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A curiosity concerning the role of coherent states in quantum field theory

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Abstract. In the usual Fock quantisation of fields in Minkowski space-time, one has the result that the expectation value of the quantum Hamiltonian in any coherent state equals the energy of the classical field at which the state is peaked. It is shown that this property can be used to *characterise* the usual Fock representation. It is also pointed out that the entire analysis goes through for a substantially more general class of systems including, in particular, Bose fields in arbitrary stationary space-times.

Keywords. Quantum field theory; Fock representations; coherent states.

1. Introduction

Consider, in Minkowski space, a box B of volume V filled with electro-magnetic radiation of frequency ω. For simplicity, let us work in the Coulomb gauge (in the rest frame of B) and assume that the vector potential A_a vanishes on the walls of B. Then, the classical energy $\mathscr{E}_c$ of this A_a is given by

$$\mathscr{E}_c = (\tfrac{1}{2}) \int_B (E_a E^a + H_a H^a) dV = \omega^2 k,$$

where E_a and H_a are the electric and magnetic parts of the given Maxwell field in the rest frame of B, and,

$$k = \int_B A^a A_a \, dV.$$

On the other hand, one may choose to describe the content of the box in the framework of quantum field theory. Then, this content is best described by a coherent state of the quantised Maxwell field (see e.g. Glauber 1963; Sudarshan 1963). Denote this state by Ψ. The energy $\mathscr{E}_q$ of this state is given by:

$$\mathscr{E}_q = \langle \Psi, \Psi \rangle^{-1} \langle \Psi, \mathcal{H} \Psi \rangle = \hbar\omega \langle \Psi, \Psi \rangle^{-1} \langle \Psi, \mathcal{N} \Psi \rangle = \hbar\omega \langle A, A \rangle = \omega^2 k,$$

where $\langle , \rangle$, $\mathcal{H}$ and $\mathcal{N}$ are respectively, the inner-product the Hamiltonian and the total-number operator on the Fock space of quantum states of the electromagnetic field in B, and where A denotes the one-photon state defined by the given vector potential A_a. Thus, $\mathscr{E}_c = \mathscr{E}_q$! That is, the classical and quantum descriptions of the content of the box assign it the same energy. Furthermore, it is only for simplicity that we restricted ourselves to electro-magnetic radiation of a given frequency; a substantially more general result can be proved. Consider any (free) Bose field in Minkowski space-time and the standard Fock quantisation thereof. Then, associated with every (suitably regular) classical solution S to field equations, there exists a unique coherent state Ψ_S in the corresponding Fock space. The pair (S, Ψ_S) has the property that the classical energy† of S equals the expectation-value of the quantum Hamiltonian in Ψ_S, for all classical fields S!

The purpose of this paper is to show that this remarkable property of coherent states can, in addition, be used to obtain a *characterisation* of the usual Poincaré invariant Fock representation. More precisely, we shall show that, among all irreducible Fock representations of the canonical commutation relations of Bose fields, only in the standard Fock representation can the coherent states satisfy the above condition on classical and quantum energies. (The restriction to ' Fock representations ' is motivated by the desire to have a well-defined particle interpretation. The precise definition of the term is given in the next section.) It is curious to note that this result enables one to single out the standard vacuum state without having to require invariance under the Poincaré group or any sub-group thereof: Poincaré invariance of the vacuum is a consequence of—rather than an input in to—the main result.

Since the notion of energy refers only to time-translations, the entire analysis goes through for Bose fields in stationary space-times. To bring out this point, throughout our analysis we shall refer only to that structure of Minkowski space (and of Bose fields thereon) which continues to be available in stationary space-times.

For simplicity, in the main body of the paper we shall work with massive scalar fields; the situation with respect to other Bose fields will only be summarised at the end. Also, several technical issues in functional analysis—particularly the ones concerning domains of various operators—have been ignored; a satisfactory treatment of these issues would require considerably more space and would mask the simple physical ideas on which the analysis rests.

