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Space-time singularities and microwave background radiation

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Abstract. A general relativistic space-time universe is considered together with a radiation such as the microwave background radiation. It is shown that if certain reasonable conditions are satisfied, then the presence of such a radiation would imply space-time singularities in the sense that all time-like curves will be incomplete in the past. The considerations provide an upperbound to the age of the universe, which is consistent with present data.

Keywords. Space-time singularity; microwave radiation; energy conditions; causal structure.

1. Introduction

The observations of radio frequencies between 20 cm and 1 mm indicate that there is a radiation background in the universe whose spectrum seems to be close to that of a black body at 2·7° K. The observed pattern is highly isotropic spatially and so it is widely believed that it must have an extragalactic origin, since we are not symmetrically placed in the plane of our galaxy. Various beliefs exist concerning the origin of this radiation, however, the general opinion is that this is a blackbody radiation left over from a hot early stage of the universe (See Sciama 1971 for more discussion on microwave radiation.)

In view of the observational support to the existence of microwave radiation, the considerations concerning the global structure of the universe must take it into account suitably. Hawking and Ellis (1968, 1973) considered the microwave background radiation in a general relativistic space-time universe and showed that this implies a possibility of singularity at the beginning of the present phase of expansion of the universe. Such a singularity would exist if together with certain physically reasonable assumptions it is taken that the microwave radiation was exactly spatially isotropic or is partially thermalised by scattering. Here the singularity exhibits itself in the form of geodesic incompleteness.

We further study here the relationship between the presence of such a radiation in the universe and space-time singularities. We consider a class of general relativistic space-times which are globally hyperbolic. The only assumption made concerning the microwave radiation is that its energy density at all events in the past of a given event p be greater or equal to that at p. This is eminently reasonable if universe had a radiation dominant region in the past or the radiation was emanated from a hot early stage, etc. It is then shown that if other matter fields have positive energy density, *all timelike curves* must be incomplete in the *past*, and also that the proper time lengths of all past-directed timelike curves from an event are finitely bounded. This implies an upperbound to the age of the universe, which is consistent with the present data.

§2 discusses certain topics concerning the structure of space-time which are required here. The energy conditions on the space-time are set in §3. The theorem is proved in §4, and the final section obtains an upperbound to the age of the universe as implied by the theorem.

2. Causal structure and conjugate points

In the course of the recent developments in space-time physics, the global structure of the space-time has been analysed in detail. The general relativistic model for the physical universe is a space-time M which is a four-dimensional real differentiable manifold with a globally defined metric tensor on it and which is time and space orientable. The causal future of any event p is defined as the set of all events q such that there is a future directed nonspacelike curve from p to q; and is denoted by $J^+(p)$. The set $J^-(p)$ can be defined similarly. An important global axiom on a space-time is global hyperbolicity. M is said to be globally hyperbolic if for all $p, q \in M$, $J^+(p) \cap J^-(q)$ is compact in the natural manifold topology on M, and if M is causal i.e. it does not contain closed nonspacelike curves. (For more details on causal structure and global axioms, we refer to Hawking and Ellis 1973).

Now the normal matter content (i.e. matter with positive energy density) of the space-time influences its causal structure, it focusses the geodesics into pairs of focal points, called conjugate points. We shall make here some remarks concerning the occurrence of conjugate points along a nonspacelike geodesic. Points $p, q \in \gamma(t)$ are said to be conjugate along $\gamma(t)$ if the expansion of the geodesic congruence containing $\gamma(t)$ becomes infinite at p and q. This expansion θ satisfies the Raychaudhuri equation

$$d\theta/dt = - R_{ij} V^i V^j - 2\sigma^2 - \frac{1}{n} \theta^2, \tag{1}$$

where V^i is the tangent vector to the geodesic, t is the parameter, $n = 3$ for timelike geodesics and $n = 2$ for null geodesics. Now if one chooses a function y by $\theta = 1/y \; dy/dt$, then two points p and q are conjugate if and only if $y=0$ at p and q (Hawking and Ellis 1973, p. 97). Tipler (1976) substituted $z^n = y$, thus defining a new function z. Then (1) becomes

$$(d^2z/dt^2) + H(t)z = 0, \tag{2}$$

where

$$H(t) = \frac{1}{n} (R_{ij} V^i V^j + 2\sigma^2).$$

Then obtaining the conjugate points along $\gamma(t)$ is equivalent to finding out the zeros of a solution of (2). We shall make use of the following sufficient condition for the occurrence of zeros in a solution to equation (2) (Hille 1969, p. 376):

Theorem A: A sufficient condition that *every* solution to equation (2) have at least n zeros in an interval (t_1, t_2) is that

$$\inf_{t_1 < t < t_2} H(t) > \left[\frac{n\pi}{t_2 - t_1}\right]^2.$$

Finally, we mention the relationship between the conjugate points along a timelike geodesic and the global hyperbolicity of a space-time.

