Pramāṇa, Vol. 15, No. 1, July 1980, pp. 45–51. © Printed in India.

On the joint eigenvalue distribution for the matrix ensembles with non zero mean

NAZAKAT ULLAH
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005, India

MS received 6 September 1979

Abstract. Exact distributions are given for the two-dimensional case when the mean of the off-diagonal element is non-zero. The joint eigenvalue distribution for the N dimensional case, derived using the volume element in the space of $N \times N$ orthogonal matrices, is checked by rederiving the exact results for $N=2$. The smooth nature of the N-dimensional joint distribution supports the claim of the method of moments that the single eigenvalue distribution is a smooth function of the ratio of mean-to-mean square deviation.

Keywords. Randon matrices; new matrix ensembles.

1. Introduction

The random matrix ensembles have been successfully used in the past (Mehta 1967) to study the average properties of the excited states in compound nucleus. Because of the random sign rule for the off-diagonal elements, the mean value of the off-diagonal elements was always taken to be zero in these studies. Later (Nazakat Ullah *et al* 1970) it was realised that for a satisfactory description of the nuclear configuration interaction problem new ensembles have to be introduced in which the mean value of the off-diagonal elements is not zero. Because of the complexity of the problem a perturbative treatment was carried out. Recently (Edwards and Jones 1976; Kota and Potbhare 1977) there has been considerable interest in the study of single eigenvalue distribution of a random matrix ensemble each element of which has a Gaussian distribution and non-zero mean. The results obtained by Kota and Potbhare (1977) using the method of moments are very different than the ones given by Edwards and Jones (1976) using what is known as $n \to 0$ trick. The main result which one is interested in is to see whether the single eigenvalue distribution is a smooth function for all values of the ratio of mean-to-mean square deviation or has some kind of discontinuity in it. In a private communication Parikh (1979) has also expressed the opinion that the method of moments argument leads one to believe that the single eigenvalue has a smooth distribution. The purpose of the present work is to give an exact joint eigenvalue distribution and see what can be predicted about the nature of single eigenvalue distribution from it.

We shall first derive the joint distribution for the two-dimensional cases in the next section. This will serve the purpose of checking the general derivation which is given in §3. In §4 we apply general formalism to the three dimensional case. Concluding remarks are presented in §5.

2. Joint distribution in two dimensions

Let us consider a real symmetric 2×2 matrix having the following distribution

$$P(\{H_{\mu\nu}\}) = K\exp\left(-\frac{1}{4\sigma^2}\left[\sum_{\mu=1}^{2}(H_{\mu\mu}-\lambda)^2+2(H_{12}-\lambda)^2\right]\right), \tag{1}$$

where λ is the mean value and σ is the variance of the off-diagonal element. We have used a slightly different distribution of $H_{\mu\nu}$ than the one used by Edwards and Jones (1976). In our distribution the variance of the diagonal element is twice that of the off-diagonal element. We have done this in order that our distribution goes over to the earlier distribution (Mehta 1967) when $\lambda=0$ for all values of the dimension of the matrix H.

The mean value λ of all the matrix elements is taken to be the same (Edwards and Jones 1976; Nazakat Ullah *et al* 1970). This is the simplest description of the actual physical situation and is almost the same what one does in the mean-field theory. In the many-nucleon system further support for this simplifying aspect of the problem comes from the diagonalisation of large shell-model matrices.

In order to get the joint eigenvalue distribution directly from the distribution of matrix elements of H, we introduce the following two variables

$$u = H_{11} + H_{22}, \tag{2a}$$

$$v = H_{11}H_{22} - H_{12}^2. \tag{2b}$$

According to the theory of probability (Kendall 1945) the unnormalised joint distribution of u, v using the distribution (1) is given by

$$P(u, v) = K\int\delta[u-(H_{11}+H_{22})]\,\delta[v-(H_{11}H_{22}-H_{12}^2)]$$

$$\exp\left\{-\frac{1}{4\sigma^2}\left[\sum_{\mu=1}^{2}(H_{\mu\mu}-\lambda)^2+2(H_{12}-\lambda)^2\right]\right\}\prod_{\mu\leqslant\nu=1}^{2} dH_{\mu}. \tag{3}$$

Carrying out the integrations, we get

$$P(u, v) = K\exp\left(-\frac{1}{4\sigma^2}[u^2-2\lambda u-2v]\right)I_0\left(\frac{\lambda}{2\sigma^2}\sqrt{u^2-4v}\right), \tag{4}$$

where K now denotes the appropriate normalisation constant.

