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# Possible realisation and generalisation of two specific 2×2 forms

S K SONI
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.
Present address: Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA

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**Abstract.** For each of a couple of two-dimensional forms for quark mass matrix, it is discussed how that form may be realised in a certain gauge scheme (one of them in the standard model and the other in a scheme based on simple rank two times U(1)) by imposing suitable discrete symmetries and how under a certain small angle approximation that form may be regarded as the simplest member of a family of higher dimensionality forms.

**Keywords.** quark mass matrix; biunitary diagonalisation; discrete symmetries; higher dimensionality forms.

## 1. Introduction

In the following, some discrete symmetries are imposed in elementary gauge schemes based on SU(2) (simple rank two) times U(1) as gauge group having four quarks (four ordinary plus four superheavy quarks, their intermixing being forbidden). This allows realisation of two specific forms for quark mass matrix, one in the former case and the other in the latter. The ucds mass matrix being a direct sum of two $2\times 2$ matrices is therefore collectively referred to as a $2\times 2$ form in the following. The $2\times 2$ form realised in the former (latter) case relates Cabibbo angle $\theta_c$ to the quark mass ratio $m_d/m_s$ as

$$\theta_c \sim \frac{m_d}{m_s}\left(\theta_c^2 \sim \frac{m_d}{m_s}\right).$$

These forms thus endowed with some physical significance are also observed to have the following property of some mathematical interest, independently of any further gauge model considerations. In analogy with a $2\times 2$ form for mixing between two negatively (and also two positively) charged fermions, one may also envision an $n\times n$ form for mixing between $n$ negatively (as well as $n$ positively) charged fermions. With a specific $n\times n$ form it is found that for arbitrary $n$ under a suitable approximation for $2n$ fermion masses $m_1^\pm, m_2^\pm, \ldots, m_n^\pm$ (which are diagonal entries resulting on biunitary diagonalisation of the $n\times n$ form) there are to leading order of approximation $n$-1 Cabibbo-like angles related to fermion mass ratios as,

$$\theta_1 \sim \frac{m_1^-}{m_2^-}; \qquad \theta_i \sim \frac{m_{i+1}^-}{m_i^-},$$

$$[\theta_1 \sim (m_1^-/m_2^-)^{1/2} - (m_1^+/m_2^+)^{1/2}\ ; \quad \theta_i \sim (m_i^-/m_{i+1}^-)^{1/2} - (m_i^+/m_{i+1}^+)^{1/2}],$$

for $2 \leqslant i \leqslant n-1$. This result suggests that these specific higher-dimensionality forms may be regarded as different-sized members (obtained when $n$ assumes values 2, 3, 4, etc.) of a common family. The family for which the former (latter) result holds includes as its first member (i.e. $n=2$) the $2\times 2$ form implying $\theta_c \sim (m_d/m_s)$ [$\theta_c = (m_d/m_s)^{1/2}$ or more precisely, $\theta_c \sim (m_d/m_s)^{1/2} - (m_u/m_c)^{1/2}$] and for the validity of this result the fermion mass approximation required in the two cases is different. To sum up, the focus of attention here is a pair of $2\times 2$ forms that may each be rea lised in a certain gauge scheme and in the sense outlined above may each also be generalised to an $n\times n$ form.

The scheme based on SU(2)×U(1) is the well-known standard model (Weinberg 1967; Salam 1968; Glashow *et al* 1970) with four quarks. The derivation of the $2\times 2$ form in this case (see § 2·1) adds little to the spirit of original derivation of Pakvasa and Sugawara (1978) who were also incidentally led to the same result for $\theta_c$, i.e. $\theta_c \sim m_d/m_s$. Since the form realised in § 2.1 is different from what they derived (*e.g.*, whereas their form constrains $m_u$ to vanish, in the following case a small nonvanishing value for $m_u$ may be consistently allowed), elucidation of some of the mathematical details becomes essential. The permutation group found useful in this context is discussed in group-theory text-books. The other scheme considered in § 2.2 has O(5) as the simple rank two group (Soni 1979). In this case, details of derivation are omitted for the reason that O(5)×U(1) effectively reduces to the left-right symmetric gauge group for which Fritzsch (1977) (see also Wilczek and Zee 1977) derived an identical form modulo some irrelevant phase factors. In addition to these papers of Fritzsch and others, literature on the subject has grown rapidly in the past two years. The interested reader may consult the references given in the talk by Illiopoulos (1979). Finally, § 3 is a discussion of diagonalisation of the two $n\times n$ forms.

## 2. Realisation of the 2×2 forms

The following is a derivation of the pair of $2\times 2$ forms. On setting the essential notation they are treated separately, the standard model form in § 2.1 and the O(5) ×U(1) form in §2.2. The negative results summarised in §2.1a originally found by Gatto and others (for many relevant references, see Illiopoulos 1979) prove instructive for the partially successful attempt at derivation of the $2\times 2$ form given in § 2.1b. Grouping quarks of given charge (subscript $\pm$ for positively and negatively charged) into a column vector, the quark mass term may be written as,

$$\mathscr{L}^{\text{mass}}_{\text{quark}} = \bar{Q}_{+L} M_+ Q_{+R} + \bar{Q}_{-L} M_- Q_{-R} + \text{h.c.} \tag{1}$$

and on diagonalisation as,

$$\mathscr{L}^{\text{mass}}_{\text{quark}} = \bar{q}_{+L} \mathrm{m}_+ q_{+R} + \bar{q}_{-L} \mathrm{m}_- q_{-R} + \text{h.c.} \tag{2}$$

$\mathrm{m}_\pm$ are diagonal matrices with quark masses as their entries. Quark states in $Q_{\pm L(R)}$ are mixtures of corresponding diagonal states in $q_{\pm L(R)}$, *i.e.*

$$Q_{\pm L(R)} = V_{\pm L(R)}\, q_{\pm L(R)}, \tag{3}$$

where $V_{\pm L(R)}$ is a unitary matrix. The mass matrix $M_+ \oplus M_-$ is diagonalised as,

$$V^{-1}_{\pm L} M_{\pm} V_{\pm R} = \mathrm{m}_{\pm}. \tag{4}$$

This biunitary transformation is equivalent to the following pair of unitary transformations,

$$V^{-1}_{\pm L} M_{\pm} M^{+}_{\pm} V_{\pm L} = \mathrm{m}^{2}_{\pm},$$

$$V^{-1}_{\pm R} M^{+}_{\pm} M_{\pm} V_{\pm R} = \mathrm{m}^{2}_{\pm}. \tag{5}$$

The left-handed charged-current matrice defined through

$\bar{Q}_{+L} Q_{-L} \equiv \bar{q}_{+L} U^{\mathrm{cc}} q_{-L}$ is given by,

$$U^{\mathrm{cc}} = V^{+}_{+L} V_{-L}. \tag{6}$$

(The superscript on $V_{+L}$ stands for hermitian disjoint).

For the well-known $2\times2$ case, this matrix is an orthogonal matrix parametrised by Cabibbo angle $\theta_c$ equal to the difference of mixing angles in the left-handed negatively and positively charged quark sectors.

