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Relativistic star clusters with high central redshift

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Abstract. Collisionless star clusters in dynamical equilibrium are of current interest in general relativity and astrophysics. A step-function distribution is chosen for star clusters. The corresponding equation of state is analogous to a Fermi-gas equation. These clusters are found to be pulsationally unstable for a central redshift of $Z_c \geqslant 0{\cdot}54$. Further, a model of clusters is developed in which the core has an extremely relativistic equation of state. These structures are unstable for $Z_c \geqslant 2{\cdot}55$ when we use Chandrasekhar's technique to study their pulsational stability.

Keywords. Star clusters; general relativity; redshift; Fermigas equation; field equations.

1. Introduction

The normal astrophysical evolution of a galactic nucleus is estimated (Sanders 1970) to lead under certain circumstances to a star cluster so dense that general relativity influences its structure and evolution. Hoyle and Fowler (1967) proposed relativistic cluster as idealized model for quasars. The theory of static, collisionless, relativistic star cluster is constructed along the same line as the corresponding Newtonian theory: A distribution function is constructed from the first integral of the motion of a 'particle', so that this distribution function automatically satisfies the kinetic equation. The Poisson equation in Newtonian case is replaced by Einstein equations relating to the curvature of stress-energy tensor of the distribution of particles. However, collisionless systems of particles cannot give rise to every conceivable equation of state (Zeldovich and Novikov 1971).

For a Newtonian star cluster if we consider the distribution function, $F(E) \sim \exp(-E/\theta)$, we find $P=\theta\rho$ (analogue of an isothermal, ideal gas). For a step-function, $[F(E)=\text{constant},\ E<E_0$ and $F(E)=0,\ E>E_0]$ we obtain $P=\text{constant}\ \rho^{5/3}$ (analogue of a Fermigas). Fackerell (1968) has given a method to obtain distribution function $F(x)$ for a relativistic spherical configuration with isotropic pressure and given internal solutions (that is, known values of ρ, e^{ν}, e^{λ} and P). Alternatively, if the distribution function is given, one can obtain the equation of state. In this paper we have chosen a step-function for the distribution function and obtained the equation of state. This equation of state is analogous to a Fermi-gas and for lower values of P/ρ the behaviour of the equation of state is similar to that of a polytropic gas of index $n=1{\cdot}5$. Further, some equations of state have been obtained which behave like polytropic gas of index n under Newtonian approximation. The distribution function of these new equations of state are positive, that is $F(x)\geqslant 0$ and $\partial F(x)\ /\ \partial x \leqslant 0$.

Ipser (1969) and Ipser and Thorne (1968) analysed the dynamical stability of relativistic clusters. According to them a bounded spherical cluster with isotropic pressure and $\partial F(x) / \partial x \leqslant 0$ is stable against collisionless, radial perturbation if the gas sphere which has the same radial pressure is stable against radial perturbation. Ipser (1969) showed that the clusters are unstable against collisionless radial perturbation for $Z_c \geqslant 0{\cdot}621$, while Fackerell (1970) obtained $Z_c \leqslant 0{\cdot}73$ by considering an extreme ratio of core-halo density distribution and a polytrope with index $n=4$. In this paper, we have used Chandrasekhar's method (1964) to obtain the limiting value of central redshift for onset of instability and obtained a value of $Z_c \leqslant 0{\cdot}54$; however, with a two-density distribution of extreme relativistic core the onset of instability occurs for $Z_c \geqslant 2{\cdot}55$. More rigorous methods (Bardeen *et al* 1966) may lead to some lower value of Z_c.

2. Field equations and equations of state

2.1 *Field equations*

The general assumptions made for solving the Einstein's field equations are the same as given by Bondi (1964). For a spherically symmetric and static metric

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2. \tag{1}$$

Here ν and λ are functions of r alone. The resulting field equations are

$$\begin{aligned} 8\pi P &= e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2, \\ 8\pi P &= e^{-\lambda}[\nu''/2 + \nu'^2/4 - \lambda'\nu'/4 + (\nu' - \lambda')/2r], \\ 8\pi\rho &= e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2, \end{aligned} \tag{2}$$

where primes denote differentiation with respect to r. These equations can be simplified to give

$$P' = -\nu'(P+\rho)/2, \tag{3}$$

$$P' = -(P+\rho)\,[m + 4\pi P r^3]/r(r-2m), \tag{4}$$

where
$$m(r) = \int_0^r 4\pi\rho r^2 dr \tag{5}$$

2.2 *Physical limitations*

The equations of state and the solutions of field equations are expected to follow one or both of the following restrictions on pressure and density:

