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Massless particles and cosmology

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Abstract. Post recombination density perturbations have been studied in a two-component matter-radiation universe. The study has been carried out for variable matter and radiation densities. The possibility of the existence of different kinds of neutrinos in addition to the electronic, muonic and taonic has been considered and conclusions have been drawn as to the upper limit of the radiation density for different values of Ω_m.

Keywords. Cosmology; density perturbations; radiation; massless particles; neutrino.

1. Introduction

Recent observations show that our Universe is homogeneous for distances $\sim 10^{10}$ light years. The deviation from absolute homogeneity being 1%–0.1% for such distances. Similarly light reaching us in the form of microwave radiation from all parts of the universe supports its isotropic character. In the background of a homogeneous and isotropic Universe, we may consider the presence of galaxies and their clusters ($\sim 10^8$ light years) to be perturbations which grew from initially small values.

Perturbations in a homogeneous and isotropic Universe were first studied by Jeans (1902, 1929) and later by Lifshitz (1946) and Lifshitz and Khalatnikov (1963), who studied the same in an expanding Universe using the framework of general relativity. Perturbations in a uniformly expanding homogeneous and isotropic Universe were also studied by Bonnor (1957) in the framework of Newtonian physics. Although Jeans assumed in his analysis a static Universe, most of his basic results hold good for one that is uniformly expanding. The discovery of cosmic microwave radiation brought about a change in our understanding of the early universe. In an expanding Universe matter and radiation densities change differently with time

$$\rho_m \propto (a_0 / a)^3, \quad \rho_r \propto (a_0 / a)^4, \tag{1}$$

where a is the scale-factor.

Thus for a two component Universe there existed a period (during the very early stages of its expansion) when radiation predominated and the Universe could be considered to be made up of one basic component, radiation. As the Universe expanded it 'cooled of' and H, He, nuclei, which had previously existed along with electrons and other elementary particles in the form of a plasma, started recombining with the electrons giving rise to neutral H, He atoms. By 'recombination period' is meant the period during which the temperature of the Universe dropped to about 4000°K and the recombination process started occurring most rapidly for the basic matter component of the Universe, hydrogen (the helium having recombined earlier). In the aftermath of this period almost all the matter in the Universe became electrically neutral. The two periods (i) recombination, (ii) the period when matter and radiation densities assume roughly equal values, partially overlap, and for the sake of simplicity we can say that recombination marks the transition from a one-component radiation filled Universe to a two-component, matter-radiation one. It was shown by Peebles (1968) that the presence of a predominating radiation prevents the formation of gravitationally bound systems during the prerecombination period. After recombination, the temperature falls below 4000°C, matter becomes ' transparent ' for radiation and the picture changes radically. Matter perturbations which previously had been confined to an accoustic regime, become gravitationally unstable and radiation can no longer prevent the formation of gravitationally bound systems but can only delay the process.

It is the purpose of this work to further study long wave perturbations in a two-component matter-radiation Universe during the post recombination period. The radiation being further made up of three components (i) photons, (ii) neutrinos, (iii) gravitational waves. The study is carried out for a large spectrum of models of the Universe with widely varying matter and radiation densities. We suppose a variable radiation density in order to consider a wider break-up of neutrino types in addition to the three kinds whose existence is assumed today and also to consider the possibility of a large overequilibrium energy density for gravitational waves.

2. Equations for the unperturbed Universe

Let us consider a sphere of radius $a(t)$ filled with a homogeneous distribution of matter and radiation. Carrying out the study in the framework of Newtonian physics we get (McCrea and Milne 1934; Zeldovich 1963)

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3} a(\rho_m + \rho_r + 3p_r/c^2),\ \rho_r = \frac{\epsilon_r}{c^2},\ p_r = \frac{\epsilon_r}{3},$$
$$\rho_m = \epsilon_m/c^2 \tag{2}$$

Here ϵ_m and ϵ_r are the energies per unit volume of matter and radiation respectively and ρ_m and ρ_r are the respective densities p_r being the radiation pressure.