2. Mathematical preliminaries

Consider real scalar fields Φ in Minkowski space-time (M, g_{ab}) satisfying the massive Klein-Gordon equation

$$\nabla^a \nabla_a \Phi - \mu^2 \Phi = 0, \tag{1}$$

where ∇ is the derivative operator on (M, g_{ab}) and μ a positive real number. Fix a time translation Killing field t^a in (M, g_{ab}). The associated conserved quantity,

†In fact, the corresponding statement holds for *any* component of energy-momentum and angular momentum. We focus on energy since other quantities are not of relevance to the main analysis.

$\mathscr{E}_c$, is the energy functional (in the rest frame defined by t^a) on the space of classical fields:

$$\mathscr{E}_c = \int_\Sigma T_{ab}\, t^a \, d\Sigma^b \tag{2}$$

with $$T_{ab} = \nabla_a \Phi \nabla_b \Phi - (\tfrac{1}{2}) g_{ab} (\nabla_c \Phi \nabla^c \Phi + \mu^2 \Phi^2),$$

where Σ is any Cauchy surface in (M, g_{ab}).

Before going on to the problem of quantisation it is convenient to introduce a number of classically available mathematical structures which play an important role in the quantum description. Denote by Γ the space of real, C^∞-solutions to (1) which induce initial data of compact support on any Cauchy surface. Let τ be the space of real C^∞-functions of compact support on M. Then, given any f in τ,

$$\Phi_f(x) = \int_M \Delta(x, y) f(y)\, dV_y, \tag{3}$$

belongs to Γ, where $\Delta(x, y)$ is the Jordan-Pauli propagator, the difference between the advanced and the retarded Green's functions associated with (1). Furthermore, every element of Γ can be obtained from some element of τ via (3). Next, there exists on Γ, a weakly non-degenerate symplectic tensor Ω, i.e., a bilinear, skew symmetric mapping Ω from $\Gamma \times \Gamma$ into the reals, such that $\Omega(\Phi, \tilde{\Phi}) = 0$ for all $\tilde{\Phi}$ if and only if $\Phi = 0$;

$$\Omega(\Phi, \tilde{\Phi}) := \int_\Sigma (\Phi \nabla_a \tilde{\Phi} - \tilde{\Phi} \nabla_a \Phi)\, d\Sigma^a, \tag{4}$$

where Σ is any Cauchy surface. (That Ω is independent of the choice of Σ follows from the divergence-free character of the integrand.) Finally, this Ω is related to the Jordan-Pauli propagator Δ via.:

$$\Omega(\Phi_f, \Phi_g) = \int_M \int_M \Delta(x, y) f(x)\, g(y)\, dV_x\, dV_y. \tag{5}$$

(For details, See Lichnerowicz 1961 or Segal 1967).

To obtain the quantum description of this field, one first introduces the operator-valued distribution $\underline{\Phi}(x)$ satisfying the field equation (1) and the canonical commutation relations (CCR):

$$[\underline{\Phi}(f), \underline{\Phi}(g)] = \hbar/i \left[\int_M \int_M \Delta(x, y) f(x)\, g(y)\, dV_x\, dV_y\right] \mathbf{I}, \tag{6}$$

for all f and g in τ, where, $\underline{\Phi}(f) \equiv \int_M \underline{\Phi}(x) f(x)\, dV_x$, and, $\mathbf{I}$ denotes the identity operator. The problem of quantisation now reduces to that of obtaining a $*$-representation of these relations. That is, one must find a pair $(\mathbf{H}, \Lambda)$ consisting of a complex Hilbert space $\mathbf{H}$ and an imbedding Λ of the $*$-algebra-generated by $\underline{\Phi}(f)$'s into the $*$-algebra of operators on $\mathbf{H}$.

Let us restrict ourselves to irreducible Fock representations of the CCR, i.e. to representations satisfying the following conditions: (i) the only sub-spaces of $\mathbf{H}$ left invariant by the action of the (image under Λ of the) algebra generated by $\underline{\Phi}(x)$ are the zero sub-space and **H** itself; and, (ii) **H** is a symmetric Fock space based on a complex Hilbert space **h** and Λ sends each $\underline{\Phi}(f)$ to the sum of a creation and an annihilation operator on **H**.

