \Omega_D\ (k^2) \,|\, q\ lm\rangle.$$

In the mixed representation equation (14) becomes

$$\psi_l^+\ (k, q, r) = \psi_{Dl}^+(k, q, r) + \int_0^\infty G_{Dl}^+(k, r, r')\ V_1\ (r')\ \psi_l^+\ (k, q, r'), \tag{19}$$

where

$$G_{Dl}^+(k, r, r') = -\frac{1}{k}\psi_{Dl}^+(k, r_<)\, f_{Dl}\ (k, r_>)\ \exp\left(-i\frac{l\pi}{2}\right). \tag{20}$$

The functions $\psi_{Dl}^+\ (k, r_<)$ and $f_{Dl}\ (k, r_>)$ are the on-shell outgoing and irregular solutions of H_D. For $r \to \infty$ equation (19) takes the form

$$\psi_l\ (k, q, r) \underset{r\to\infty}{\sim} \psi_{Dl}(k, q, r) - \left[\exp\left(-i\frac{l\pi}{2}\right)\Big/\ k\ \right] f_{Dl}\ (k, r) \int_0^\infty dr'$$

$$\psi_{Dl}^+(k, r')\ V_1(r')\ \psi_l^+\ (k, q, r'). \tag{21}$$

Using the asymptotic form of ψ_l's and f_{Dl} (Fuda and Whiting 1973) in equation (21) we get

$$T_l(k,q,k^2) = T_{Dl}(k,q,k^2) + \frac{2}{\pi kq}\int_0^\infty dr'\ \psi_{Dl}^+(k,r')\, V_1(r')\, \psi_l^+\ (k,q,r'). \tag{22}$$

Equation (22) represents the half-off-shell generalization of the two-potential formula.

To look for the fully off-shell case we multiply equation (19) by $u_l(pr)V(r)$ and integrate. We thus obtain

$$T_l\ (p, q, k^2) = T_{Dl}\ (p, q, k^2) + \frac{2}{\pi pq}\int_0^\infty dr'\Big[u_l\ (pr') +$$

$$\int_0^\infty dr\ G_{Dl}^+(k, r, r')\ V_D\ (r)\ u_l\ (pr)\Big]\ V_1\ (r')\ \psi_l^+\ (k, q, r'), \tag{23}$$

where $u_l(x)$ is the Ricatti-Bessel function written as $u_l(x) = x j_l(x)$. In deriving equation (23) we have used the matrix elements of equations (9) and (10) between states designated by the off-shell momenta p and q. In view of equation (8) it is easy to see that the object in the square bracket of equation (23) stands for the off-shell wave function $\psi^+_{Dl}(k, p, r')$. Thus we write (23) in the form

$$T_l(p, q, k^2) = T_{Dl}(p, q, k^2) + \frac{2}{\pi pq}\int_0^\infty dr'\, \psi^+_{Dl}(k, p, r')\, V_1(r')\, \psi^+_l(k, q, r'). \tag{24}$$

Equation (24) is clearly the off-shell two-potential formula, which we shall use in the next section to write an interpolating equation for the T matrix.

3. Interpolating equation for the off-shell T matrix

Consider the case of scattering by a total potential $V+\Delta V$, V being the distorting potential. We omit the subscript D. Instead we introduce $V+\Delta V$ and V in the arguments of the appropriate T matrices. Thus equation (24) reads

$$T_l(p, q, k^2;\, V+\Delta V) - T_l(p, q, k^2;\, V)$$

$$= \frac{2}{\pi pq}\int_0^\infty \psi^+_l(k, p, r;\, V)\, \Delta V(r)\, \psi^+_l(k, q, r;\, V+\Delta V)\, dr. \tag{25}$$

Expanding the T matrix and wave function about the distorting potential and retaining terms to first order only we get $\Delta T_l(p, q, k^2;\, V)$

$$= \frac{2}{\pi pq}\int_0^\infty \psi^+_l(k, p, r;\, V)\, \Delta V(r)\, \psi^+_l(k, q, r;\, V)\, dr. \tag{26}$$

If the potential V depends on some parameter α, equation (26) can be written in the form (Tikochinsky 1970, 1977)

$$\frac{dT_l(p, q, k^2;\, \alpha)}{d\alpha} = \frac{2}{\pi pq}\int_0^\infty \psi^+_l(k, p, r;\, \alpha)\, \frac{dV(r;\, \alpha)}{d\alpha}\, \psi^+_l(k, q, r;\, \alpha). \tag{27}$$

Following the usual practice in the VPA we consider the potential (Calogero 1967)

$$V(r, \alpha) = V(r)\, \theta(\alpha - r). \tag{28}$$

The step function θ tells us that the potential is cut off at the point $r = \alpha$. From equations (27) and (28) we get

$$\frac{dT_l(p, q, k^2;\, \alpha)}{d\alpha} = \frac{2}{\pi pq}\, V(\alpha)\, \psi^+_l(k, p, \alpha)\, \psi^+_l(k, q, \alpha), \tag{29}$$

with $$\psi^+_l(k, x, \alpha;\, \alpha) = \psi^+_l(k, x, \alpha), \tag{30}$$

the values of the wavefunction in equation (30) beyond the cut-off radius of the potential. We thus write* $\psi_l^+(k, x, a)$ in the form