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3c^2} a\ (\epsilon_m + 2\ \epsilon_r),\ \epsilon_m = \epsilon_{om}\ (a_0/a)^3,\ \epsilon_r = \epsilon_{or}\ (a_0/a)^4 \tag{3}$$

$$d^2a/dt^2 = -\frac{4\pi G}{3c^2} a\ [\epsilon_{om}\ (a_0/a)^3 + 2\ \epsilon_{or}\ (a_0/a)^4]. \tag{4}$$

Integrating once

$$\tfrac{1}{2}\ (da/dt)^2 = \int \frac{d^2a}{dt^2}\, da = \frac{4\pi G}{3c^2} \epsilon_{om} \frac{a_0^3}{a} + \frac{4\pi G}{3c^2} \epsilon_{or} \frac{a_0^4}{a^2} + \text{const.} \tag{5}$$

and normalising to the present-day conditions

$$t = t_0,\ a = a_0,\ \left.\frac{da}{dt}\right|_{t=t_0} = a_0\, H_0,\ \rho = \rho_{om} = \Omega_m \frac{3H_0^2}{8\pi G} = \Omega_m\, \rho_{ocr} \tag{6}$$

one obtains (Rozgacheva 1980):

$$H(Z) = H_0\,(1+Z)\left[1 + \Omega_m\, Z\, \left\{1 + \frac{\epsilon_{or}}{\epsilon_{om}}\,(2+Z)\right\}\right]^{1/2} \tag{7}$$

where $\quad (1+Z) = \frac{a_0}{a},\ \Omega_m = \frac{\rho_{om}}{\rho_{ocr}},\ \rho_{ocr} = \frac{3H_0^2}{8\pi G},$

for $\quad H_0 \simeq 50$ km $|s|$ Mps., $\rho_{ocr} \simeq 4{\cdot}7 \cdot 10^{-30}$ g/cm^3. (8)

Differentiating $a_0/a = 1+Z$ with reference to t, we get

$$da/dt = -\frac{a_0}{(1+Z)^2}\frac{dZ}{dt} = -\frac{a}{(1+Z)}\frac{dZ}{dt}, \tag{9}$$

$$\frac{dZ}{dt} = -H\,(1+Z) = -H_0\,(1+Z)^2\left[1 + \Omega_m\, Z\, \left\{1 + \frac{\epsilon_{or}}{\epsilon_{om}}\,(2+Z)\right\}\right]^{1/2} \tag{10}$$

It may be noted that putting $\epsilon_{or} = 0$ we get

$$H(Z) = H_0\,(1+Z)\,(1+\Omega_m\, Z)^{1/2}, \tag{11}$$

$$\frac{dZ}{dt} = -H_0\,(1+Z)^2\,(1+\Omega_m\, Z)^{1/2}, \tag{12}$$

which correspond to the standard equations for a radiationless Universe. (Zeldovich and Novikov (1975)). Since

$$\Omega_m \frac{\epsilon_{or}}{\epsilon_{om}} \ll \Omega_m$$

hence for small Z the radiation term is considerably smaller than the term corresponding to matter in equation (7). Thus with standard assumptions about neutrinos, etc. equations (11) and (12) hold good for $Z < 10^3$ in spite of the microwave radiation.

Alpher and Hermann (1953), as well as Peebles (1966 a, b) can be followed in obtaining an expression for the present-day neutrino energy density. Considering the Universe in its infancy, when $T > 10^{10}$K we find an abundant number of quantum processes taking place including the conversion of electron-positron pairs to corresponding pairs of neutrino-antineutrino

$$e^+ + e^- \rightarrow \nu_e + \bar{\nu}_e \tag{13}$$

It may be shown (Weinberg 1972) that after $\tau = 0{\cdot}2$ seconds the equilibrium between e^+, e^- and ν_e, $\bar{\nu}_e$ is disturbed and the neutrinos ' switch off '. It may be shown likewise, that the switch-off time for taonic and muonic neutrino is basically the same as that of the electronic neutrino. As $T_\tau \simeq 2\text{MeV} > m_e c^2$ the photons and the electron positron pairs will be in equilibrium with each other at this instant and their energy densities will almost be the same. Evaluating the respective entropies at that moment we get

$$S_\gamma + S_{e+e-} = \tfrac{4}{3}(\sigma T_\tau^3 + \tfrac{7}{4}\sigma T_\tau^3)V_\tau \tag{14}$$

$$S_{\nu_e \bar{\nu}_e} = \tfrac{4}{3}\tfrac{7}{8}\sigma T_\tau^3 V_\tau. \tag{15}$$