The second condition implies that the pair (**H**, Λ) is completely determined by the complex Hilbert space **h**, while, together, the two conditions imply that **h** has the same *real* vector space structure as (the Cauchy completion of) Γ. Hence, to obtain an irreducible Fock representation of the CCR, we only need to endow on Γ an appropriate complex pre-Hilbert space structure; i.e., a real-linear operator J satisfying $J^2 = -I$, and an hermitian inner product $\langle \cdot , \cdot \rangle$ on the resulting complex vector space (Γ, J). (Using J, we can 'multiply' an element Φ of Γ by a complex number $a+ib$ as follows: $(a+ib) \cdot \Phi = a\Phi + bJ \cdot \Phi$. Note that the result is again in Γ; we have *not* enlarged Γ in order to define the 'multiplication by i'.) The choice of $(J, \langle \cdot , \cdot \rangle)$ is constrained somewhat by the requirement that Λ must satisfy $[\Lambda \cdot \underline{\Phi}(f), \Lambda \cdot \underline{\Phi}(g)] = \Lambda \cdot [\underline{\Phi}(f), \underline{\Phi}(g)]$: $(\Gamma, \Omega, J, \langle \cdot , \cdot \rangle)$ must be a Kähler space. More precisely, J must be compatible with the symplectic structure Ω in the sense that $\gamma(,) := \Omega(, J)$ is a positive definite metric on Γ, and, $\langle , \rangle$ must have the form $\langle , \rangle = (1/2\hbar)\, \gamma(,) + (i/2\hbar)\, \Omega(,)$. (For details, See Segal 1967 or Ashtekar and Magnon 1975). Given such a 'Kählerisation' of (Γ, Ω), we can construct the representation (**H**, Λ) as follows. Denote by **h** the Cauchy completion of the complex pre-Hilbert space $(\Gamma, J, \langle , \rangle)$. This **h** serves as the one-particle Hilbert space. The representation space **H** is the symmetric Fock space based on **h**. Finally, the representation map Λ has the following action: $\Lambda \cdot \underline{\Phi}(f) = \hbar \mathbf{C}(\Phi_f) + \hbar \mathbf{A}(\Phi_f)$ where, Φ_f is the element of Γ- and hence also of **h**- defined by f *via* (3) and $\mathbf{C}(\Phi_f)$ and $\mathbf{A}(\Phi_f)$ are, respectively, creation and annihilation operators on **H** associated with the one-particle state Φ_f. (The factors $\hbar$ ensure that the inner product as well as the creation and annihilation operators are dimensionless, properties essential to guarantee that probabilities are pure numbers and that all elements of **h** have the same physical dimensions. Throughout, we set $c=1$). That every irreducible Fock representation arises in this manner is easy to establish.

To summarise, each irreducible Fock representation of the CCR is characterised by a 'Khälerisation' of the classical phase space (Γ, Ω). In general, distinct Kählerisations give rise to unitarily *inequivalent* representations. To select a particular Kählerisation, one needs to impose additional conditions. The standard Fock representation results if one invokes Poincaré invariance, i.e., if one demands that the natural action of the Poincaré group on Γ be unitarily implemented on **h**, or, equivalently, that the vacuum state in **H** be Poincaré invariant. (The resulting complex structure has the following action:

$J\Phi = i\Phi^+ + (-i)\Phi^-$, where Φ^+ and Φ^- are, respectively, the positive and the negative frequency parts of Φ; $\Phi = \Phi^+ + \Phi^-$. The scalar product is therefore given by

$$\langle \Phi, \tilde{\Phi} \rangle = (1/2\hbar)\,(\Omega(\Phi, J\tilde{\Phi}) + i\Omega(\Phi, \tilde{\Phi})) = (i/2\hbar)\, \Omega(\Phi^-, \tilde{\Phi}^+)$$
$$= (i/2\hbar) \int_\Sigma (\bar{\Phi}^+ \nabla_a \tilde{\Phi}^+ - \tilde{\Phi}^+ \nabla_a \bar{\Phi}^+)\, d\Sigma^a.$$

Note that, since Φ^+ and Φ^- are complex conjugates of each other, $J\Phi$ is real.) In the next section, we shall show that one can also select the standard Fock representation by exploiting the interplay between the classical fields in Γ and the coherent states in **H**, without *any* reference to the Poincaré action on the algebra generated by $\Phi(f)$'s.