## 2.1 *The standard model form*

In the standard model, the left-handed and right-handed gauge multiplets, isodoublets and isosinglets respectively, may be grouped into three column vectors $\psi_L$, $\psi^{\pm}_R$ (see eq. (8)) so that the most general gauge invariant Yukawa interaction with a Higgs isodoublet $\Phi \equiv \begin{pmatrix}\varphi^0\\ \varphi^-\end{pmatrix}$ may be written as,

$$\mathscr{L}^{\mathrm{Yuk}} = \bar{\psi}_L\, \Gamma^{+}\, \Phi\, \psi^{+}_R + \bar{\psi}_L\, \Gamma^{-}\, \Phi^{c}\, \psi^{-}_R + \text{h.c.} \tag{7}$$

Here $\Phi^c$ is the charge conjugate of $\Phi$ and,

$$\bar{\psi}_L = ((\bar{U}\,\bar{D})_L, (\bar{C}\,\bar{S})_L),$$

$$\psi^{+}_R = \begin{pmatrix}U_R\\ C_R\end{pmatrix} \qquad \psi^{-}_R = \begin{pmatrix}D_R\\ S_R\end{pmatrix} \tag{8}$$

$\Gamma^{\pm}$ is a $2\times2$ matrix with the arbitrary Yukawa coupling constants as entries, *e.g.*, the first term in (7) is $\Gamma^{+}_{11}\,(\bar{U}\,\bar{D})_L \begin{pmatrix}\varphi^0\\ \varphi^-\end{pmatrix} U_R$. Denoting the vacuum expectation value (vev) of $\varphi^0$ as $\langle\varphi^0\rangle$, the mass-matrix is $(\Gamma^{+}\oplus\Gamma^{-})\,\langle\varphi^0\rangle$. In the case of a number of similar higgs isodoublets,

$$\Phi_i \equiv \begin{pmatrix}\phi^0_i\\ \phi^-_i\end{pmatrix}, \quad i=1, 2, \text{etc.}$$

equation (7) is correspondingly a sum of similar terms, one for each Higgs isodoublet.

2.1a *Summary of negative results.* To begin with, consider a single Higgs. Let it be multiplied by an arbitrary phase factor simultaneously with multiplication of $\psi_L$, $\psi_R^{\pm}$ by arbitrary $2\times 2$ unitary matrices denoted (upto overall phase factors) by $S_L$, $S_R^{\pm}$ each with unit determinant. Under this arbitrary discrete transformation, the gauge boson interaction with fermions remains invariant because for any gauge boson (suppressing weak isospin indices) this interaction is simply of form,

$$\sum_{i=1}^{2} \overline{\psi_{Li}}\, A\, \psi_{Li} + \overline{\psi_{Ri}^{+}}\, A^{+}\, \psi_{Ri}^{+} + \overline{\psi_{Ri}^{-}}\, A^{-}\, \psi_{Ri}^{-},$$

where the appropriate $2\times 2$ matrices $A$, $A^{\pm}$ act in weak isospin space. Under this transformation, the Yukawa coupling matrix becomes

$$S_L \Gamma^{+} (S_R^{+})^{-1} \oplus S_L\, \Gamma^{-} (S_R^{-})^{-1}.$$

In arriving at this expression the overall phases multiplying $S_L$, $S_R^{\pm}$ are adjusted to absorb the phase factors with which $\Phi$ and $\Phi^c$ are multiplied. For invariance of Yukawa couplings,

$$S_L\, \Gamma^{\pm} (S_R^{\pm})^{-1} = \Gamma^{\pm}.$$

For a nontrivial solution (*i.e.* $\Gamma^{\pm}$ does not vanish identically), $S_R^{\pm}$ is related to $S_L$ by a similarity transformation. $S_L$ may be parametrised as

$$\begin{pmatrix} \cos\alpha \exp(i\eta_{11}) & \sin\alpha \exp(i\eta_{12}) \\ -\sin\alpha \exp(-i\eta_{12}) & \cos\alpha \exp(-i\eta_{11}) \end{pmatrix},$$

considering separately the following three ranges for values of parameters

(i) $\alpha=0,\ 0 < \eta_{11} < \pi$

(ii) $-\frac{\pi}{2} < \alpha < +\frac{\pi}{2} (\alpha \neq 0),\quad 0 < \eta_{11} < \pi,\quad 0 < \eta_{12} < \pi$

(iii) $\alpha = \pm\frac{\pi}{2},\ 0 < \eta_{12} < \pi.$

The resulting nontrivial solution for $\Gamma^{\pm}$ correspondingly implies constraints on form of mass matrix and hence on $\theta_c$. For case (i) ((iii)), $\theta_c$ equals 0 (90°) and for case (ii) $\theta_c$ is indeterminate (*i.e.* not expressible as a mass-ratio and may lie anywhere between 0 and 90°). Equivalently, in place of requiring invariance of Yukawa coupling matrix under arbitrary discrete transformations on $\psi_L$, $\psi_R^{\pm}$ as done above, the same result (*i.e.* Cabibbo mixing is absent, indeterminate or maximal) follows with the requirement that Yukawa interaction transforms as identity representation of any discrete group under which $\psi_L$, $\psi_R^{\pm}$ transform as its two-dimensional representations.

The two Higgs case that is now considered proves instructive for the final attempt with three Higgs discussed in § 2.1b. In this case, Higgs potential $V(\Phi_1, \Phi_2)$ is an arbitrary superposition of bilinears $\Phi_i^+ \Phi_j$ $(i, j=1, 2)$ and biquadratic terms like $(\Phi_k^+ \Phi_l)$, $(\Phi_m^+ \Phi_n)$ $(k, l, m, n = 1, 2)$. Its form is constrained by requiring invariance under following pair of discrete transformations (i) interchange of labels 1 and 2, $\Phi_1 \longleftrightarrow \Phi_2$ (ii) multiplication of $\Phi_1$ and $\Phi_2$ by arbitrary phase factors,

$$\Phi_1 \to \exp(i\eta)\,\Phi_1 \text{ and } \Phi_2 \to \exp(-i\eta)\,\Phi_2,$$

where $\eta$ can have any value between 0 and $\pi$. Since the invariant form for potential does not depend on the precise value of $\eta$, this suggests the following. $\Phi_1$ and $\Phi_2$ may be regarded as top and bottom members of the two-dimensional irreducible representation,

$$a = \begin{pmatrix} \exp(2\pi i/n) & 0 \\ 0 & \exp(-2\pi i/n) \end{pmatrix}, \; b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

of the nonabelian discrete group which is semi-direct product of the $n$-element cyclic group $C_n=(e, a, a^2, \ldots, a^{n-1})$; $a^n=e$ and the reflection group $P=(e, b)$; $b^2=e$. Here $n \geqslant 3$ and, by definition, $bg\,b^{-1}=g^{-1}$ for each element $g$ of $C_n$. The action of $b$ is the transformation (i) above and that of $g(\neq e)$ is the one labelled (ii) above with a specific value of $\eta$, *e.g.* $\eta=2\pi/n$ for action of $a$. Hence invariance under this group also results in a form for potential identical to that obtained above. Having seen this equivalence, the latter viewpoint is adopted henceforth. Without loss of generality $n$ may be assumed equal to 3. The nonabelian group for this special choice of $n$ is the familiar permutation group $S_3$ which is also the simplest nonabelian group. Having decided how $\Phi_1$ and $\Phi_2$ transform under $S_3$ from invariance considerations of Higgs potential, a variety of $S_3$-invariant forms for Higgs-fermion Yukawa interaction are permissible depending on representation content of