During the expansion process the part of the entropy corresponding to neutrino-antineutrino pairs and the part corresponding to γ, e^+, e^- will be separately conserved. At the instant when $T = m_e C^2 = 5{\cdot}10^9\,°\text{K}$, $t \sim 10$ sec. the e^+, e^- pairs annihilate each other and are transformed to photons. Therefore the entropy previously contained in the e^+, e^- pairs is now transferred to the photons. Thereafter no substantial changes occur and each of the corresponding entropies is conserved separately. Denoting today's situation by the letter '*o*' we get

$$S_o(\gamma) = S_\tau(\gamma) + S_\tau(e^+,e^-) \text{ or } \tfrac{4}{3}\sigma T_{o\gamma}^3 V_o = \tfrac{4}{3}\tfrac{11}{4}\sigma T_\tau^3 V_\tau, \tag{16}$$

$$S_o(\nu_e, \bar{\nu}_e) = S_T(\nu_e, \bar{\nu}_e) \text{ or } \tfrac{4}{3}\cdot\tfrac{7}{8}\sigma T_{o\nu}^3 V_o = \tfrac{4}{3}\cdot\tfrac{7}{8}\sigma T_\tau^3 V_\tau. \tag{17}$$

Here $T_{o\gamma} = 2{\cdot}7°\text{K}$, from these equations, we obtain

$$T_{o\nu_\tau} = T_{o\nu_\mu} = T_{o\nu_e} = (4/11)^{1/3}\,T_{o\gamma} = 2°\text{K}, \tag{18}$$

$$\epsilon_o(\nu_T, \bar{\nu}_T) = \epsilon_o(\nu_\mu, \bar{\nu}_\mu) = \epsilon_o(\nu_e, \bar{\nu}_e) = 0{\cdot}23\,\epsilon_{or}, \tag{19}$$

ϵ_{or}-present-day photon energy density,

$$\epsilon_{or} = 4{\cdot}5{\cdot}10^{-13}\ \text{erg/cm}^3,\ \rho_{or} = 5{\cdot}10^{-34}\ \text{g/cm}^3. \tag{20}$$

Thus if we consider the present Universe to be composed of (apart from matter and photons) n different kinds of neutrino-antineutrino we shall get for the total energy

$$\epsilon_o(\nu, \bar{\nu}) = \sum_{k=1}^{n} \epsilon_o(\nu_k, \bar{\nu}_k) = n.0{\cdot}23\,\epsilon_{or} \tag{21}$$

The total energy due to radiation is therefore given by

$$\epsilon_0(\gamma, \nu, \bar{\nu}) = \epsilon_{or} + \epsilon_o(\nu, \bar{\nu}) = \epsilon_{or}(1 + 0{\cdot}23n). \tag{22}$$

We introduce the variables b and b_r which are defined as follows: b is the ratio of the total massless particle density to the matter density, b_r is the ratio of the photon density to the matter density.

$$b = \frac{\epsilon_o(\gamma, \nu, \bar{\nu})}{\epsilon_{om}} = \frac{\epsilon_{or}}{\epsilon_{om}}(1 + 0{\cdot}23n) = b_r(1 + 0{\cdot}23n)$$

$$b_r\,\Omega_m = \frac{\rho_{or}}{\rho_{om}}\cdot\frac{\rho_{om}}{\rho_{ocr}} = 1{\cdot}06.10^{-4},\quad b = 1{\cdot}06{\cdot}10^{-4}\frac{(1 + 0{\cdot}23n)}{\Omega_m} \tag{23}$$

for $\rho_{or} = 5{\cdot}10^{-34}\text{g/cm}^3$, $\rho_{ocr} = 4{\cdot}7{\cdot}10^{-30}\ \text{g/cm}^3$.

In the foregoing we have assumed neutrinos to be massless particles. This, however, need not necessarily be true. Laboratory experiments impose a definite upper limit on the neutrino mass $m_{\nu_e} < 100eV/c^2$ and $m_{\nu_\mu} < 1{\cdot}5$ MeV/c^2. Evaluations using cosmological arguments, however, put $m_{\nu_\mu} < 500$ eV/c^2. (Gerstein and Zeldovich 1966). For neutrinos having a finite mass $E_\nu^2 = p_\nu^2 c^2 + m_\nu^2 c^4$ and their behaviour will be massless for $kT \gg m_\nu c^2$ and massive for $kT \leqslant m_\nu c^2$. By virtue of its having a finite mass the neutrino ceases to be a spiral particle which leads to the doubling of its statistical weight. Thus for $kT \leqslant m_\nu c^2$, $\rho_\nu \propto (a_0/a)^3$, and clearly a large break-up of neutrino types is not very likely as it would give rise to a matter density much greater than what experimental observations seem to indicate.