(i) The trace of energy-momentum tensor is non-negative. For this (Bondi 1964)

$$3P \leqslant \rho. \tag{6}$$

(ii) The speed of sound nowhere exceeds the speed of light (Zeldovich 1962), that is

$$dP/d\rho \leqslant 1. \tag{7}$$

The suitability of equation of state will be discussed in view of these physical restrictions.

2.3 *Step distribution function and the corresponding equation of state*

Fackerell (1968) has shown that

$$\int_b^\infty F(x)\,(x - b)^{3/2}\,dx = G(b) \tag{8}$$

where $F(x)$ = distribution function, $G(b) = Pb^2$ and $b = e^\nu$. Since $P = 0$ for $r \geqslant a$ (where a = radius of the entire configuration), $G(b)$ vanishes for $b \geqslant \beta$ (where, $\beta = 1 - 2m/a$, is the value of b at the surface $r = a$) and as step function $F(x)$ = constant for $b \leqslant x \leqslant \beta$ and $F(x) = 0$ for $b > \beta$, we obviously obtain

$$\int_b^\beta F(x)\,(x - b)^{3/2}\,dx = G(b), \tag{9}$$

or $$G(b) = \text{constant}\,(\beta - b)^{5/2},$$

and $$P = \text{constant} \times b^{-2}\,(\beta - b)^{5/2}. \tag{10}$$

Expressing b in terms of β and pressure in the units which make the resulting constant equal to unity, we obtain

$$P = b^{-2}\,(1 - b)^{5/2}. \tag{11}$$

Equations (3) and (11) yield

$$\rho = b^{-2}\,(1 - b)^{3/2}\,(3 + 2b), \tag{12}$$

and $$P = (5)^{5/3}\,\rho^{5/3}\,(1 - 3P/\rho)^{4/3}\,(1 + 2P/\rho)^{-5/3}. \tag{13}$$

2.4 *Nature of the equation of state*

For a step-function distribution in a collisionless cluster, the equation of state is given by equation (13). In Newtonian approximation P/ρ is small and (13) reduces to (Fermi-gas equation)

$$P \propto \rho^{5/3}. \tag{14}$$

This is also the equation of state for a polytrope of index, $n=1\cdot5$. On comparing (13) with the relativistic Fermi-gas equation (Oppenheimer and Volkoff 1939)

$$\rho = K(\sinh t - t), \tag{15}$$

$$P = K/3\,(\sinh t - 8\sinh t/2 + 3t),$$

we see that

(i) at lower values of P/ρ, both (13) and (15) lead to $P \propto \rho^{5/3}$ which is the equation of a polytropic gas with index 1·5.

(ii) For both the equations of state the maximum attainable value of P/ρ is 1/3. Hence, the trace of energy-momentum tensor will always be positive.

(iii) The maximum value for speed of propagation of any signal is $c/\sqrt{3}$.

Thus we see that the equation of state obtained for a collisionless cluster with a step-function distribution is very much similar to the equation of a Fermi-gas. For the values of P/ρ used in this paper equation (13) resembles very much a polytropic gas with index 1·5. The advantages of equation (13) over a polytropic equation are (i) From equations (11) and (12), $b \to 0$ (and $Z \to \infty$) as $P/\rho \to 1/3$. Hence, without violating any of the physical restrictions [equations (6) and (7)], one can obtain arbitrarily large redshift. This is not possible for a polytropic gas sphere. (ii) For any value of central redshift $F(x) \geqslant 0$ and $F'(x)=0$. But for polytropes with index 1·5, the value of $F(x) \geqslant 0$ only when $dP/d\rho \geqslant 1$ (this corresponds to $Z_c \leqslant 3\cdot46$) and throughout the structure $F'(x)$ is not $\leqslant 0$.

In this paper, the solution of equation (13) has been discussed in detail. Also, we have used equation (13) in the envelope of a two-density structure with extreme relativistic core to avoid getting a negative value of $F(x)$ near the surface (Das 1976).