3. Coherent states

Fix any complex structure J, compatible with Ω, on Γ. Denote the resulting Fock representation by (**H**, Λ). Since **H** is a Fock space based on the one-particle Hilbert space **h**, one can naturally introduce the notion of coherent states: given any Φ in **h**, $\Psi_{\Phi} := \exp C(\Phi)\cdot v$ is a coherent state associated with Φ, where v is the vacuum state in **H**. It is convenient to represent elements of **H** by normalisable (entire) holomorphic functions on **h**: The complex structure J enables one to introduce the notion of holomorphicity and the Hilbert space structure of **h** gives rise to a natural Gaussian pro-measure on **h** (See, e.g., Bargmann 1962, Segal 1963, Choquet-Bruhat *et al* 1977). Thus, a complex-valued function $\Psi(h)$ on **h** defines an element of **H** if and only if $J \urcorner d\Psi = i\, d\Psi$, where $d\Psi$ denotes the gradient of Ψ and $\urcorner$ denotes the operation of contraction, and $\int_{\mathbf{h}} |\Psi|^2\, d\mu < \infty$, where $d\mu$ is the Gaussian promeasure on **h**. (The physical interpretation associated with Ψ is the following: a holomorphic function $\Psi(h)$ which is a n-nomial on **h** represents a n-particle state. For details, see e.g. Segal 1967 or Ashtekar and Magnon-Ashtekar 1980). The (normalised) coherent state corresponding to the classical field Φ is now represented by the function $\Psi_{\Phi}(h) := (\exp -(1/2) \langle \Phi, \Phi \rangle) \exp \langle \Phi, h \rangle$, where $\langle , \rangle$ denotes the hermitian inner product on **h**.

Next, we introduce the Hamiltonian operator $\mathcal{H}$ on **H** associated with the time-translation Killing field t^a on Minkowski space. Clearly, $\mathcal{H}$ must be self-adjoint and satisfy:

$$i\, [\mathcal{H}, \Lambda \cdot \underline{\Phi}(f)] = \Lambda \cdot \underline{\Phi}\, (T \cdot f), \tag{7}$$

on (**H**, Λ); where $T \cdot f = t^a\, \nabla_a f$. It turns out that these two requirements suffice to determine $\mathcal{H}$ up to a multiple of identity. To see this, consider two operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on **H** which are self-adjoint and satisfy (7). Then, $\mathcal{H}_1 - \mathcal{H}_2$ must commute with all field operators. Since creation and annihilation operators on **H** can be expressed as linear combinations of (images under Λ of) field operators, it follows that $\mathcal{H}_1 - \mathcal{H}_2$ must also commute with each creation and annihilation operator, and hence, also with the total-number operator. Therefore, there exists a complex number λ such that $(\mathcal{H}_1 - \mathcal{H}_2)\, v = \lambda v$ where v is the vacuum state in **H**. Since $(\mathcal{H}_1 - \mathcal{H}_2)$ commutes with all creation operators and since one can generate a dense sub-space of **H** by the repeated action of creators on v, it follows that $(\mathcal{H}_1 - \mathcal{H}_2) = \lambda\, \mathbf{I}$ on **H**. Next, let us eliminate the freedom to add a multiple of identity to the Hamiltonian by requiring that its vacuum expectation value be zero. These conditions ensure the uniqueness of the Hamiltonian. Finally, we display an operator $\mathcal{H}$ which has the required properties:

$$\mathcal{H} \cdot \Psi(h) := \hbar/2\, \mathcal{L}_V\, \Psi(h) + \hbar/4\, (\langle h, (TJ - JT)h \rangle + \text{H.C.})\, \Psi(h) \tag{8}$$

where, V is the vector field on **h** defined by $V|_h = (JT + TJ) \cdot h$; $\mathcal{L}_V$ denotes the

Lie-derivative with reference to the vector field V and H.C. stands for 'Hermitian Conjugate.' (Note: $\mathscr{L}_V \Psi(h) := \lim_{\epsilon \to 0} (1/\epsilon)(\Psi(h+\epsilon V) - \Psi(h))$.)