Finally, the interaction of gravitation alwaves with matter is weaker still, and therefore it is possible for their density to be larger than the equilibrium one.

3. Perturbation equations in a two-component medium

The fact that the structure of the Universe corresponds to very large density perturbations

$$\delta_m = \left|\frac{\rho - \rho_0}{\rho_0}\right| \gg 1$$

makes the advancement of any complete theory which attempts to study their growth all the more difficult.

The effect of radiation on perturbations is however strongest during the period close to recombination when the latter assume relatively small values. During this period and until $Z \sim 10$ $\delta_m < 1$ and the problem may be studied in the framework of a linear theory. For perturbations to acquire their present magnitude it is necessary that $\delta_m \sim 1$ for $Z \sim 3 \div 10$ after which recourse has to be made to a nonlinear perturbation theory such as the one put forward by Zeldovich (1970).

In the present paper we confine ourselves to a study of the problem in the framework of the linear theory.

The following second order linear differential equation was obtained by Bonnor (1957) in his study of density perturbations in a uniformly expanding homogeneous and isotropic medium.

$$\frac{d^2\delta_m}{dt^2} + 2H\frac{d\delta_m}{dt} - (4\pi G\rho_m - a^2k^2)\delta_m = 0. \tag{24}$$

Perturbations are assumed to be of the form

$$\frac{\rho - \rho_0}{\rho_0} = \delta(t) \exp (i\vec{k}(t)\vec{r} \tag{25}$$

Where $\vec{k}(t)$ the wave vector determines the wavelength of the perturbation and a is the speed of sound. For long wavelengths (24) reduces to ($a \ll c$)

$$\frac{d^2\delta_m}{dt^2} + 2H\frac{d\delta_m}{dt} - 4\pi G\rho_m\delta_m = 0, \tag{26}$$

the two linearly independent solutions of this equation for a radiationless flat Universe $(\Omega_m = 1, b = 0)$ are

$$\delta_i \propto t^{2/3}\ , \quad \delta_d \propto t^{-1}, \tag{27}$$

corresponding respectively to increasing and decreasing modes, in terms of the parameter z

$$t = \frac{1}{H_0}(1+Z)^{-3/2}, \quad \delta_i \propto (1+Z)^{-1}, \quad \delta_d \propto (1+Z)^{3/2}. \tag{28}$$

A particular solution of (26) for initial conditions

$$\delta_m\ (Z = 1400) = 1, \quad \frac{d\delta_m}{dZ}(Z = 1400) = 0 \tag{29}$$

gives
$$\delta_m(Z) = \frac{3}{5}\left(\frac{1401}{1+Z}\right) + \frac{2}{5}\left(\frac{1+Z}{1401}\right)^{3/2} \tag{30}$$

or in particular $\delta_m(0) \simeq 841$. That is in a radiationless flat Universe, density perturbations grow to over 800 times of their initial values, over the period ranging from recombination $(z = 1400)$ to the present day $(z = 0)$. Substituting (7) and (10) in (26) (omitting lengthy intermediate steps) we obtain

$$\begin{aligned}(1+Z)\ [1 + \Omega_m Z\{1 + b\,(2+Z)\}]\ \frac{d^2\delta_m}{dZ^2}\\ + (1+Z)\,\frac{\Omega_m}{Z}\,\{1 + b\,(2+2Z)\}\,\frac{d\delta_m}{dZ} - \frac{3}{2}\Omega_m\,\delta_m = 0,\end{aligned} \tag{31}$$

where $b = \epsilon_0\,(\gamma, \nu, \bar{\nu})/\epsilon_{om}$.

The basic effect of radiation during the post recombination period is qualitatively clear (Guyot and Zeldovich 1970). By increasing the rate of expansion of the Universe, radiation indirectly retards the growth of perturbations.

This stands in marked contrast to its more direct pre-recombination role when it was in thermodynamic equilibrium with ionised matter and thereby damped the growth of density perturbations in the latter.

We wish to study quantitatively the effect of massless particles, neutrinos and gravitons, whose density even if small now could be larger than the matter density in the past, near recombination.

For the sake of convenience, taking into account the linearit y fo the differential equation (31) we assume the initial conditions

$$\delta_0 = \delta_m\,(Z = 1400) = 1, \frac{d\,\delta_m}{dZ}\,(Z = 1400) = 0.$$

Actually $\delta_m\,(Z = 1400) \ll 1$ and $\delta_m\,(Z \sim 3 \div 10) \simeq 1.$

We assume

$$\frac{d\delta_m}{dZ}(Z = 1400) = 0$$

since density perturbation for which recombination is not instantaneous, grow very slowly during the recombination period due to the large damping effect of radiation.