$$\begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \; \begin{pmatrix} \psi_{R1}^{\pm} \\ \psi_{R2}^{\pm} \end{pmatrix}$$

under $S_3$. Each of these two-component objects may transform according to a representation that is equivalent to $\Gamma^{(1)} \oplus \Gamma^{(1)}$, $\Gamma^{(1)} \oplus \Gamma^{(2)}$ or $\Gamma^{(3)}$. In the following, '$\sim$' stands for 'transforms according to' and $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are the identity, signature and two-dimensional (here, $a$ is diagonal) irreducible representations of $S_3$. To give a simple illustration, when

$$\psi_L \sim \Gamma^{(3)} \text{ and } \psi_R^{\pm} \sim \Gamma^{(1)} \oplus \Gamma^{(2)}$$

the $S_3$-invariant Yukawa interaction is of form,

$$(\bar{\psi}_{L1}\;\bar{\psi}_{L2}) \left[ \begin{pmatrix} a^+ \Phi_1 & -\beta^+ \Phi_1 \\ a^+ \Phi_2 & \beta^+ \Phi_2 \end{pmatrix} \begin{pmatrix} \psi_{R1}^+ \\ \psi_{R2}^+ \end{pmatrix} + \begin{pmatrix} a^- \Phi_2^c & -\beta^- \Phi_2^c \\ a^- \Phi_1^c & \beta^- \Phi_1^c \end{pmatrix} \begin{pmatrix} \psi_{R1}^- \\ \psi_{R2}^- \end{pmatrix} \right] + \text{h.c.},$$

where $\alpha^{\pm}$, $\beta^{\pm}$ are the only nonvanishing Yukawa couplings. The resulting quark mass matrices involve $\langle \varphi_1^0 \rangle$ and $\langle \varphi_2^0 \rangle$. As a consequence of the constrained form for potential, the extremum conditions on it (*i.e.* $\partial V/\partial \Phi_i = \partial V/\partial \Phi_i^+ = 0, = 1, 2$) are such that for consistency they require $|\langle \varphi_1^0 \rangle| = |\langle \varphi_2^0 \rangle|$. Using this result, it is found that irrespective of how $\psi_L$, $\psi_R^{\pm}$ are chosen to transform under $S_3$ diagonalisation of $M_{\pm} M_{\pm}^+$ implies $\theta_c$ is zero, indeterminate or maximal. In the following three Higgs case, $\Phi_1$, $\Phi_2$ and $\Phi_3$ have definite transformation properties under $S_3$ with which invariance considerations of Higgs potential allow vev of only two of the three neutral fields to be independent.

2.1b *Partially successful attempt.* Let

$$\Phi_1 \sim \Gamma^{(2)} \text{ and } \begin{pmatrix} \Phi_2 \\ \Phi_3 \end{pmatrix} \sim \Gamma^{(3)}$$

As in the previous case of two Higgs, an invariant form for Higgs potential is constructed. The following form is invariant under the two-element subgroup $(e, b)$ of $S_3$.

$$V(\Phi_1 \Phi_2 \Phi_3) = V_{\rm I} (\Phi_1 \Phi_2 \Phi_3) + V_{\rm II} (\Phi_1 \Phi_2 \Phi_3)$$

$$\begin{aligned} V_{\rm I} &= a(\phi_1^+ \phi_1)^2 + b\,\phi_1^+ \phi_1 + c\,[(\phi_2^+ \phi_2)^2 + (\phi_3^+ \phi_3)^2] \\ &+ d(\phi_2^+ \phi_2)(\phi_3^+ \phi_3) + e(\phi_2^+ \phi_3)(\phi_3^+ \phi_2) + f\,[\phi_2^+ \phi_2 \\ &+ \phi_3^+ \phi_3] + g\,\phi_1^+ \phi_1 (\phi_2^+ \phi_2 + \phi_3^+ \phi_3) + h\,[(\phi_2^+ \phi_1)(\phi_1^+ \phi_2) \\ &+ (\phi_3^+ \phi_1)(\phi_1^+ \phi_3)] + i\,[(\phi_1^+ \phi_3)(\phi_2^+ \phi_3) - (\phi_1^+ \phi_2)(\phi_3^+ \phi_2)] + \text{h.c.} \end{aligned}$$

$$\begin{aligned} V_{\rm II} &= j\,[\phi_2^+ \phi_3 + \phi_3^+ \phi_2] + k\,\phi_1^+ (\phi_2 - \phi_3) + l\,\phi_1^+ \phi_1 [\phi_2^+ \phi_3 + \phi_3^+ \phi_2] \\ &+ m\,\phi_1^+ \phi_1 (\phi_2^+ \phi_2 - \phi_1^+ \phi_3) + n\,[(\phi_1^+ \phi_2)(\phi_3^+ \phi_3) - (\phi_1^+ \phi_3)(\phi_2^+ \phi_2)] \\ &+ o\,[(\phi_1^+ \phi_3)(\phi_3^+ \phi_3) - (\phi_1^+ \phi_2)(\phi_2^+ \phi_2)] + p\,[(\phi_1^+ \phi_3)(\phi_3^+ \phi_2) \\ &- (\phi_1^+ \phi_2)(\phi_2^+ \phi_3)] + q\,[\phi_2^+ \phi_3]\,[\phi_2^+ \phi_2 + \phi_3^+ \phi_3] + \text{h.c.} \end{aligned} \tag{9}$$

The potential thus written has two parts. $V_{\rm I}$ is an arbitrary superposition of bilinear and biquadratic terms each invariant under $S_3$ and $V_{\rm II}$ that of such terms as are invariant only under its $\{e, b\}$ subgroup. Arguing as in the previous case, the extremum conditions

$$\frac{\partial V}{\partial \phi_2} = \frac{\partial V}{\partial \phi_3} = 0 \left( \frac{\partial V}{\partial \phi_2^+} = \frac{\partial V}{\partial \phi_3^+} = 0 \right)$$

are consistent if and only if $\langle\varphi_2^0\rangle = -\langle\varphi_3^0\rangle$. Thus the above constraints force vev of $\varphi_2^0$ and $\varphi_3^0$ to differ only by a phase difference of $\pi$. In contrast, vev developed by $\varphi_1^0$ is not related to that of $\varphi_2^0$. If $V_{\text{II}}$ is omitted (*i.e.* potential is $S_3$-invariant) and Yukawa interaction is required to transform as identity representation of $S_3$, it is found impossible to relate Cabibbo angle to quark mass-ratios, irrespective of the choice for $\psi_L$, $\psi_R^{\pm}$ representations. The line of reasoning leading to this conclusion is essentially that which was outlined in the previous section. With the inclusion of $V_{\text{II}}$ (this maintains the constraint $\langle\varphi_2^0\rangle = -\langle\varphi_3^0\rangle$ that proves crucial), it is now discussed that the following choice for Yukawa interaction survives the test of invariance under $\{e, b\}$ subgroup of $S_3$ with a suitable choice for representation content of $\psi_L$, $\psi_R^{\pm}$.