3. Some new solutions and their distribution functions

Fackerell (1968) has given a method to obtain distribution function $F(x)$ for a spherical configuration with isotropic pressure and given internal solutions (that is, known values of ρ, e^{ν}, e^{λ} and P). The distribution function is given by

$$F(x) = -(4/3\pi)\int_x^{\beta} G'''(b)\,(b-x)^{-1/2}db + (1/\pi)G(\beta)\,(\beta-x)^{-5/2}H(\beta-x)$$

$$+ (2/3\pi)G'(\beta)\,(\beta-x)^{-3/2}\,H(\beta-x) + (4/3\pi)\,G''(\beta)\,(\beta-x)^{-1/2} \tag{16}$$

Here primes denote differentiation with respect to b and $H(\beta-x)$ is Heavyside unit function, equal to 1 for $x \leqslant \beta$. Fackerell (1968) further showed that for polytropic gas spheres $F(x) \geqslant 0$ if $dP/d\rho \leqslant 1$ but $F'(x)$ is not $\leqslant 0$ for the entire domain of positive $F(x)$. The values of b/β for which both the conditions are satisfied in polytropes are: $F'(x)$ is not $\leqslant 0$ for $n=1$ and 1·5; $F(x) \geqslant 0$ and $F'(x) \leqslant 0$ for $x \geqslant 0\cdot78$, 0·62 and 0·48 for $n=2\cdot0$, 2·5 and 3·0 respectively.

We can obtain such equations of state which behave like polytropic equations for lower values of P/ρ and which satisfy the conditions $F(x) \geqslant 0$ and $F'(x) \leqslant 0$ for a larger range of central redshifts. For this, we choose a general relation

$$P = b^m (1-b)^{n+1}$$

[Equation (13) is a particular case of this with $m = -2$ and $n = 1{\cdot}5$]. Using equations (3) and (17), we get

$$\rho = b^m (1-b)^n [\{2(n+m)+3\} b-(2m+1)], \tag{18}$$

or

$$P = \frac{\rho^{1+1/n}}{[2(n+1)]^{1+1/n}} \left[1+\frac{(2m+1)P}{\rho}\right]^{-m/n} [1+\{2(n+m)+3\}P/\rho]^{1+(m+1/n)}. \tag{19}$$

For lower values of P/ρ, equation (19) behaves like a polytrope of index $n(P \propto \rho^{1+1/n})$. The physical limitations and the nature of the distribution function $F(x)$ for different m and n values have been discussed below

(i) *For* $m = -2$: The values of P/ρ and $dP/d\rho$ are given by

$$P/\rho = (1-b) / [(2n-1)b+3],$$

and

$$dP/d\rho = (1-b)[(n-1)b+2] / [(n-1)(2n-1)b^2+(5n-7)b+6]. \tag{20}$$

The maximum values of P/ρ and $dP/d\rho$ can be 1/3 when $b \to 0$. Thus, the equation of state satisfies both the restrictions [equations (6) and (7)] for arbitrarily large redshifts. Further, $G(\beta)=G'(\beta)=G''(\beta)=0$

and

$$G'''(b) = -n(n-1)(n+1)(1-b)^{n-2} \text{ for } n > 1{\cdot}5. \tag{21}$$

Equations (16) and (21) give

$$F(x) = \frac{2}{3\sqrt{\pi}} \frac{\Gamma(n+2)}{\Gamma(n-1/2)} (\beta-x)^{n-3/2},$$

$$F'(x) = -\frac{2}{3\sqrt{\pi}} \frac{\Gamma(n+2)}{\Gamma(n-3/2)} (\beta-x)^{n-5/2}. \tag{22}$$

For $n \geqslant 1{\cdot}5$, we get $F(x) \geqslant 0$ and $F'(x) \leqslant 0$ for all the values of $x \leqslant \beta$.