Equation (31) has been solved numerically for different values of

$$\Omega_m, n : 0{\cdot}02 \leqslant \Omega_m \leqslant 1{\cdot}5, \quad 0 \leqslant n \leqslant 100.$$

The results obtained for intermediate values of Ω_m, b have been plotted in figures 1 to 5. For each of $\Omega_m = 0{\cdot}03$, 0·1. 0·5, 1, 1·5, four corresponding cases have been considered for the radiation:
(a) radiation absent ($b = 0$), (b) radiation consisting only of photons ($b \neq 0$, $n = 0$), (c) radiation = photons + 10 different kinds of neutrino-antineutrino ($n = 10$), (d) radiation = photons + 100 different kinds of neutrino-antineutrino ($n = 100$). The different cases are labelled (1), (2), (3), and (4) respectively. Although the contribution of the graviton has not been separately considered, its clear that (31) does not undergo a basic change in this case and the results obtained for the case of a variable neutrino density may also be used to obtain a qualitative picture of a Universe with a variable graviton density.

Analysing the figures we find that as Ω_m increases the difference between the corresponding curves decreases and therefore $\Omega_m = 1$, 1·5 three and two curves have been plotted, corresponding respectively to cases (1), (3), (4) and (1), (4). Finally the complete results for the current ($Z = 0$) density perturbations have been plotted for the entire range $0{\cdot}02 \leqslant \Omega_m \leqslant 1{\cdot}5$ for two extreme cases, (a) in the absence

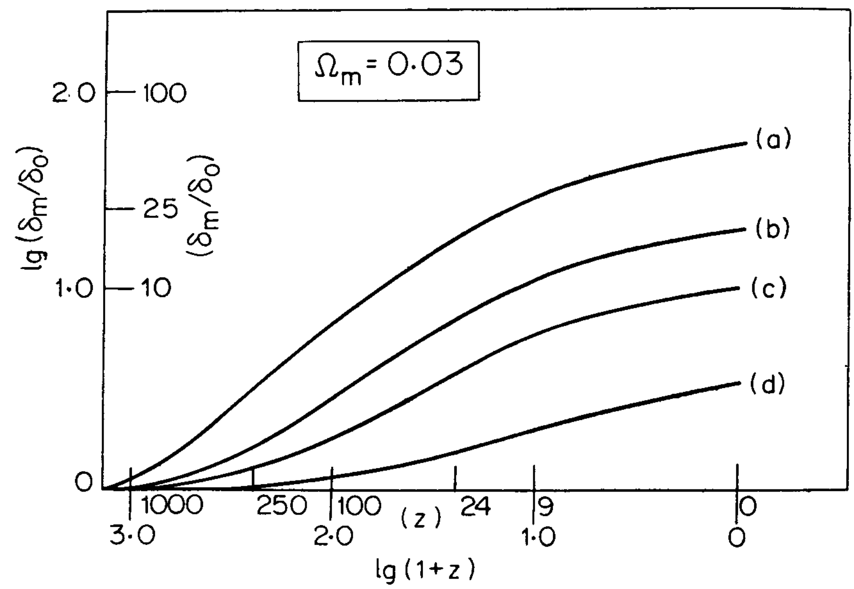

Figure 1.