Using this Hamiltonian, we can now compute the energy $\mathscr{E}_q$ of (normalised) coherent states $\Psi_\Phi(h)$. We have:

$$\begin{aligned}
\mathscr{E}_q := & \langle \Psi_\Phi, \mathscr{H} \cdot \Psi_\Phi \rangle, \\
= & [\exp -\tfrac{1}{2} \langle \Phi, \Phi \rangle] \cdot \langle v, \exp A(\Phi) \cdot \mathscr{H} \cdot \Psi_\Phi \rangle, \\
= & [\exp -\tfrac{1}{2} \langle \Phi, \Phi \rangle] \cdot \mathrm{Re} \langle v, \exp A(\Phi) \cdot [\hbar/2\, \mathscr{L}_V + \hbar/2 \langle h, \\
& (TJ - JT)h \rangle] \cdot \exp \langle \Phi, h \rangle \rangle, \\
= & [\exp -\tfrac{1}{2} \langle \Phi, \Phi \rangle] \cdot \mathrm{Re} \langle v, \exp A(\Phi) \cdot [\hbar/2 \langle \Phi, (JT + TJ)h \rangle + \\
& \hbar/2 \langle h, (TJ - JT)h \rangle] \exp \langle \Phi, h \rangle \rangle, \\
= & [\exp -\tfrac{1}{2} \langle \Phi, \Phi \rangle] \cdot \mathrm{Re} \langle v, [\hbar/2 \langle \Phi, (JT + TJ)(\Phi + h) \rangle + \\
& \hbar/2 \langle \Phi + h, (JT - TJ)(\Phi + h) \rangle] \times \exp \langle \Phi + h, \Phi + h \rangle \rangle, \\
= & \mathrm{Re}\, \hbar \langle \Phi, (T.J) \Phi \rangle, \\
= & \tfrac{1}{2} \Omega(\Phi, JTJ\Phi), \qquad (9)
\end{aligned}$$

where, $\langle \Psi, X \rangle \equiv \int_h \Psi^* X \, d\mu$ is the inner product on H and v, the vacuum state in H. Thus, irrespective of the choice of the complex structure J, in the resulting Fock representation (H, Λ), the energy $\mathscr{E}_q$ in any coherent state is *independent* of the value of Planck's constant $\hbar$: in *every* irreducible Fock representation, coherent states 'behave like classical fields' as far as energy is concerned. The precise value of $\mathscr{E}_q$ on the other hand does depend on the choice of the complex structure.†

We now impose the energy requirement: $\mathscr{E}_c = \mathscr{E}_q$ for all classical fields Φ in Γ and the associated coherent states Ψ_Φ in H, where $\mathscr{E}_c$ is given by (2) and $\mathscr{E}_q$ by (9). Since $\mathscr{E}_c$ makes no reference at all to the complex structure introduced in the transition from the classical to the quantum description, the requirement is a further restriction on the choice of complex structures.

To analyse its consequences, let us first re-express $\mathscr{E}_c$ in terms of the symplectic tensor Ω and the operator T. A simple calculation yields:

$$\mathscr{E}_c \equiv -\tfrac{1}{2} \Omega(\Phi, T\Phi). \qquad (10)$$

Hence the requirement on J now reduces to:

$$\Omega(\Phi, T\Phi) = -\Omega(\Phi, JTJ\Phi), \qquad (11)$$

†Consider an example. Fix a 3-plane Σ orthogonal to the Killing field t^a. Let (ϕ, π) denote the Cauchy data on Σ of Φ. Define J by requiring that the initial data of $J.\Phi$ be $(-\pi, \phi)$. It is easy to check that this J is compatible with Ω. In the irreducible Fock representation of the CCR determined by this J—which, incidently is unitarily inequivalent to the standard one—the coherent state state peaked at Φ has the energy $\mathscr{E}_q = \int_\Sigma (\phi^2 + D^a\pi D_a\pi + \mu^2\pi^2)\, dV$, where D denotes the derivative operator induced on Σ by ∇. Clearly, $\mathscr{E}_c \neq \mathscr{E}_q$ in this representation.