(ii) *For* $m = -1$: We have

$$P/\rho = (1-b) /[(2n+1)b+1],$$

and

$$dP/d\rho = (1-b)(1+nb) / [n(2n+1)b^2+(n-1)b+1]. \tag{23}$$

As $b \to 0$, both P/ρ and $dP/d\rho$ approach to unity. Thus for the entire range of b we completely fulfil only one restriction [equation (7)]. Further

$$F(x) = \frac{2}{3\sqrt{\pi}} \frac{\Gamma(n+2)}{\Gamma(n+1/2)} (\beta-x)^{n-3/2} [(n-1/2)\beta-(n+2)(\beta-x)]$$

$$F'(x) = -\frac{2}{3\sqrt{\pi}} \frac{\Gamma(n+2)}{\Gamma(n-1/2)} (\beta-x)^{n-5/2} [(n-3/2)\beta-(n+2)(\beta-x)] \tag{24}$$

The condition that $F(x)$ is positive and $F'(x)$ is $\leqslant 0$ is satisfied if

$$x \geqslant 7/2(n+2).$$

(iii) *For* $m=0$: We have

$$P/\rho = (1-b) \,/\, [(2n+3)b-1],$$

and
$$dP/d = (1-b) \,/\, [(2n+3)b-3] \tag{25}$$

If $dP/d\rho \leqslant 1$, we get $b \geqslant 2/(n+3)$ and $P/\rho = n/(3n+4)$. We see that $P/\rho \leqslant 1/3$ for any value of n. Thus both the restrictions [equations (6) and (7)] are satisfied simultaneously. Further,

$$F(x) = \frac{2}{3\sqrt{\pi}} \frac{\Gamma(n+2)}{\Gamma(n-1/2)} [(2n+1)(2n-1)\beta^2 - 4(2n+1)(n+2)\beta(\beta-x)$$

$$+4(n+2)(n+3)(\beta-x)^2],$$

$$F'(x) = -\frac{2}{3\sqrt{\pi}} \frac{\Gamma(n+2)}{\Gamma(n-3/2)} [(2n-1)(2n-3)\beta^2 - 4(2n-1)(n+2)\beta(\beta-x)$$

$$+4(n+2)(n+3)(\beta-x)^2]. \tag{26}$$

The minimum values of $\chi = (x/\beta)$ for which $F(x) \geqslant 0$ and $F'(x) \leqslant 0$ for each value of n are shown in table 1.

For $m=-2$, $F(x) \geqslant 0$ and $F'(x) \leqslant 0$ for the entire range of redshifts and both the restrictions on pressure and density [equations (6) and (7)] are followed, but the equations of state with $m=0$, -1, etc. do not have any perceptible advantage over the polytropic equation of state. The later equations thus have only a mathematical significance.

However, we have dealt the cases with $m=-2$ in detail and particularly the equation of state (13) which corresponds to a cluster structure with step-function distribution.

Table 1. Values of $\chi = x/\beta$ for which $F(x) = +$ ve and $F'(x) = -$ ve.

n	χ	n	χ
2	0·9291	11	0·3701
3	0·8038	12	0·3460
4	0·7141	13	0·3255
5	0·6250	14	0·30697
6	0·51125	15	0·2904
7	0·5090	16	0·2756
8	0·4655	17	0·2620
9	0·4287	18	0·2500
10	0·3971	19	0·2389

4. Stability of structures and central redshift

It was shown in §2 that the step distribution function fulfils both the restrictions on pressure and density [equations (6) and (7)] for arbitrarily large central redshifts. However, these structures may not be stable under radial perturbations. The criterion for stability of a relativistic cluster was given by Ipser and Thorne (1968): A bounded spherical relativistic cluster with isotropic pressure and with $\partial F/\partial x \leqslant 0$ is stable against collisionless, radial perturbation if the gas sphere which has the same radial distribution of density and pressure is stable against radial perturbation for which the adiabatic index is given by

$$\Gamma_1 = (dP/d\rho)\,(P+\rho)/P\,|_{\text{constant entropy}}. \tag{27}$$

In this paper, we have chosen only those solutions in which $F(x)$ is positive and $\partial F/\partial x \leqslant 0$. The pulsational stability of these structures is determined by using the method given by Chandrasekhar (1964) and Harrison *et al* (1965). According to this method a spherical configuration is pulsationally stable if the integral Ω is positive, that is

$$\Omega = \int_0^a [e^{(\lambda+3\nu)/2}\{9(P+\rho)\,(dP/d\rho) + 4r\,(dP/dr) - r^2\,(dP/dr)^2/(P+\rho)\} + 8\pi\, e^{3(\lambda+\nu)/2}\, P(P+\rho)\, r^2]\, r^2\, dr \geqslant 0. \tag{28}$$