Theorem B (Avez-Hawking): Let p and q be events in a globally hyperbolic set N such that there is a timelike curve from p to q. Then there exists a timelike geodesic from p to q of maximal proper time length. Further, a nonspacelike curve of maximal length from p to q must have no conjugate points between p and q. (For further details and proof see Hawking and Ellis 1973, p. 213).

3. Energy conditions

The matter content of the universe is represented by a second rank symmetric tensor T_{ij} on M, which is called the stress-energy tensor of the spacetime. The essential feature that we would like to make use of concerning the microwave radiation is that it contributes to the stress-energy density of the spacetime. The dominant contribution to this comes from the matter and electromagnetism; essentially from the rest mass of galaxies. However, the microwave photons also contribute to this stress energy, which is about 10^{-4} of the galactic contribution on the cosmological scale.

The essential energy condition which we shall require to hold for the microwave radiation is that the gravitational force created by its energy density is always attractive and that it has a nonzero value at a given event of the space-time. Also if this radiation came from a hot early stage of the universe, it would be reasonable to assume that if it is present at an event p, then so is the case at all events in $J^-(p)$. Further, we shall assume that all the other matter fields on the spacetime obey the strong energy condition (Hawking and Ellis 1973, p. 95). Thus we assume the following

Energy Condition: For any $p \in M$ and timelike vectors

$$V \in T_p, \quad \underset{\text{(MBR)}}{R_{ij}} \quad V^i V^j \geqslant k > 0; \; k = \text{constant}.$$

and the same holds for all $q \in J^-(p)$. Also, all the rest of the matter fields on M obey

$$(T_{ij} - 1/2\, g_{ij} T) V^i V^j \geqslant 0$$

for all timelike vectors $V^i \in T_p$ (i.e. the strong energy condition holds.) Here $\underset{\text{(MBR)}}{R_{ij}}$ is the Ricci tensor associated through the field equations with the microwave background radiation.

In view of the fact that the microwave radiation is a known type of field whose stress-energy tensor should be of type I, and that its presently observed energy density is about 10^{-34} g/cm^3; the above condition is fully reasonable. The motivation for the strong energy condition on other matter fields is well-known.

4. The theorem

Now we prove the following

Theorem: Let (M, g) be a spacetime in which the following hold:

(a) (M, g) obeys Einstein's field equations,
(b) the energy condition mentioned above,
(c) (M, g) is globally hyperbolic.

Then all the timelike curves in M are past incomplete.

Proof: Let $p \in M$. Consider some past directed, endless, timelike curve γ from p. Suppose γ is complete in the past. Let $\gamma(0) = p$, and suppose at p we have

$$\underset{\text{(MBR)}}{R_{ij}} \quad V^i V^j \geqslant k > 0, \; k = \text{constant}.$$

Then conditions (a) and (b) imply that we have $R_{ij}V^iV^j \geqslant k$ at p and also at all points $q \in J^-(p)$. Now consider a point q such that $q = \gamma(2\pi\sqrt{3/k})$. The point q exists since γ is past complete. Consider the space of all nonspacelike curves from p to q. Then M being globally hyperbolic, theorem (B) implies the existence of a maximal past-directed timelike geodesic γ' from p to q, the length of which is greater or equal to $2\pi\sqrt{3/k}$. Now the sufficient condition for γ' to contain at least two conjugate points in the interval $(0, 2\pi\sqrt{3/k})$, which is according to theorem A

$$\inf_{0 < t < 2\pi\sqrt{3/k}} \frac{1}{3}(R_{ij}V^iV^j + \sigma^2) \geqslant \frac{4\pi^2}{4\pi^2\, 3/k} = k/3,$$

is seen to be satisfied since $R_{ij}V^iV^j \geqslant k$ and σ, the shear is intrinsically positive. Hence γ' contains at least two conjugate points in the interval $(0, 2\pi\sqrt{3/k})$. However, this is not possible according to theorem B; a maximal geodesic cannot contain conjugate points. Hence the timelike curve γ cannot be extended in the past beyond the proper time length $2\pi\sqrt{3/k}$; i.e. γ is incomplete in the past. Since the above argument holds for any $p \in M$ and any other endless, past-directed curve from p, the theorem is established.