$$\mathscr{L}^{\text{Yuk}} = \overline{(\psi_{L1}\,\psi_{L2})}\left[\begin{pmatrix}\alpha\,\Phi_2 & \beta\,\Phi_3\\ \alpha\,\Phi_1 & \beta\,\Phi_1\end{pmatrix}\begin{pmatrix}\psi_{R1}\\ \psi_{R2}\end{pmatrix}+\right.$$

$$\left.+\begin{pmatrix}\gamma\Phi_2^c & \gamma\Phi_3^c\\ \gamma\Phi_1^c & \gamma\Phi_1^c\end{pmatrix}\begin{pmatrix}\psi_{R1}^-\\ \psi_{R2}^-\end{pmatrix}\right] + \text{h.c.} \tag{10}$$

Besides its yielding readily the result $\theta_c \sim m_d/m_s$, the following invariance argument for its justification leaves much to be desired. Let,

$$\psi_{L1} \sim \Gamma^{(1)}; \qquad \psi_{L2} \sim \Gamma^{(2)},$$

$$\begin{pmatrix}\psi_{R1}^{\pm}\\ \psi_{R2}^{\pm}\end{pmatrix} \sim \Gamma^{(3)},$$

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix} \sim \Gamma^{(3)}\ ;\ \gamma \sim \Gamma^{(1)}. \tag{11}$$

Then the LHR type blocks in (10) have definite transformation properties under $S_3$ as given by

$$\begin{pmatrix}\alpha\,\psi_{L1}\,\Phi_2\,\psi_{R1}^+\\ \beta\,\psi_{L1}\,\Phi_3\,\psi_{R2}^+\end{pmatrix} \sim \Gamma^{(3)},$$

$$\alpha\,\bar{\psi}_{L2}\,\Phi_1\,\psi_{R1}^+ + \beta\,\bar{\psi}_{L2}^+\,\Phi_1\,\psi_{R2}^+ \sim \Gamma^{(1)},$$

$$\gamma\,\bar{\psi}_{L1}\,(\Phi_2^c\,\psi_{R1}^- + \Phi_3^c\,\Psi_{R2}^-) \sim \Gamma^{(1)},$$

$$\begin{pmatrix}\gamma\,\bar{\psi}_{L2}\,\Phi_1^c\,\psi_{R1}^-\\ \gamma\,\bar{\psi}_{L2}\,\Phi_1^c\,\psi_{R2}^-\end{pmatrix} \sim \Gamma^{(3)}. \tag{12}$$

Equation (10) being direct sum of these blocks is thus invariant under $\{e, b\}$ subgroup of $S_3$. As seen in (11), $\alpha$ and $\beta$ transform nontrivially under $S_3$.

However, we cannot say what the implications of this unavoidable choice are. Replacing neutral fields by their vev, $\langle\varphi_2^\circ\rangle = -\langle\varphi_3^\circ\rangle \equiv x_1$ and $\langle\varphi_1^\circ\rangle \equiv x_2$ the result for $M_\pm M_\pm^+$ is,

$$M_+M_+^+ = \begin{pmatrix} \lambda_1^+ |x_1|^2 & \lambda_2 x_1 x_2^* \\ \lambda_2 x_2^* x_1 & \lambda_1^+ |x_2|^2 \end{pmatrix}, \quad M_-M_-^+ = \lambda_1^- \begin{pmatrix} |x_1|^2 & O \\ O & |x_2|^2 \end{pmatrix} \tag{13}$$

where $\lambda_1^+ = |\alpha|^2 + |\beta|^2$, $\lambda_2 = |\alpha|^2 - |\beta|^2$ and $\lambda_1^- = |\gamma|^2$. Cabibbo mixing arises solely from mixing in positively charged sector (obtained by diagonalising $M_+M_+^+$). In terms of quark masses,

$$\lambda_2/\lambda_1^+ = \left[1 - \left(\frac{m_u m_c}{m_d m_s} \cdot \frac{m_d^2 + m_s^2}{m_u^2 + m_c^2}\right)^2\right]^{1/2}$$

$$\tfrac{1}{2}\tan 2\theta_c = \frac{\lambda_2}{\lambda_1^+} \cdot \frac{m_d m_s}{m_s^2 - m_d^2} \tag{14}$$

clearly $\lambda_2/\lambda_1^+ \approx 1$ and $\theta_c \sim m_d/m_s$. Consistent with this result for $\theta_c$, a small value of $m_u$ is allowed. This completes the discussion of realisation of the $2 \times 2$ form given by (13) in the standard model by imposing suitable discrete symmetries.

## 2.2 The $O(5) \times U(1)$ *Form*

The motivation for this scheme has been given elsewhere. The following salient features of this scheme may be recalled here for the present discussion. The ordinary left-handed fermionic pairs $(U, D)_L$, $(C, S)_L$, $(ve, e)_L$ and $(\nu_\mu\ \mu^-)_L$ are placed in four quartets partnered by the pairs $(U'_{2/3}, D'_{-1/3})_L$ $(C'_{2/3}\ S'_{-1/3})_L$ $(\nu'_e\ e')_L$ and $(\nu'_\mu\ \mu')_L$, respectively of superheavy fermions (their charges are labelled by the subscripts). The left-handed charged-current interactions between ordinary fermions are mediated by a single $W^\pm$ associated with generator(s) of an SU(2) subgroup of O(5) under which the above ordinary (superheavy) fermions transform as doublets (singlets). Their corresponding right-handed components are also arranged in four quartets by interchanging their O(5) assignments in going from the left-handed to the corresponding right-handed quartet of identical weak hypercharge assignment *e.g.* for the right-handed quartet of

$$U_R\ D_R\ (U'_{2/3}\ D'_{-1/3})_R$$

the O(5) assignments tally with

$$(U'_{+2/3}\ D'_{-1/3})_L\ (U,\ D)_L$$

respectively of the corresponding left-handed quartet with identical U(1) assignment. The fermions acquire arbitrary masses by Yukawa coupling with $\underline{10}+\underline{5}$ dimensional Higgs multiplet with null weak hypercharge assignment. An $R\overline{U}$ type of discrete symmetry forbids weak mixing between ordinary and superheavy fermions of equal charge. Consequently the $8\times 8$ quark mass matrix is a direct sum of four $2\times 2$ matrices (two for positively and two for negatively charged sector). Furthermore, in this case the gauge-boson interaction with fermions is invariant under interchange of the corresponding left-handed and right-handed quartets provided this is accompanied by an appropriate relabelling of the gauge boson indices. In view of the absence of light-superheavy mixing and this left-right symmetry, the realisation of the following $2\times 2$ form in this case reduces effectively to the derivation of that form in the $SU_L(2)\times SU_R(2)\times U(1)$ scheme considered by Fritzsch (1977).

$$M_{\pm} = \begin{pmatrix} 0 & \beta_1^{\pm} \\ \beta_1^{\pm} & \beta_2^{\pm} \end{pmatrix}. \tag{15}$$

Here the nonvanishing elements are unique only up to multiplication by an arbitrary phase factor because all these phases are absorbable by redefinition of the phases of basis states. $\beta_1^{\pm}$ and $\beta_2^{\pm}$ are chosen real and the latter in addition to be positive definite, $\beta_2^{\pm} > 0$. This symmetric (but negative, det $M_{\pm} < 0$) form yields the well-known result,

$$\theta_c = \tan^{-1} (m_d/m_s)^{1/2} - \tan^{-1} (m_u/m_c)^{1/2} \tag{16}$$

Here since $M_+$ and $M_-$ have identical structure, the functional dependence of $\theta_-$ (*i.e.* mixing in negatively charged sector) on $m_d/m_s$ is identical with that of $\theta_+$ on $m_u/m_c$, Cabibbo angle being their difference. $\theta_c \sim (m_d/m_s)^{1/2} - (m_u/m_c)^{1/2}$. This outlines how the $2\times 2$ form given by (15) may be realised in the $O(5)\times U(1)$ scheme by imposing suitable discrete symmetries, one of them being the left-right symmetry peculiar to this scheme.