The square of pulsational frequency is obtained by dividing by the following integral T;

$$T = \int_0^a \exp\,[(3\lambda+\nu)/2]\,(P+\rho)\, r^4\, dr. \tag{29}$$

For equations of state with $m=0$ and -1 we do not get any significantly important values of the central redshift, if we wish to satisfy all the restrictions on the solutions, *viz.* (i) $F(x) \geqslant 0$, (ii) $F'(x) \leqslant 0$ and (iii) $\Omega \geqslant 0$. However, we get a central redshift $Z_c \leqslant 0{\cdot}54$ for the case when $m=-2$ and $n=1{\cdot}5$ (the equation of state for step distribution function).

The above method to check the pulsational stability is sufficient to predict the onset of instability. However, it is not rigorous enough for making a categorical statement about stability. It is quite probable that the conclusions regarding stability may change when more rigorous techniques (Bardeen *et al* 1966) for detailed calculations of the eigenfunctions are used. However, such a calculation is beyond the scope of this paper.

5. Solutions of the field equations

5.1 *Solution for equation with m=−2*

For $m=-2$ the equation of state is given by

$$P = b^{-2}\,(1-b)^{n+1},$$
$$\rho = b^{-2}\,(1-b)^n \cdot [(2n-1)\,b+3]. \tag{30}$$

Using the coupled equations (4), (5) and (30), we can numerically find the solutions of the spherical configuration corresponding to different values of n. At the centre, that is at $r=0$, we have the initial values of $m(0)=0$, $e^{-\lambda}=1$ and $b=b_0$ (some assigned value). The solutions are carried till pressure becomes zero. The value of r where pressure vanishes corresponds to the radius a of the spherical configuration. Within physically reasonable limits we can get any central redshift if the pulsational stability is not taken into account. However, if we wish to keep $\Omega \geqslant 0$ the limiting value of Z_c comes out to be $Z_c \leqslant 0{\cdot}54$ for $n=1{\cdot}5$ and $Z_c=0{\cdot}37$ for $n=2{\cdot}0$ (figures 1 and 2).

Further, it is to be remembered that the right side is dimensionless and $b=e^{\nu}/e^{\nu_a}$; P and ρ are in some arbitrary units which can be adjusted by assigning some value to the central density, ρ_c,

$$\rho_c = b_0^{-2} (1-b_0)^n [(2n-1)b_0+3]. \tag{31}$$

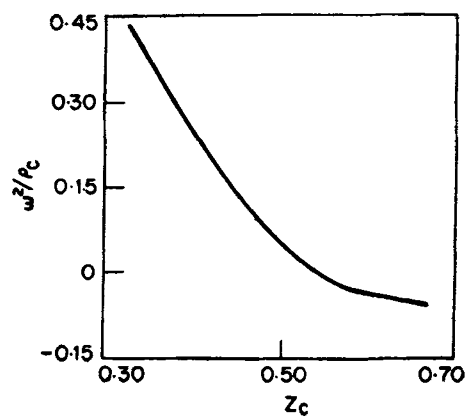

Figure 1. Variation of ω^2/ρ_c with Z_c for $n=1{\cdot}5$

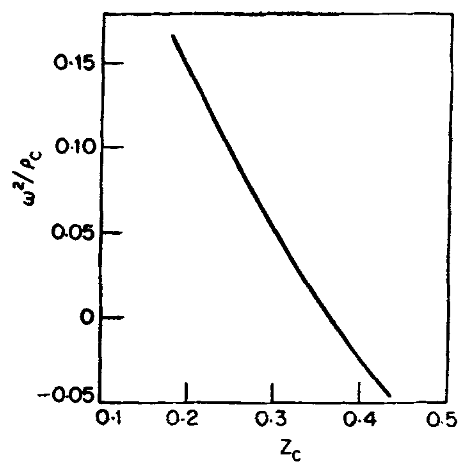

Figure 2. Variation of ω^2/ρ_c with Z_c for $n=2{\cdot}0$.