or, equivalently, to

$$\gamma(\Phi, (JT - TJ)\Phi) = 0, \tag{12}$$

for all Φ in Γ. It is easy to verify that T is an infinitesimal canonical transformation on (Γ, Ω); i.e., $\Omega(T.\,,.) = -\Omega(.\,, T.)$. This property, together with the compatibility of J with Ω imply that $(J.T - T.J)$ is symmetric on the real Hilbert space obtained by the Cauchy completion of (Γ, γ). Hence, $J.T = T.J$ on Γ. Finally, since

$$\Omega(\Phi, JT\Phi) = \Omega(TJ\Phi, \Phi) = -\Omega(\Phi, TJ\Phi), \tag{13}$$

we have:

$$\Omega(\Phi, JT\Phi) = 0 \tag{14}$$

for all Φ in Γ. A similar argument shows that (14) implies (11). Thus, to obtain an irreducible Fock representation of the CCR in which coherent states satisfy the energy requirement, it is necessary and sufficient to introduce on Γ a complex structure J which is compatible with Ω and which satisfies (14). However, such complex structures were analysed in detail by Ashtekar and Magnon (1975): according to the theorem in the Appendix of this reference, there exists a unique complex structure satisfying these conditions. An explicit calculation (or, alternatively, the uniqueness result itself) implies that this J is precisely the one selected by Poincaré invariance: $J\cdot\Phi = i\Phi^+ + (-i)\Phi^-$, where, Φ^+ and Φ^- are, respectively, the positive and negative frequency parts of Φ. (Note incidently that since this J commutes with T, the expression (8) of the Hamiltonian simplifies. We have: $\mathcal{H}\cdot\Psi(h) = i\hbar\mathcal{L}_{T\cdot h}\Psi(h)$, which is the usual Schrödinger equation.)

To summarise, the only irreducible Fock representation of the CCR in which the expectation value of the quantum Hamiltonian in any coherent state equals the classical energy of the field at which that state is peaked, is the usual one in which the vacuum state is Poincaré invariant.

From a mathematical viewpoint, the crucial step in the argument is the theorem on uniqueness of complex structures proved before. However, the physical interpretation of (14) given at that time—based on the relation between classical fields and one-particle states—is incorrect. (It is curious to note that this conceptually important error came about because of an apparently minor oversight. It turns out that the energy of the *one-particle state*, defined by the classical field Φ is given by $-\langle\Phi, \Phi\rangle^{-1}\Omega(\Phi, JTJ\Phi) = \langle\Phi, \Phi\rangle^{-1}\mathcal{E}_p$, where, $\mathcal{E}_q$ is, as before, the energy of the coherent state associated with Φ! In the previous work, the normalisation factor $\langle\Phi, \Phi\rangle^{-1}$ was overlooked. We were therefore led to the incorrect requirement that the energy of the one-particle state associated with Φ should equal the classical energy $\mathcal{E}_c$.) The correct interpretation of (14), as shown here, involves the interplay between classical fields and the *coherent states*. A secondary improvement is that the present introduction of the quantum Hamiltonian—on which the energy requirement depends sensitively—is more systematic: whereas in the previous work the Hamiltonian was simply assumed to be $i\hbar\mathcal{L}_{T\cdot h}$, the present analysis uses only the more general expression (8) which can be—and has been—derived from first principles. (Note that (8) reduces to $i\hbar\mathscr{L}_{T\cdot h}$ only if $J.T=T.J$; in *general* Fock representations, the Hamiltonian is given by (8) and *not* by $i\hbar\mathscr{L}_{T\cdot h}$.)

4. Discussion

It is straightforward to extend the analysis of the previous two sections to fields with higher spins. Furthermore, zero rest mass fields can be incorporated; infrared problems do not arise. In this sense, the characterisation of the vacuum state obtained here is more general than several others available in the literature. In another sense, however, our characterisation is substantially weaker than, say, that due to Segal (1962) and Sudarshan (1963): right at the outset, we restricted ourselves to Fock representations. Although this restriction is a severe one, it is quite essential to the present approach since in a general representation, one cannot even introduce the notion of coherent states in a meaningful way. In any case, the principal goal here is *not* that of obtaining a generalisation of the various available characterisations of the Poincaré invariant vacua. Rather, we wish to draw attention to the curious fact that a simple and physically motivated requirement on the interplay between classical fields and coherent states naturally leads to a unique representation of the CCR and that the corresponding vacuum state is automatically Poincaré invariant.