From the theory of matrices we know that the variable u and v are related to the eigenvalues E_1, E_2 in the following way:

$$u = E_1 + E_2, \tag{5a}$$

$$v = E_1E_2. \tag{5b}$$

Using expressions (4), (5) and the Jacobian of the transformation (Kendall 1945) from the variables u, v to E_1, E_2 we obtain the following expression for the joint eigenvalue distribution of H

$$P(E_1, E_2) = K$$

$$\exp\left(-\frac{1}{4\sigma^2}[E_1^2+E_2^2-2\lambda(E_1+E_2)] \mid E_1-E_2 \mid I_0\left(\frac{\lambda}{2\sigma^2}(E_1-E_2)\right)\right), \quad (6)$$

where K is fixed by the normalisation condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(E_1, E_2)dE_1\, dE_2 = 1,$$

and I_0 is the modified Bessel function (Abramowitz and Stegun 1965).

Comparing the distribution (6) with the earlier distribution when $\lambda = 0$, we find that a non-zero mean value for all the elements of H has given rise to the additional factor I_0 apart from shifting each eigenvalue by λ.

From the joint distribution (6) it is easy to see that the spacing distribution $P(S)$, $S = \mid E_1-E_2 \mid$ for the non-zero mean ensemble is given by

$$P(S) = (4\sigma^2)^{-1}\left[\exp\left(-\frac{\lambda^2}{2\sigma^2}\right)\right] S \exp\left(-\frac{S^2}{8\sigma^2}\right) I_0\left(\frac{\lambda}{2\sigma^2}S\right). \quad (7)$$

Next integrating out one of the eigenvalues in (5) we get the following expression for the distribution of single eigenvalue

$$P(E) = K\exp\left(-\frac{1}{2\sigma^2}(E^2-2\lambda E)\right)\int_0^{\infty} dy\, y\exp\left(-\frac{y^2}{4\sigma^2}\right)$$

$$I_0\left(\frac{\lambda}{2\sigma^2}y\right)\cosh\left[\frac{(E-\lambda)}{2\sigma^2}y\right]. \quad (8)$$

The integral

$$G = \int_0^{\infty} dy\; y\exp\left(-\frac{y^2}{4\sigma^2}\right) I_0\left(\frac{\lambda}{2\sigma}y\right)\cosh\left[\frac{(E-\lambda)}{2\sigma^2}y\right], \quad (9)$$

in expression (8) can be integrated in the form of the series using the expansion of I_0

$$G = \sum_{k=0}^{\infty}\frac{(2k+1)!}{(k!)^2\, 2^{2k+2}}(\lambda/2\sigma^2)^{2k}(4\sigma^2)^{k+1}\exp\left(\frac{1}{8}\frac{(E-\lambda)^2}{\sigma^2}\right)$$

$$\left[U\left(2k+\frac{3}{2}, \frac{E-\lambda}{\sqrt{2}\sigma}\right)+U\left(2k+\frac{3}{2}, -\frac{E-\lambda}{\sqrt{2}\sigma}\right)\right], \quad (10)$$

where U are the functions related to parabolic cylinder functions and tabulated by Abramowitz and Stegun (1965).

3. General distribution

It is obvious that the method of §2 cannot be generalised easily to N dimensions. We now give a prescription for finding the N-dimensional joint distribution.

Let us consider a real symmetric $N\times N$ matrix H having the following distribution

$$P(\{H_{\mu\nu}\}) = K \exp\left\{-\frac{1}{4\sigma^2}\left[\sum_{\mu=1}^{N} (H_{\mu\mu}-\lambda)^2+2\sum_{\mu<\nu=1}^{N} (H_{\mu\nu}-\lambda)^2\right]\right\}, \tag{11}$$

The matrix H is related to the eigenvalues E_k and orthogonal matrix T by the following relation

$$H = \tilde{T} E T. \tag{12}$$

The eigenvalue distribution is obtained by integrating out over T. Using (11) and (12) this is given by

$$P(\{E_\mu\}) = K \exp\left[-\frac{1}{4\sigma^2}\sum_{\mu=1} (E_\mu^2 - 2\lambda)\, E_\mu)\right] \prod_{\mu<\nu} |E_\mu - E_\nu|$$

$$\int \exp\left[\frac{\lambda}{\sigma^2}\sum_k E_k\left(\sum_{\mu<\nu} T_{k\mu}\, T_{k\nu}\right)\right] dT, \tag{13}$$