5.2 *A two-density distribution*

In order to obtain a higher value of central redshift for the onset of instability we have chosen a two-density distribution of matter. We have made use of Bondi's method (1964) for constructing a model with an extreme relativistic core. The difference from Bondi's work lies in choosing the equation of state of the core. Bondi had chosen a core equation with $P=\rho$, while we have used the core equation as $dP/d\rho=1$. Because of the continuity of P, ρ, e^{ν} and e^{λ} at the core-envelope boundary we have $F(x)=F'(x)=0$ for the core as $G'''(b)=0$ for $dP/d\rho=1$. Hence for the entire structure we may write

$$F(x)=0, \qquad 0\leqslant r\leqslant r_b$$

$$F(x)=\frac{2}{3\sqrt{\pi}}\frac{\Gamma(n+2)}{\Gamma(n-1/2)}(\beta-x)^{n-3/2} \quad r_b\leqslant r\leqslant a,$$

where r_b is the radius of the core.

5.3 *Method for solving core-envelope equation*

The core-envelope boundary is obtained when (Bondi 1964)

$$H=2v-(u^2+v^2+6uv)=0, \tag{32}$$

where $v=4\pi Pr^2$, $u=m(r)/r$.

For $dP/d\rho=1$,

$$P=\rho-\alpha, \tag{33}$$

where α is a constant. The coupled equations (4), (5) and (33) are solved for different values of α. Figure 3 shows (u, v) tracks for various α values. Beyond the curve $H=0$, the $(u-v)$ tracks follow the equation of state (30) with $n=1{\cdot}5$.

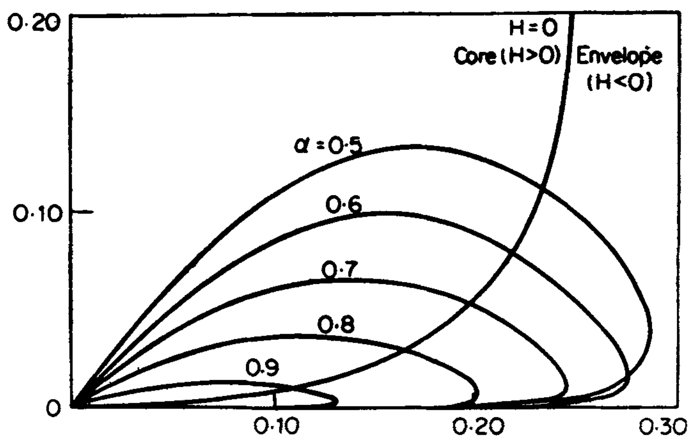

Figure 3. $(u-v)$ tracks for extremely relativistic core.

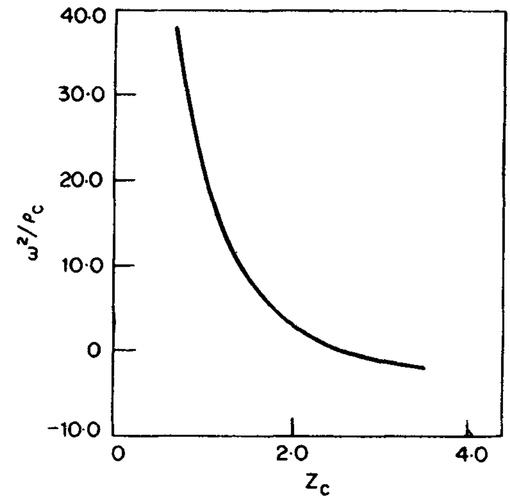

Figure 4. Variation of ω^2/ρ_c with Z_c for extremely relativistic core.

For this model $F(x) \geqslant 0$ and $F'(x) = 0$ throughout the configuration. There is an onset of instability for $Z_c \geqslant 2{\cdot}55$. Figure 4 shows the variation of ω^2/ρ_c with Z_c for this model.

6. Results and discussion

In this paper we have used a step-function distribution for relativistic star clusters. The equation of state for such a cluster is analogous to Fermi-gas equation or a polytropic gas with index 1·5. For central redshifts $Z_c \geqslant 0{\cdot}54$ these structures are found to be unstable. When this distribution of cluster is used in the envelope of a spherical configuration with an extremely relativistic core ($dP/d\rho = 1$) the onset of instability is obtained for $Z_c \geqslant 2{\cdot}55$. Further, we have obtained a set of equations of state which behave like polytropic gas equations but for which the value of $F(x) \geqslant 0$ and $F'(x) \leqslant 0$ for a wider range of x.

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