Thus it appears that if certain reasonable conditions are satisfied, then the microwave radiation is closely connected with the occurrence of spacetime singularities. Some remarks on the above theorem are in order:

(i) All the known singularity theorems have been proving the incompleteness of nonspacelike *geodesics*. However, the theorem above proves the incompleteness of *any timelike curve* in the *past*, which is an improvement in view of the singularity theory. One would not expect the observers to be moving freely, especially in the early universe. Then the theorem says that *any* observer, either accelerated or freely falling, must encounter a singularity in the past.

(ii) The global hyperbolicity assumption used in proving the above result implies a strong form of predictability in the space-time and it is conceivable that the real universe may not be globally hyperbolic (e.g. see Hawking and Ellis 1973, pp. 205-206). The point here is this: The Friedmann universes have a universal singularity for all past-directed nonspacelike geodesics at $r=0$. They are globally hyperbolic and the t=constant hypersurfaces are global Cauchy surfaces. Finally, a suitable energy condition is also required to deduce the universal singularity. Here we have generalised the situation to all globally hyperbolic universes obeying suitable energy condition and in which the microwave radiation is present.

(iii) Also it is possible here to have information concerning the nature of these singularities. From the results of Clarke (1975), these are very likely to be curvature singularities.

(iv) The theorem offers a possibility to determine an upperbound to the age of the universe, which is discussed in the next section.

5. Upperbound to the age of universe

Since the observable universe is supposed to be represented by Robertson-Walker universes which are globally hyperbolic, we can think ourselves as being represented at the present epoch by some event p on some Cauchy surface. Now if at the present epoch we have

$$\underset{\text{(MBR)}}{R_{ij}} \; V^i V^j \geqslant k,$$

then as we have seen, the theorem sets a limit of proper time extension $2\pi\sqrt{3/k}$ for any past-directed timelike curve into the past.

Here the point is that the information concerning the constant k can be obtained by the observation of the energy density of the microwave radiation at the present epoch.

Consider the microwave radiation energy momentum tensor which is given by

$$T_{ij} = [\tfrac{4}{3}\rho \; U_i \; U_j - \tfrac{1}{3} \rho g_{ij}].$$

Now Einstein's equations are

$$R_{ij} = 8\pi \, (T_{ij} - \tfrac{1}{2} g_{ij} \, T),$$

which gives

$$\underset{\text{(MBR)}}{R_{ij}} \; V^i V^j = 8\pi \, [\tfrac{4}{3} \, \rho(U \cdot V)^2 - \tfrac{1}{3}\rho]$$

$$= 8\pi \, [\tfrac{4}{3} \, \rho_0 \, (V^4)^2 - \tfrac{1}{3} \, \rho_0 \,] \geqslant 8\pi\rho_0.$$

So to obtain an upperbound to the age of the universe, one could set $k \equiv 8\pi\rho_0$. Then the upperbound to the propertime lengths would be $2\pi(3/k)^{1/2} \equiv 2\pi(3/8\pi\rho_0)^{1/2}$. Substituting here for $\rho \equiv 4{\cdot}4 \times 10$ g/cm^3 and $G \equiv 6{\cdot}67 \times 10^{-8}$ dyn cm^2/g^2, one gets an upperbound for the age of the universe as

$$1{\cdot}24 \times 10^{13} \text{ years.}$$

Now the presently believed age of the universe based on a Hubble constant of ~ 50 km^{-1} Mpc^{-1} and a deceleration parameter $q_0 \approx 0$ is approximately $1{\cdot}8\times 10^{10}$ years. Cosmological models with $q_0 > 0$ will give ages shorter than the above value. Although a downward revision of Hubble's constant may increase the above value, the present astronomical observations indicate that Hubble's constant may in fact be higher than the above value. Hence it turns out that the upperbound to the age of universe, as provided by our theorem is consistent with the present data. It is interesting to note that the *local, here and now* observations on microwave background provide this upperbound.

6. Conclusions

We have shown that the presence of a radiation field such as the microwave background radiation does have important implication towards the global structure of the universe in the sense that every observer encounters a singularity in the past. Further considerations on the occurrence of conjugate points could improve the upperbound to the age of the universe which we have obtained here.

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