$$\psi_l^+(k, x, a) = -\tfrac{1}{2}\pi x\, T_l(k, x, k^2; a)\, \omega_l^+(kx) + u_l^+(x\, a), \tag{31}$$

where $T_l(k, x, k^2; a)$ is the interpolating half-off-shell T matrix and $\omega_l(k\, a)$ is a Ricatti-Hankel function with the phase convention used by Fuda and Whiting (1973). In view of equation (31), equation (29) reads

$$\frac{dT_l(p, q, k^2; a)}{da} = \frac{2}{\pi pq} V(a)\left[u_l(p\, a) - \frac{\pi p}{2} T_l(k, p, k^2; a)\, \omega_l^+(ka)\right]$$

$$\times \left[u_l(qa) - \frac{\pi q}{2} T_l(k, q, k^2; a)\, \omega_l^+(ka)\right]. \tag{32}$$

Equation (32) represents a linear differential equation with the boundary conditions that $T_l(p, q, k^2; 0) = 0$ and $T_l(p, q, k^2; \infty)$, the off-shell T matrix. Here the interpolating half-off-shell T matrices $T_l(k, p, k^2; a)$ and $T_l(k, q, k^2; a)$ enter as input. Thus knowing $T_l(k, x, k^2; a)$ the off-shell T matrix can be obtained from equation (32) by means of a single quadrature.

A useful check on equation (32) consists in showing that in the half-off-shell limit $p \to k$, our off-shell equation yields the equation of Sobel (1968). Needless to say that in the on-shell limit we get the well-known equation of Calogero (1967). To facilitate comparison with the Sobel's work we restrict ourselves to the s-wave case and omit the subscript $l = 0$. Thus in the half-off-shell limit equation (32) becomes (Talukdar and Das 1979)

$$\frac{dT(k, q, k^2; a)}{da} = \frac{2}{\pi kq} V(a)\left[\sin ka - \frac{\pi k}{2}\exp(ika)\, T(k; a)\right]$$

$$\times \left[\sin q\, a - \frac{\pi q}{2}\exp(ika)\, T(k, q, k^2; a)\right], \tag{33}$$

where the on-shell T matrix function

$$T(k; a) = T(k, k, k^2; a). \tag{34}$$

Equation (33) is a linear differential equation with boundary condition similar to that for the off-shell T matrix function. Following Sobel (1968) we introduce the quasi phase parameter $\Delta(k, q, a)$ such that

$$\Delta(k, q, a) \underset{q \to k}{\sim} \sin \delta(a). \tag{35}$$

*An intuitive way of deriving equation (31) is to assume that beyond the cut-off radius $f_l(k, q, r)$ behaves as $\omega_l^+(qr)$ rather than its asymptotic behaviour. However, for a rigorous derivation one can look at Bahethi and Fuda (1971).

The half-off-shell T matrix function written in terms of the quasi phase parameter reads

$$T(k, q, k^2; a) = -\frac{2}{\pi k} \Delta(k, q, a) \exp[i\delta(a)]. \tag{36}$$

In these equations $\delta(a)$ is the phase function of Calogero (1967) which satisfies the nonlinear differential equation

$$\frac{d\delta(a)}{da} = -\frac{V(a)}{k} \sin^2[ka+\delta(a)]. \tag{37}$$

Equations (33) to (37) can be combined to write the equation for quasi phase

$$\frac{d\Delta(k, q, a)}{da} = -V(a) \sin(ka+\delta(a)) \times \left[\frac{\sin qa}{q} + \frac{1}{k} \Delta(k, q, y) \cos(ky+\delta(y))\right]. \tag{38}$$

In the on-shell limit equation (33) reads

$$\frac{dT(k; a)}{da} = \frac{2}{\pi k^2} V(a) \left[\sin ka - \frac{\pi k}{2} \exp(ika) T(k; a)\right]^2. \tag{39}$$

In comparing equation (39) with that of Calogero one should note that our on-shell normalization of the T matrix function is

$$T(k, a) = -\frac{2}{\pi k} \exp[i\delta(a)] \sin \delta(a), \tag{40}$$

while that of Calogero is

$$T(k; a) = \exp[i\delta(a)] \sin \delta(a). \tag{41}$$

Solving the coupled equations (33) and (39) by any appropriate numerical method, the spatial behaviour of the off-shell T matrix can be studied by integrating equation (32) with different values of the upper limit. Investigation of the nature of $T(p, q, k^2, r)$ as a function of r is expected to play a significant role in nuclear few-body problems.

4. Conclusion

In this work we have sought an off-energy-shell generalization of the two-potential formula, which we use in turn to develop an interpolating equation for the fully off-shell T matrix. The numerical study of this equation will delineate, on the one hand,

the spatial behaviour of the T matrix and, on the other, meet the basic requirement of the Faddeev equation. Further comments on applications of the method developed are also in order. For a separable residual interaction $V_1 = |v\rangle\langle v|$, equations (23) and (24) remain effectively unchanged except that one needs to replace the single quadrature occurring here by appropriate double integrals. Off-shell Coulomb functions are now available in the literature (van Haeringen 1978). Thus one would expect to treat the Coulomb plus Yamaguchi potential (Yamaguchi 1954) in a relatively noncomplicated manner by using the present approach.

There are two important channels for deuteron which are excited with relatively large probability—the stripping and break-up channels. The r-space formalism developed here may be utilized to deal with break-up channels in a realistic way.

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