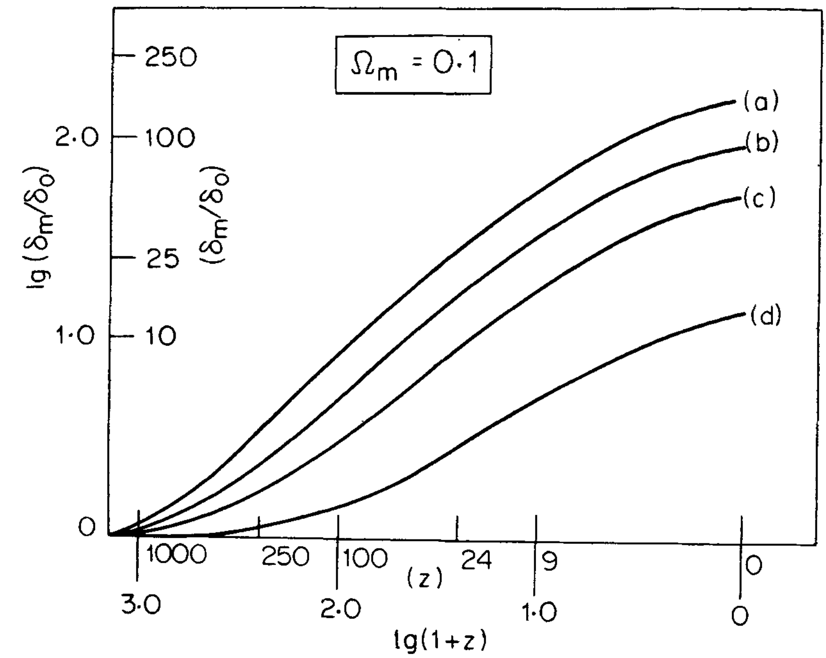

Figure 2.

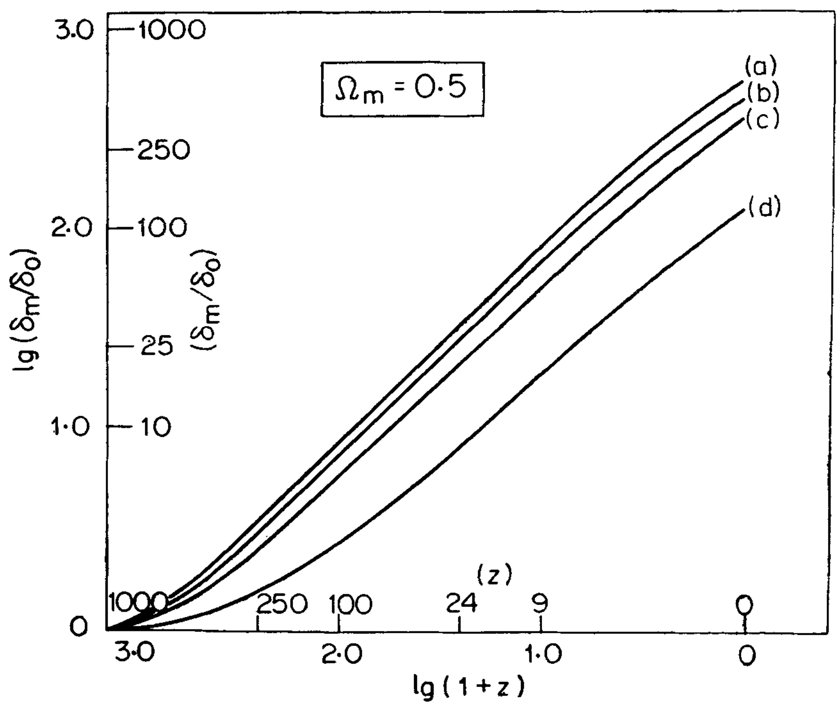

Figure 3.

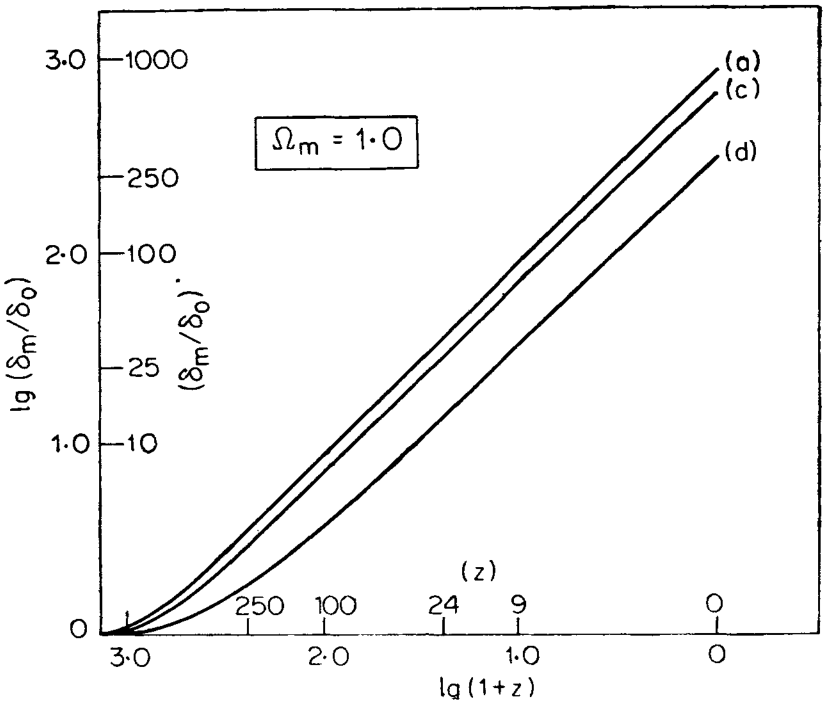

Figure 4.