The factor $\prod_{\mu<\nu} |E_\mu - E_\nu|$ arises from the Jacobian of the transformation (12) and dT denotes the volume element in the space of $N\times N$ orthogonal matrices. Using δ functions (Nazakat Ullah 1964) this can be written as

$$dT = \prod_j \delta\left(\sum_i T_{ij}^2 - 1\right) \prod_{j<k} \delta\left(\sum_i T_{ij}\, T_{ik}\right) \prod dT_{ij}. \tag{14}$$

Equivalently this can also be written as

$$dT = \Pi_i\, \delta\left(\sum\nolimits_j T_{ij}^2 - 1\right) \prod_{i<k} \delta\left(\sum\nolimits_j T_{ij}\, T_{kj}\right) \prod dT_{ij}. \tag{15}$$

Using (14) the integral in (13) can be written as

$$\int \exp \frac{\lambda}{\sigma^2}\left(\sum_{K=1}^{N-1} (E_k - E_N) \sum_{i<j} T_{ki}\, T_{kj}\right) dT. \tag{16}$$

Now since

$$\sum_{i<j} T_{ki} T_{kj} = \frac{1}{2}\left[\left(\sum_i T_{ki}\right)^2 - \sum_i T_{ki}^2\right],$$

Using (15), (16) can be written as

$$\exp\left(-\frac{\lambda}{2\sigma^2}\right)\left[\sum_{k=1}^{N-1} E_k - (N-1)\, E_N\right] \int \exp \frac{\lambda}{2\sigma^2} \sum_{k=1}^{N-1} (E_k - E_N)$$

$$\left(\sum_j T_{kl}\right)^2 dT. \tag{17}$$

We now apply an orthogonal transformation to T

$$T' = R\, T, \tag{18}$$

and choose the elements in the first row of R as $1/\sqrt{N}$. This gives $\Sigma_l\, T_{kl} = \sqrt{N}\, T'_{k1}$ and since under an orthogonal transformation the volume element (14) or (15) remains invariant, expression (17) can be rewritten as

$$\exp\left[-\frac{\lambda}{2\sigma^2}\left(\sum_{k=1}^{N-1} E_k - (N-1)\, E_N\right)\right] \exp\left[\frac{\lambda_N}{2\sigma^2} \sum_{k=1}^{N-1} (E_k - E_N)\, T'^2_{k1}\right] dT. \tag{19}$$

Since in the exponent only the elements of the first column appear we can formally integrate out over the remaining $N-1$ columns and get the following expression for the joint probability of the eigenvalues

$$P(\{E_\mu\}) = K \exp\left[-\frac{1}{4\sigma^2}\sum_{\mu=1}^{N} (E_\mu^2 - 2\lambda\, E_\mu)\right] \prod_{\mu<\nu} |E_\mu - E_\nu|$$

$$\exp\left\{-\frac{\lambda}{2\sigma^2}\left[\sum_{k=1}^{N-1} E_k - (N-1)\, E_N\right]\right\} \int \exp(\lambda_N/2\sigma^2)$$

$$\sum_{k=1}^{N-1} (E_k - E_N)\, T'^2_{k1}\, \delta\left(\sum_k T'^2_{k1} - 1\right) \prod_k dT'_{k1}. \tag{20}$$

Integrating over T'_{N1} we can also write the joint distribution as

$$P(\{E_\mu\}) = K \exp\left[-\frac{1}{4\sigma^2} \sum_{\mu=1}^{N} (E_\mu^2 - 2\lambda\, E_\mu)\right] \prod_{\mu<\nu} |E_\mu - E_\nu|$$

$$\exp -\frac{\lambda}{2\sigma^2}\left(\sum_{k=1}^{N-1} E_k - (N-1)\, E_N\right)$$

$$\int \exp \frac{\lambda N}{2\sigma^2} \sum_{k=1}^{N-1} (E_k - E_N)\, T'^2_{k1}\left[1 - \sum_{k=1}^{N-1} T'^2_{k1}\right]^{-1/2} \prod_{k=1}^{N-1} dT'_{k1}, \tag{21}$$

where the integration is to be carried out over the region

$$0 \leqslant \sum_{k=1}^{N-1} T_{k1}'^2 \leqslant 1$$

and K is the appropriate normalisation constant. Expression (21) is the desired expression for the joint distribution of E_μ. We see that the presence of λ gives rise to an additional factor which is fairly complicated. But free from any discontinuity for all values of λ/σ. Further integration over the remaining variables of T is not easy. However for $N=3$, it can be done fairly easily. This is done in the next section. We also note from expression (21) that to get the single eigenvalue distribution the integration over $N-1$ eigenvalues is also quite involved for general N and can be done explicitly for small values of N only. However as mentioned in §1 our main interest was to see whether the single eigenvalue distribution is a smooth function of λ/σ or has some discontinuity for some value of λ/σ. It is fairly obvious from the structure of expression (21) and also the explicit integrations for $N=2$, 3 that further integration over $N-1$ eigenvalues cannot give rise to any discontinuity in the single eigenvalue distribution.