The approach adopted here is also of interest because it encompasses physical systems which are not necessarily tied down to Minkowski space-time. (Compare, e.g. with Sudarshan 1963.) Since we have made use only of the presence of a time-translation isometry, the entire argument can be repeated for linear Bose fields in (globally hyperbolic) stationary space-times whose conserved energy is positive. More generally, consider a physical system which is described classically by a linear symplectic space $(\Gamma.\ \Omega)$, with Ω weakly non-degenerate. Then, one can introduce the $*$-algebra of quantum operators associated with this system. (See, e.g. Ashtekar and Magnon-Ashtekar 1980). Fix an infinitesimal canonical transformation T on Γ – i.e. a linear operator on Γ satisfying $\Omega(T.u, v) = -\Omega(u, T.v)$ for all u, v in Γ — whose generating function, $\mathscr{E}_c^T(u) = -(1/2)\ \Omega(u, T.u)$ is positive for all non-zero u in Γ. Then, using the same arguments as in the previous two sections, it is straightforward† to show that the $*$-algebra of quantum operators admits a unique irreducible Fock representation $(\mathbf{H}, \Lambda)$ in which the expectation value $\mathscr{E}_q^T := \langle\!\langle \Psi_u, \mathscr{H}\ \Psi_u \rangle\!\rangle$ of the quantum Hamiltonian operator $\mathscr{H}$ on H (defined by (8)) in any coherent state Ψ_u equals $\mathscr{E}_c^T$, for all u in Γ where, $\langle\!\langle\,,\rangle\!\rangle$ denotes the inner-product on $\mathbf{H}$. This result is useful in the quantum theory of fields on space-times which are not necessarily stationary: although an exact time-translation isometry may not be available, a preferred canonical transformation T is often made available by the presence of a conformal temporal isometry in the cosmological contexts and by asymptotic iso-

†Actually, T must also satisfy the following technical condition: On the real Hilbert space obtained by the Cauchy completion of $(\Gamma, (\cdot\,,\cdot) := \Omega(T\cdot\,, \cdot))$, the operator T with $D(T) = \Gamma$ must be essentially self-adjoint. With the choice of Γ as in § 2, this condition is automatically satisfied by linear Bose fields in globally hyperbolic stationary space-times.

metries in the asymptotically flat contexts.[††] The result may also be relevant to statistical mechanics.

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References

Ashtekar A and Magnon A 1975 *Proc. R. Soc. (London)* **A346** 375
Ashtekar A and Magnon-Ashtekar A 1980 *J. Gen. Rel. Grav.* (in press)
Bargmann V 1962 *Proc. Natl. Acad. Sci.* **48** 199
Choquet-Bruhat Y, DeWitt-Morette C and Dillard-Bleick M 1977 *Analysis, manifolds and physics* (Amsterdam: North Holland) Ch. 7. D
Glauber R J 1963 *Phys. Rev.* **130** 2539
Glauber R J 1963 *Phys. Rev.* **131** 2766
Lichnerowicz A 1961 Propagateurs et Commutateurs en Relativite generale; IHES Publication No. 10 (Bures sur Yvette)
Segal I E 1962 *Ill. J. Math.* **6** 500
Segal I E 1963 *Mathematical problems in relativistic physics* (Providence: Am. Math. Soc.)
Segal I E 1967 in *Carges lecture notes in theoretical physics* (ed.) F. Lurcat, (New York: Gordon & Breach)
Sudarshan E C G 1962 *J. Math. Phys.* **4** 1029
Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277

††In general, different choices of T lead to distinct complex structures. Two complex structures, J_1 and J_2 lead to unitarily equivalent representations of the CCR if and only if (J_1-J_2) is a Hilbert-Schmidt operator on the real Hilbert space obtained by Cauchy completion of (Γ, γ_1) or, of (Γ, γ_2). For details, as well as for applications of these results to the S-matrix theory of linear Bose fields in curved space-times, see Ashtekar and Magnon-Ashtekar (1980).