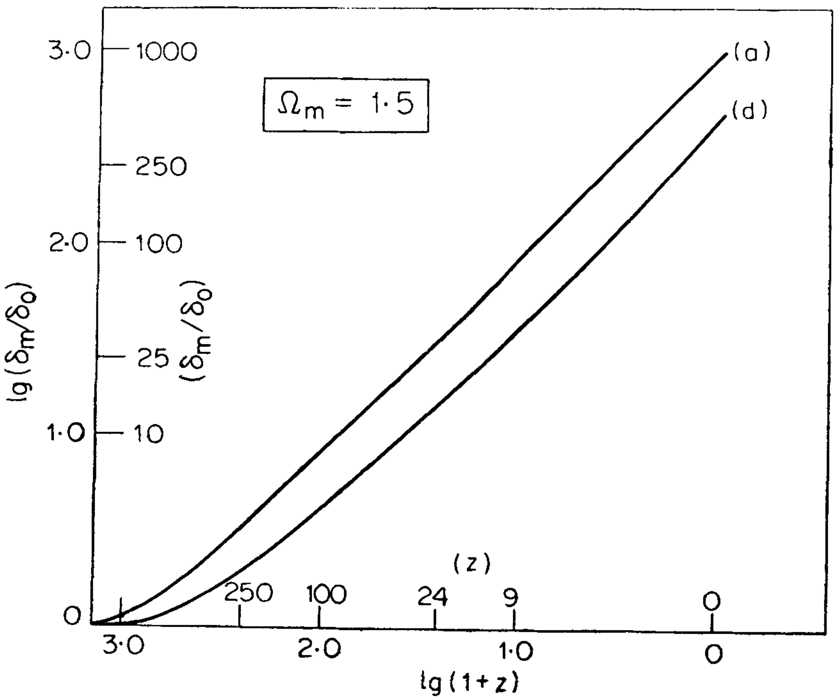

Figure 5.

Figures 1 to 5. Plot of lg (δ_m/δ_0) vs lg $(1+z)$. (a) radiation absent, (b) radiation consisting of only photons, (c) radiation consisting of photons and 10 kinds of neutriono, antineutrino, (d) radiation consisting of photons and 100 kinds of neutrino, antineutrino.

of any radiation, and (b) in the presence of radiation consisting of photons and a hundred kind of neutrinos and antineutrinos (figure 6). It will be seen that whereas in case (a) $d\,\delta_m/d\,\Omega_m$ is large for small Ω_m and gradually diminishes as Ω_m increases, in case (b) $d\delta_m/d\Omega_m$ is small for small Ω_m and increases to a roughly constant value for $\Omega_m \simeq 0{\cdot}5$. This clearly demonstrates the damping effect of radiation on density perturbations for small Ω_m. We would like to point out that the results obtained by us for a radiationless, flat universe, $b=0$, $\Omega_m=1$ coincide with those arrived at analytically in the last paragraph.

We have mentioned earlier the possibility of the neutrino possessing a small but finite mass. This would lead to Ω_m taking a much larger value, which along side the fact that neutrino perturbations will be relatively free from damping, points to a very rapid growth for δ_m. If, further, Ω_m is accurately known as study of density perturbations could give an accurate value for the density of gravitational waves.

The recent discovery of the τ meson gives rise to the feeling that there may be more hitherto undiscovered kinds of neuitrno existing in the Universe alongside the three kinds mentioned earlier. It was shown however by Schwartzmann (1969), and later by Yang *et al* (1979) and Steigman *et al* (1979), that the existence of a large density for massless particles would, by influencing the rate of expansion, indirectly affect the processes of nuclear synthesis occuring during the former's early stages. This would lead to the Universe's chemical composition being noticeably different from what it is today. (More specifically by increasing the rate of expansion a large massless particle density would cause a greater relative concentration of helium than