## 3. Generalisation of the 2×2 forms

Generalisation of the $2\times 2$ forms is discussed (one in § 3.1 and the other in § 3.2) by constructing approximate solutions to the exact equations derived in Appendix B for the $n\times n$ case.

### 3.1 *The n×n form of first kind*

When those symmetric $n\times n$ forms are examined which all correctly reduce on putting $n=2$ to the symmetric $2\times 2$ form given by (15), the following form referred to as the $n\times n$ form of first kind is found such a candidate. In addition to this form there are a couple of other symmetric forms that also satisfy the same above requirement. One of them may be obtained from the form of first kind by grouping all its off-diagonal entries, $\beta_1^{\pm}, \beta_2, \ldots, \beta_{n-1}^{\pm}$ into the corresponding positions of its first row (column) (*i.e.* 12 (21) element is $\beta_1^{\pm}$, 13 (31) element is $\beta_2^{\pm}$, etc.) and the other obtains by similarly grouping these $\beta$s in its last row (column). Going to the next higher case of $n = 3$, however, makes it manifest on diagonalisation that these two alternate forms are unsatisfactory: the former is inconsistent with small angle approximation and the latter constrains one of the quark masses in either sector to vanish. For this reason only the form of first kind is discussed at length which is given by,

$$M_{\pm} = A_n(\beta_1^{\pm}, \beta_2^{\pm}, \ldots, \beta_n^{\pm}),$$

$$A_n(\beta_1\beta_2\ldots\beta_n) = \begin{pmatrix} 0 & \beta_1 & 0 & 0 & & 0 & 0 \\ \beta_1 & 0 & \beta_2 & 0 & & 0 & 0 \\ 0 & \beta_2 & 0 & \beta_3 & & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & 0 & & 0 & \beta_{n-1} \\ 0 & 0 & 0 & 0 & & \beta_{n-1} & \beta_n \end{pmatrix} \tag{17}$$

Phases of the nonvanishing entries are absorbable by a redefinition of the phases of basis vectors. For this reason, all $\beta$s may be chosen real and, for reasons explained later, $\beta_n$ is also chosen positive definite, $\beta_n > 0$

3.1*a Identification of fermion masses*

Characteristic polynomial for $A_n$ *is*

$$\det(A_n - \lambda E_n) = \sum_{i=0}^{n} a_i(-\lambda)^{n-i},$$

$$a_0 = 1,$$

$$a_1 = \beta_n,$$

$$a_2 = -\beta_{n-1}^2 - \beta_{n-2}^2 - \beta_{n-3}^2 - \ldots - \beta_1^2,$$

$$a_3 = (-\beta_n)(-\beta_{n-1}^2) + \beta_n a_2,$$

$$a_4 = (-\beta_{n-1}^2)(-\beta_{n-3}^2 - \beta_{n-4}^2 - \ldots - \beta_1^2) + (-\beta_{n-2}^2)(-\beta_{n-4}^2 - \beta_{n-5}^2 - \ldots - \beta_1^2) + \ldots + (-\beta_3^2)(-\beta_1^2),$$

$$a_5 = -\beta_n(-\beta_{n-1}^2)(-\beta_{n-3}^2 - \beta_{n-4}^2 - \ldots - \beta_1^2) + \beta_n a_4$$

$$\vdots$$

$$a_n = \begin{cases} (-\beta_{n-1}^2)(-\beta_{n-3}^2)\ldots(-\beta_3^2)(-\beta_1^2) & n \text{ even} \\ \beta_n \quad (-\beta_{n-2}^2)\ldots(-\beta_3^2)(-\beta_1^2) & n \text{ odd.} \end{cases} \tag{18}$$

Here $E_n$ denotes identity matrix of dimension $n$. This may be verified by induction on using the identity,

$$\det(A_n(\beta_1, \beta_2, \ldots, \beta_n) - \lambda E_n) = (-\lambda)\det(A_{n-1}(\beta_2, \beta_3, \ldots, \beta_n) - \lambda E_{n-1}) + (-\beta_1^2)\det(A_{n-2}(\beta_2, \beta_3, \ldots, \beta_n) - \lambda E_{n-2}). \tag{19}$$

Let the eigenvalues of symmetric form denoted by $\lambda_1, \lambda_2 \ldots \lambda_n$ be ordered as

$$|\lambda_n| > |\lambda_{n-1}| > |\lambda_{n-2}| > \ldots > |\lambda_2| > |\lambda_1|.$$

It may also be verified inductively that if

$$(\text{for } | 1 \leqslant i \leqslant n)\ \lambda_i = (-)^{n-i}\, m_i$$

where $m_i > 0$, then the signs of coefficients of the different powers of $\lambda$ in the expansion of $\prod_{i=1}^{n} (\lambda - \lambda_i)$ match correctly the signs of coefficients of corresponding powers of $\lambda$ in equation (18), remembering that $\beta_n$ is chosen positive. Thus the characteristic equation for $A_n$ may be taken as,

$$\det(A_n - \lambda E_n) = \prod_{i=1}^{n} [(-)^{n-i}\, m_i - \lambda]\ \ (m_i > 0,\ \ 1 \leqslant i \leqslant n) \tag{20}$$

This equation also shows that each $\beta_i$ is uniquely expressible (apart from sign ambiguity that is inconsequential) in terms of the $n$ positive definite quantities $m_i$ introduced above, *e.g.*,

$$\beta_n = a_1,\ \ \beta_{n-1}^{2} = \frac{a_3 - a_2\, a_1}{a_1},\ \text{etc.}$$

where $a_1$ is coefficient of $(-\lambda)^{n-i}$ on the right-hand side of this equation. Clearly $m_n > m_{n-1} > m_{n-2} > \ldots > m_2 > m_1$. When the symmetric matrix $A_n$ is diagonalised by an orthogonal matrix $V$, *i.e.* $V^{-1} A_n V$ is a diagonal, the diagonal elements are $\lambda_1, \lambda_2, \ldots, \lambda_n$ (the order in which they are arranged on the diagonal becomes clear later when an explicit form for $V$ is given). However, if the orthogonal transformation acting on right of $A$ is taken $V_R = V_K$ where $K$ is a diagonal matrix such that $K^2 = E$, then $K$ may be chosen such that diagonal elements of $V^{-1} A_n V_R$ are $m_1, m_2, \ldots, m_n$. This is nothing but biunitary diagonalisation of $A_n$ to a positive definite diagonal form. For this reason, $m_1^{\pm}, m_2^{\pm}, \ldots, m_n^{\pm}$ are identified with fermion masses in the picture when $M_{\pm}$ describes mixing of $n$ positively charged ($n$ negatively charged) fermions among themselves.