4. Three-dimensional distribution

Before we work out the three-dimensional distribution we would like to show that expression (21) reproduces the result of §2 when $N=2$. For $N=2$ we get from expression (21)

$$P(E_1, E_2) = K \exp\left[-\frac{1}{4\sigma^2}\sum_{\mu=1}^{2}(E_\mu^2-2\lambda E_\mu)\right] | E_1-E_2 |$$

$$\exp\left[-\frac{\lambda}{2\sigma^2}(E_1-E_2)\right]\int_{-1}^{1} \exp(\lambda/\sigma^2)(E_1-E_2)\, T_{11}'^2 (1-T_{11}'^2)^{-1/2}\, dT'_{11}.$$

Carrying out the integration over dT'_{11} we get the same expression as (5). This provides a check on expression (21).

For the three-dimensional case let us write

$$A = \exp\left[-\frac{\lambda}{2\sigma^2}(E_1+E_2-2E_3)\right]\int \exp\frac{3\lambda}{2\sigma^2}[(E_1-E_3)\, T_{11}'^2+(E_2-E_3)\, T_{21}'^2]$$

$$\frac{d\,T'_{11}\, d\,T'_{21}}{\sqrt{1-(T_{11}'^2+T_{21}'^2)}}. \tag{22}$$

Introducing polar co-ordinates $T'_{11}=\rho\cos\theta$, $T'_{21}=\rho\sin\theta$, we get

$$A = \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi} \exp(3\lambda\rho^2/4\sigma^2)(E_1+E_2-2E_3)\exp(3\lambda\rho^2/4\sigma^2)(E_1-E_2)\cos 2\theta$$

$$\frac{\rho\, d\rho}{\sqrt{1-\rho^2}}\, d\theta.$$

Carrying out the integrations over ρ and θ we get,

$$A = \pi^{3/2} \exp\left[-\frac{\lambda}{2\sigma^2}(E_1+E_2-2E_3)\right] \sum_{m,\,n} \frac{\Gamma(m+2n+1)}{m!\,(n!)^2\, 2^{2n+1}\,\Gamma(m+2n+\frac{3}{2})}$$

$$[(3\lambda/4\sigma^2)(E_1+E_2-2E_3)]^m\,[(3\lambda/4\sigma^2)(E_1-E_2)]^{2n}. \quad (23)$$

Combining expressions (21) and (23) the three-dimensional distribution is given by

$$P(\{E_\mu\}) = K \exp\left[-\frac{\lambda}{4\sigma^2}\sum_{\mu=1}^{3}(E_\mu^2 - 2\lambda\, E_\mu)\right] \prod_{\mu<\nu=1}^{3} |E_\mu - E_\nu|\, A. \quad (24)$$

5. Concluding remarks

Exact analytical distributions of various quantities have been derived for $N=2$ and for non-zero mean of the off-diagonal element. An exact joint distribution for N dimensions is derived using δ-function technique. Its validity is checked by re-deriving the two-dimensional distribution from it. Looking at the N-dimensional joint distribution which is written in the form of integral over an N-dimensional unit vector, we see that it does not show any discontinuity for any value of the ratio of mean-to-mean square deviation. We therefore conclude that the single eigenvalue distribution, will be a smooth distribution, thus supporting the result which is also arrived at by the argument based on the method of moments.

References

Abramowitz M and Stegun I A (eds) 1965 *Handbook of mathematical functions* (New York: Dover publications)

Edwards S F and Jones R C 1976 *J. Phys.* **A9** 159

Kendall M G 1945 *The advanced theory of statistics* (London: Charles Griffin) Vol. 1

Kota V K B and Potbhare V 1977 *J. Phys.* **A10** L183

Mehta M L 1967 *Random matrices and statistical theory of energy levels* (New York: Academic Press)

Nazakat Ullah 1964 *Nucl. Phys.* **58** 65

Nazakat Ullah, Wong S S M and Trainor L E H 1970 *Can. J. Phys.* **48** 842

Parikh J C 1979 Private Communication