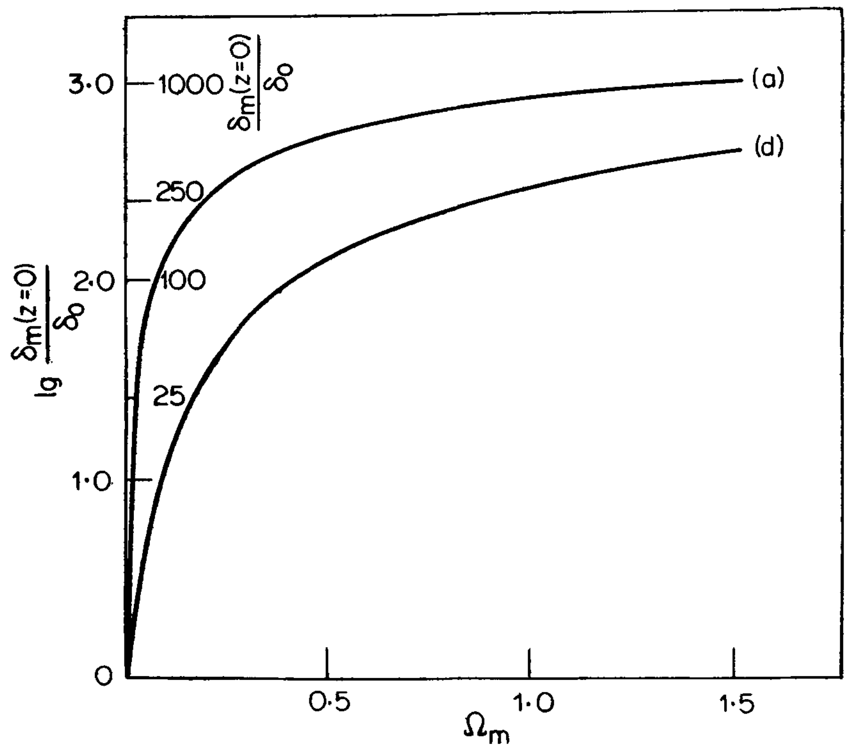

Figure 6. Plot of lg $(\delta_m\,(z=0)/\delta_0)$ *vs* Ω_m (a) radiation absent, (b) radiation consisting of photons and 100 kinds of neutriono, antineutrino.

what experimental observations seem to indicate.) This does not rule out however the existence of a moderately large massless particle density $\rho_{m=0} = 2\,\rho_\gamma$ for small $\Omega_m \simeq 0{\cdot}03$, since the percentage of helium formed during the nuclear synthesis decreases for decreasing Ω_m and the presence of massless particles only serves to compensate this process.

The above analysis is based however on the assumption that the density of the electronic neutrino and antineutrino is the same. It was pointed out by Fowler (1970) that this need not necessarily be true. In such an eventuality the theory regarding the early neucleosynthesis has to be modified and the percentage of helium formed in the framework of the modified theory is found to depend on the relative abundances of neutrino and antineutrino as well as on the total density of massless particles which serve to alter the rate of expansion of the Universe. If for instance the neutrino concentration greatly exceeds that of the antineutrino calculations show the percentage of helium formed to be extremely small and in order that the Universe possess its observed chemical composition one has to presuppose a very high density for massless particles.

An experimental verification of the relative concentration of neutrino and antineutrino is however a very difficult task and lies beyond our scope today.

In order to be able to use the present analysis in determining the density of massless particles it is moreover necessary to have an accurate knowledge both of Ω_m and of the magnitude of density perturbations during recombination. The value of the former is still the subject of much controversy and the strong isotropy of cosmic microwave radiation expresses the latter only by means of an inequality.

The results obtained by us for small $\Omega_m \sim 0{\cdot}03$ show that for $\rho_{m=0} = 2\rho_r$ density perturbations grow to only about 10 times their value during recombination. If we further assume that $\delta_m = \Delta\rho_m/\rho_m \sim 1$ for $z \sim 30$ (which is essential for the formation of gravitationally-bound systems such as galaxies and their clusters) we find that for

$$\Omega_m = 0.03 \text{ and } \rho_{m=0} = 2\rho_r,\ \delta_m(z=1400) = \frac{\Delta\rho_m}{\rho_m}(z=1400) \simeq 0.27$$

and $$\frac{\Delta T}{T}(z=1400) = 0.09.$$

Thus we find that fairly large density perturbations are needed during recombination to give rise to the present small scale inhomogeneity of the Universe. In a future work we wish to study the possibility of the existence of such initial density perturbations and also the vital role played by damping energy both on perturbations and on the background microwave radiation. Such an analysis will be a sensitive test for Ω_r.

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