3.1*b* *Small angle approximation.* This paragraph is a discussion of what is referred to as small angle approximation under which the following results for $A_n$ hold, for $1 \leqslant i \leqslant n-1$,

(a) $|\theta_i| \sim |\beta_i/\beta_{i+1}|$,

(b) $\beta_n \sim m_n$, $|\beta i| \sim (m_i\, m_{i+1})^{1/2}$,

(c) $|\theta_i| \sim (m_i/m_{i+1})^{1/2}$, (21)

where $\theta_1, \theta_2, \ldots, \theta_{n-1}$ are the only angles of the diagonalising orthogonal matrix that are nonvanishing to leading order of this approximation.

The diagonalising orthogonal matrix $V$ is the product $V_1 V_2 V_3, \ldots, V_{n-1}$ of orthogonal matrices where the orthogonal matrix $V_{n-1}$ is parametrised by $n-1$ angles denoted by $\theta_1, \theta_2, \ldots, \theta_{n-1}$, $V_{n-2}$ by $n-2$ angles denoted by $\theta_n, \theta_{n+1}, \theta_{n+2}, \ldots, \theta_{2n-3}$, etc. Under small angle approximation, it will become clear that each of $V_{n-2}$, $V_{n-3} \ldots, V_3, V_2, V_1$ reduces simply to the identity matrix. The first $n-1$ angles are in turn related to elements of the symmetric matrix being diagonalised through $n-1$ coupled equations of the form,

$$\tfrac{1}{2} \tan 2\theta_1 = N_1/D_1,$$

$$\tan \theta_1 = \frac{\partial N_1/\partial \theta_i}{\partial D_1/\partial \theta_i} \quad 2 \leqslant i \leqslant n-1 \tag{22}$$

The details may be found in appendices A and B.

To motivate small angle approximation, let $\beta_1 \equiv 0$. This also constraints $m_1$ and $\theta_1$ to vanish. In this special case, the following obvious looking recursion relations may be verified inductively using equations (22) and (20). (i) $\theta_{n-1}, \theta_{n-2}, \ldots, \theta_3, \theta_2$ associated with diagonalisation of the $n \times n$ form $A_n$ $(\beta_1 \equiv 0, \beta_2, \beta_3, \ldots, \beta_n)$ are given respectively by $\theta_{n-2}, \theta_{n-3}, \ldots\ldots, \theta_1$ of the $n-1 \times n-1$ form $A_{n-1}$ $(\beta_2, \beta_3, \ldots, \beta_n)$. (ii) The expressions for $\beta_2, \beta_3, \ldots, \beta_n$ of the $n \times n$ form $A_n(\beta_1 \equiv 0, \beta_2, \ldots, \beta_n)$ in terms of $(m_1 \equiv 0)$, $m_2\, m_3 \ldots m_n$ are identical with corresponding expressions for $\beta_1, \beta_2, \ldots \beta_{n-1}$ of $n-1 \times n-1$ form $A_{n-1}(\beta_1, \beta_2, \ldots\ldots, \beta_{n-1})$ in terms of $m$'s with $m_i$ replaced by $m_{i+1}$ *i.e.* $m_1$ is replaced by $m_2$, $m_2$ by $m_3$, etc. In the case when $\beta_1$ does not vanish identically, but $|\beta_1| \ll |\beta_i|$ for $2 \leqslant i \leqslant n$ these recursion relations hold approximately and the small value of $\theta_1$ $(\beta_1)$ in this case may be obtained by substituting the values of $\theta_2\ \theta_3 \ldots \theta_{n-1}$ $(\beta_2 \beta_3 \ldots \beta_{n-1})$ into the first (last) equation of the set of equations (22) and (18). Next beginning with $n=2$ and following the inductive route, the recursions relations, equations (22) and (18) may be used repeatedly to establish (a), (b), and (c) under $|\beta_1| \ll |\beta_2| \ll |\beta_3| \ll \ldots \ll \beta_n$ or, equivalently, $m_1 \ll m_2 \ll m_3 \ll \ldots \ll m_n$. With $\theta$'s thus related to $m$'s through (c), $V_n$ diagonalises the symmetric matrix and it is seen that the subscript $i$ (on $m$ or $\lambda$) corresponds to $ii$ th position on diagonal. This means that all angles except the first $n-1$ angles are negligible.

The result (c) holds for $M_\pm$ under fermion mass hierarchy

$$m_n^\pm \gg m_{n-1}^\pm \gg m_{n-2}^\pm \ldots \gg m_1^\pm.$$

The $n-1$ Cabibbo like angles which appear in the charged current matrix $U^{cc}$ (equation (6)) are given by the differences

$$\theta_i = \theta_i^- - \theta_i^+, \; |\theta_i^\pm| \sim (m_i^\pm / m_{i+1}^\pm)^{1/2}$$

There is ambiguity in the sign of $\theta_i^\pm$. If they are all assumed positive

$$\theta_i \sim (m_i^- / m_{i+1}^-)^{1/2} - (m_i^+ / m_{i+1}^+)^{1/2}.$$

This result suggests that the higher dimensionality forms given by equation (17) may be regarded as members of a single family that includes the $2 \times 2$ form given by equation (15) as its first member.

### 3.2 $n\times n$ *form of second kind*

For this form,

$$M_- M_-^+ = \lambda_- \operatorname{diag}\,(|x_1|^2, |x_2|^2, \ldots, |x_n|^2)$$

$$M_+ M_+^+ = \begin{bmatrix} \lambda_1|x_1|^2 & \lambda_2 x_1^* x_2 & \lambda_2 x_1^* x_3 & \ldots & \lambda_2 x_1^* x_n \\ \lambda_2 x_2^* x_1 & \lambda_1|x_2|^2 & \lambda_3 x_2^* x_3 & \ldots & \lambda_3 x_2^* x_n \\ \lambda_2 x_3^* x_1 & \lambda_3 x_3^* x_2 & \lambda_1|x_3|^2 & \ldots & \lambda_4 x_3^* x_n \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ \lambda_2 x_{n-1}^* x_1 & \lambda_3 x_{n-1}^* x_2 & \lambda_4 x_{n-1}^* x_3 & \ldots & \lambda_n x_{n-1}^* x_n \\ \lambda_2 x_n^* x_1 & \lambda_3 x_{n-1}^* x_2 & \lambda_4 x_n^* x_3 & \ldots & \lambda_1|x_n|^2 \end{bmatrix} \tag{23}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_n$ are $n$ real parameters and $x_1, x_2, \ldots, x_n$ are $n$ complex parameters. From equation (5), eigenvalues of $M_\pm M_\pm^+$ are $(m_1^\pm)^2$, $(m_2^\pm)^2$ $(m_n^\pm)^2$. These $2n$ $(\text{mass})^2$ terms involve the above $2n$ independent parameters. To us equation (23) seems the only obvious candidate for an $n\times n$ form of $2n$ parameters that correctly reduces to the $2\times 2$ form given by equation (13) on taking $n=2$. This form is diagonalised under the following small angle approximation. $M_+M_+^+$ reduces to a real symmetric matrix because the arrangement of $x$'s is such that their phases are absorbable by a mere redefinition of phases of the bases vectors. Hence the results given in appendix B may be used. The orthogonal matrix $V^{(+)}$ diagonalizing $M_+M_+^+$ is the product

$$V^{(+)} = V_1^{(+)} V_2^{(+)} \ldots V_{n-2}^{(+)} V_{n-1}^{(+)}$$

of orthogonal matrices. $V_{n-1}^{(+)}$ involves the first $n-1$ angles $\theta_1^+$, $\theta_2^+, \ldots, \theta_{n-1}^+$ and $V_{n-2}^{(+)}$ the next $n-2$ angles

$$\theta_n^+,\ \theta_{n+1}^+,\ \ldots,\ \theta_{2n-3}^+$$

and so on. It is straightforward to show the following exact results in the special case when $\lambda_i = \lambda_1$ *i.e.* We simply have $(M_+M_+^+)_{ij} = \lambda_i(x_i x_j)$

(a) of the $n$ masses $m_i^+$ all except $m_2^+$ vanish.

(b) All except the first $n-1$ angles $\theta_1^+$, $\theta_2^+$, …, $\theta_{n-1}^+$ vanish. For these angles,

$$-s_1 = \mu x_1,$$

$$C_1 C_2 = \mu x_2,$$

$$C_1 s_2 C_3 = \mu x_3,$$

$$C_1 s_2 s_3 \ldots s_{n-3} C_{n-2} = \mu x_{n-2},$$

$$C_1 s_2 s_3 \ldots s_{n-2} C_{n-1} = \mu x_{n-1},$$

$$C_1 s_2 s_3 \ldots s_{n-2} s_{n-1} = \mu x_n, \qquad (24)$$

where $\mu$ is an undetermined parameter, $C_i \equiv \cos\theta_i$ and $s_i \equiv \sin\theta_i$. This exact result suggests the following approximation. Let $\lambda_i \approx \lambda_1$ $(2 \leqslant i \leqslant n)$. This means that $m_2^+$ is the dominant mass in positively charged sector and the set of expressions (24) hold to a good approximation. Furthermore, if $m_1^- \ll m_2^-$ and $m_2^- \gg m_3^- \gg m_4^- \gg \ldots \gg m_n^-$ (or equivalently $|x_1| \ll |x_2|$, $|x_i| \gg |x_{i+1}|$ for $2 \leqslant i \leqslant n-1$), the angles $\theta_1^+, \theta_2^+, \ldots, \theta_{n-1}^+$ are given by,

$$-\theta_1^+ \sim x_1/x_2,$$

$$\theta_i^+ \sim x_{i+1}/x_i \quad 2 \leqslant i \leqslant n-1. \qquad (25)$$

The $n-1$ Cabibbo-like angles that appear in $U^{cc}$ are simply $\theta_i = -\theta_i^+$ because the negatively charged sector is already diagonal. There is ambiguity in the sign of $\theta_i^+$. Assuming that all the mixing angles have the same sign as $\theta_1^+$ (which is taken positive), it follows that to a leading order in this small angle approximation, the $n-1$ Cabibbo like angles are related to mass ratios as

$$\theta_1 \sim m_1^-/m_2^-, \quad \theta_i \sim m_{i+1}^-/m_i^- \ \ 2 \leqslant i \leqslant n-1.$$

This observation makes it plausible that the higher-dimensionality forms in equation (23) form a single family of whose simplest member is the $2 \times 2$ form given by equation (13). Note that the fermion mass hierarchy underlying the small angle approximation in this case is entirely different from that involved in § 3.1.

## 4. Conclusions

The above discussion has many loose ends. For each of a pair of $2 \times 2$ forms, it was discussed how this might be realised in a certain gauge scheme by imposing suitable discrete symmetries and how in a sense this may be regarded as the smallest member of a family of higher dimensionality forms. Whereas realisation of the specific forms adds nothing new to the spirit of original derivations found in the literature, another obvious limitation of the present work is that no physical significance has been attached to any higher dimensionality form in the face of the growing evidence for more than four quarks.

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**Appendix A**

The following is an explicit parametrization of an orthogonal $n \times n$ matrix in terms of angles $\theta_i$, $1 \leqslant i \leqslant n(n-1)/2$. This is followed by the same for an $n \times n$ unitary matrix in terms of $\frac{1}{2}n(n-1)$ angles and $\frac{1}{2}(n-1)(n-2)$ non-absorbable phases. The following $m \times m$ matrix is defined,

$$U_m(\theta_1, \theta_2, \ldots, \theta_{m-1}) = \begin{pmatrix} c_1 & s_1c_2 & s_1s_2c_3 & s_1s_2s_3c_4 & \cdots & s_1s_2\ldots s_{m-2}c_{m-1} & s_1s_2\ldots s_{m-1} \\ -s_1 & c_1c_2 & c_1s_2c_3 & c_1s_2s_3c_4 & \cdots & c_1s_2s_3\ldots s_{m-2}c_{m-1} & c_1s_2\ldots s_{m-1} \\ 0 & -s_2 & c_2c_3 & c_2s_3c_4 & \cdots & c_2s_3\ldots s_{m-2}c_{m-1} & c_2s_3\ldots s_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & c_{m-2}c_{m-1} & c_{m-2}s_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & -s_{m-1} & c_{m-1} \end{pmatrix} \tag{A1}$$

where $c_i \equiv \cos\theta_i$ and $s_i \equiv \sin\theta_i$. The above ordering of $c$ and $s$ makes this an orthogonal matrix. However, this is not the most general $m \times m$ orthogonal form. The most general $n \times n$ orthogonal form $V$ involving $\frac{1}{2}n(n-1)$ angles is the product $V_1 V_2 \ldots V_{n-1}$ of $n-1$ orthogonal matrices each of dimensionality $n$ and given by,

$$\begin{aligned} V_{n-1} &= U_n(\theta_1, \theta_2, \ldots, \theta_{n-1}), \\ V_{n-2} &= E_1 \oplus U_{n-1}(\theta_n, \theta_{n+1}, \ldots \theta_{2n-3}), \\ V_{n-3} &= E_2 \oplus U_{n-2}(\theta_{2n-2}, \theta_{2n-1}, \ldots, \theta_{3n-6}) \\ &\vdots \\ V_2 &= E_{n-3} \oplus U_3(\theta_{M-2}, \theta_{M-1}), \\ V_1 &= E_{n-2} \oplus U_2(\theta_M), \end{aligned} \tag{A2}$$

where $E_K$ is identity matrix of dimension $K$ and

$$M = \tfrac{1}{2}n(n-1) = \sum_{m=2}^{n} (m-1).$$

Thus $V_{n-1}$ involves the first $n-1$ angles, $V_{n-2}$ the next $n-2$ angles and so on, $V_1$ involving the last angles. Equivalently, equation (A2) may also be interpreted by the following action on $n$ vectors $e_1, e_2, \ldots, e_n$.

$$e'_{i'} = \sum_{j=1}^{n} (U_n)_{i'j}\, e_j$$

$$e''_1 = e'_1;\quad e''_{i''} = \sum_{j'=2}^{n} (U_{n-1})_{i''j'}\, e'_{j'} \quad 2 \leq i'' \leq n$$

$$e'''_{1,2} = e''_{1,2};\quad e'''_{i'''} = \sum_{j''=3}^{n} (U_{n-2})_{i'''j''}\, e''_{j''} \quad 3 \leq i''' \leqslant n$$

$$\vdots$$

$$\begin{aligned} e^{(n-1)}_{1,2,\ldots,n-2} &= e^{(n-2)}_{1,2,\ldots n-2};\ e^{(n-1)}_{n-1} = c_M\, e^{(n-2)}_{n-1} + s_M\, e^{(n-2)}_n, \\ e^{(n-1)}_n &= -s_M\, e^{(n-2)}_{n-1} + c_M\, e^{(n-2)}_n \end{aligned} \tag{A3}$$

To introduce explicitly the $\frac{1}{2}n(n-1)(n-2)$ non-absorbable phases in an $n \times n$ unitary form, define the $n$-dimensional matrix,

$$\widetilde{U}_m(\theta_1, \theta_2, \ldots, \theta_{m-1}, \eta_1, \eta_2, \ldots \eta_{m-1}),$$

which obtains from $\tilde{U}_m$ on multiplying each element of its $i$th column by $\exp(i\eta_{i-1})$, $2 \le i \le m$. For this reason $\tilde{U}_m$ is unitary. The phases thus introduced are (the only) non-absorbable possible, the vectors on which $\tilde{U}_m$ acts have all their phases predetermined. The $n \times n$ unitary form $\tilde{V}$ is the product $\tilde{V}_1 \tilde{V}_2 \ldots \tilde{V}_{n-1}$ of $n$-dimensional unitary matrices: $\tilde{V}_{n-1} = \tilde{U}_n$ involving the first $n-1$ angles, $\tilde{V}_{n-2} = E_1 \oplus \tilde{U}_{n-1}$ involving the next $n-2$ angles and the first $n-2$ phases, $\tilde{V}_{n-3} = E_2 \oplus \tilde{U}_{n-2}$ involving the next $n-3$ angles and the next $n-3$ phases and so on until $V_1 = E_{n-2} \oplus \tilde{U}_2$ involves the last angle and last phase. The preceding remarks make it apparent that the phases thus introduced (number $\sum_2^{n-1} m-1 = \frac{1}{2}(n-1)(n-2)$ ) are non-absorbable.

**Appendix B**

In the following the diagonalization equations for a symmetric matrix $H$ to be diagonalised by an orthogonal matrix $V$ are developed. For this purpose, the $n \times n$ orthogonal form parametrised in appendix A is written as

$$\begin{pmatrix} {}^{(n-1)}e_1^T \\ {}^{(n-1)}e_2^T \\ \vdots \\ {}^{(n-1)}e_n^T \end{pmatrix}$$

where $T$ (for transpose) indicated that the entry is a row-vector ($n$ components) and row vectors in $n-1$th set are related to those of $n-2$th set as,

$${}^{(n-1)}e^T_{1,2,\ldots n-2} = {}^{(n-2)}e^T_{1,2,\ldots n-2};\quad {}^{(n-1)}e^T_{n-1} = C_M\,{}^{(n-2)}e^T_{n-1} + s_M\,{}^{(n-2)}e^T_n;$$

$${}^{(n-1)}e^T_n = -s_M\,{}^{(n-2)}e^T_{n-1} + C_M\,{}^{(n-2)}e^T_n \tag{B1}$$

Similarly, vectors in $i$th set are related to those in $i-1$th set as,

$${}^{(i)}e^T_{1,2,\ldots,i-1} = {}^{(i-1)}e^T_{1,2,\ldots,i-1}\ (i \neq 1)$$

$${}^{(i)}e^T_k = (U_{n-i+1})_{kl}\,{}^{(i-1)}e^T_l\ \ 1 \le i \le k,\ l \le n \tag{B2}$$

where $\quad {}^{(0)}e_1^T = 1, 0, 0, \ldots 0),\ {}^{(0)}e_2^T = (0, 1, 0, \ldots, 0)$ etc.

$$(V H V^{-1})_{ij} = {}^{(n-1)}e_i^T H\,{}^{(n-1)}e_j \tag{B3}$$

Diagonalisation of $H$ by $V$ means that off-diagonal elements of (B3) must vanish. Using equation (B1) this amounts to,

$${}^{(n-2)}e_i^T H\,{}^{(n-2)}e_j = 0 \quad 1 \leqslant i \leqslant n-2,\ i+1 \leqslant j \leqslant n,$$

$$\tan 2\theta_M = -\frac{{}^{(n-2)}e^T_{n-1} H\,{}^{(n-2)}e_n}{-\,{}^{(n-2)}e^T_{n-1} H\,{}^{(n-2)}e_{n-1} + {}^{(n-2)}e_n^T H\,{}^{(n-2)}e_n} \tag{B4}$$

Using equation (B3) ($i=n-2$) and equation (B4) it also follows that,

$${}^{(n-3)}e_i^T H\,{}^{(n-3)}e_j = 0 \quad 1 \leqslant i \leqslant n-3,\ i+1 \leqslant j \leqslant n,$$

$$\tan 2\theta_{M-2} = N_{n-2}/D_{n-2},$$

$$\tan \theta_{M-2} = \frac{\partial N_{n-2}/\partial\theta_{M-1}}{\partial D_{n-2}/\partial\theta_{M-1}},$$

$$-N_{n-2} \equiv {}^{(n-3)}e_{n-2}^{T}\, H\, y_2,$$

$$2\, D_{n-2} = -e_{n-2}^{T}\, H\, e_{n-2} + y_2^{T}\, H\, y_2,$$

$$y_2 \equiv c_{M-1}\, {}^{(n-3)}e_{n-1}^{T} + s_{M-1}\, {}^{(n-3)}e_{n}^{T}. \tag{B5}$$

This procedure may be continued iteratively until for the first $n-1$ angles the expressions are,

$$\tan 2\theta_1 = N_1/D_1,$$

$$\tan \theta_1 = \frac{\partial N_1/\partial \theta_i}{\partial D_1/\partial \theta_i}\; 2 \leqslant i \leqslant n-1,$$

$$-N_1 \equiv {}^{(0)}e_1^{T}\, H\, y_{n-1},$$

$$2D_1 \equiv -{}^{(0)}e_1^{T}\, H\, {}^{(0)}e_1 + y_{n-1}^{T}\, H\, y_{n-1},$$

$$y_{n-1} \equiv c_2\, {}^{(0)}e_2 + s_2 c_3\, {}^{(0)}e_3 + s_2 s_3 c_4\, {}^{(0)}e_4 + \ldots + s_2 s_3 \ldots s_{n-2}\, c_{n-1}\, {}^{(0)}e_{n-1} + s_2 s_3 \ldots s_{n-1}\, {}^{(0)}e_n. \tag{B6}$$

Since ${}^{(0)}e_i^{T}\, H\, {}^{(0)}e_j$ is $ij$th element of $H$, these equations express the first $n-1$ angles in terms of the matrix elements of $H$. These equations proved useful in § 3.

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