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# Steady low shear rate cholesteric flow normal to the helical axis

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**Abstract.** Steady cholesteric flow at low shear rate normal to the helical axis is studied analytically for shear flow and plane Poiseuille flow on the basis of Leslie's continuum theory. For general asymmetric solutions the angle made by the director at the sample centre with the primary flow is found to profoundly affect the oscillations of the apparent viscosity with pitch for pitches of the order of the sample thickness. The velocity and orientation profiles are also found to change drastically. These considerations may be important in flow experiments on long pitch cholesterics.

**Keywords.** Cholesteric; liquid crystals; shear flow; plane Poiseuille flow; apparent viscosity.

## 1. Introduction

The steady shear flow of cholesterics normal to the helical axis was first studied by Leslie (1969) on the basis of the continuum theory (Leslie 1968). He derived the differential equations for homogeneous deformations and showed that the apparent viscosity $\eta$ is a function of gap width and shear rate. These theoretical predictions were found to be in qualitative agreement with the experiment of Candau *et al* (1973) on Poiseuille flow of a twisted nematic. Leslie's equations were solved (Kini 1977, 1979a, b) analytically at low shear rates and numerically at general shear rates. At low shear rates $\eta$ is independent of shear rate but depends on pitch and sample thickness, exhibiting oscillations with pitch variation. This has been verified qualitatively by Bhattacharya *et al* (1978) in their flat capillary shear experiments on a compensated cholesteric mixture.

In the absence of decisive experimental observations two types of boundary conditions have been proposed for the cholesteric twist angle $\phi$. According to the one proposed by Leslie (1969) the pitch is fixed at the boundaries and the director can slip (BC1). But in flows involving twisted nematics it may be more meaningful to fix $\phi$ at suitable values at the boundaries (Kini 1979a). This is the second type of boundary condition (BC2). Both the boundary conditions are found to yield identical results at low shear rates. In the earlier paper (Kini 1979a) a simple subset of solutions for which $\phi$ is antisymmetric (with reference to the sample centre) was treated. The antisymmetry of $\phi$ can be ensured provided that the director can be firmly anchored at the boundaries at suitable angles (BC2). On the other hand if the material obeys BC1, $\phi$ need not be antisymmetric; it can be asymmetric with a non-zero value $\phi_0$ at the centre of the sample.

In the present paper the Leslie equations have been analyticaly solved at low shear rates for simple shear flow and plane Poiseuille flow of cholesterics for asymmetric boundary conditions of $\phi$ ignoring thermo-mechanical coupling. The oscillations of $\eta$ with pitch are found to be profoundly affected by $\phi_0$ for pitches of the order of the sample thickness. Complexities that can arise in the theoretical treatment of plane Poiseuille flow are described. The relevance of this study to flow experiments involving long pitch cholesterics is discussed.

## 2. Shear flow

The cholesteric of pitch $P$ is assumed to be sheared between a pair of parallel infinite plates $z = \pm h/2$, with the plate $z = +h/2$ moving with respect to the other with a constant velocity $V$ along $x$. The cholesteric helical axis is assumed to be along $z$ before flow is induced. Following Leslie (1969) solutions are sought for the director and velocity fields in the form

$$n_x = \cos\theta(z)\cos\phi(z), \quad n_y = \cos\theta(z)\sin\phi(z), \quad n_z = \sin\theta(z),$$
$$v_x = u(z), \quad v_y = v(z), \quad v_z = 0. \tag{1}$$

On ignoring thermo-mechanical coupling one gets the following differential equations:

$$2f_1\theta'' + (\theta')^2 df_1/d\theta - (\phi')^2 df_2/d\theta - 4K_2 S_\theta C_\theta \phi' + (\lambda_1 + \lambda_2\cos 2\theta)(u'C_\phi + v'S_\phi) = 0, \tag{2}$$

$$2f_2\phi'' + 2\theta'\phi' df_2/d\theta + 4K_2 S_\theta C_\theta \theta' + (\lambda_1 - \lambda_2) S_\theta C_\theta (u'S_\phi - v'C_\phi) = 0. \tag{3}$$

$$(H_1 + H_2)(u'C_\phi + v'S_\phi) = 2(aC_\phi + bS_\phi), \tag{4}$$

$$H_1(u'S_\phi - v'C_\phi) = 2(aS_\phi - bC_\phi), \tag{5}$$

where $\theta' = d\theta/dz$ etc., $S_\theta = \sin\theta$, $C_\phi = \cos\phi$, $f_1 = K_{11}C_\theta^2 + K_{33}S_\theta^2$,

$f_2 = C_\theta^2(K_{22}C_\theta^2 + K_{33}S_\theta^2)$, $\lambda_1 = \mu_2 - \mu_3$, $\lambda_2 = \mu_5 - \mu_6$,

$H_1 = \mu_4 + (\mu_5 - \mu_2)S_\theta^2$, $H_2 = (2\mu_1 S_\theta^2 + \mu_3 + \mu_6)C_\theta^2$, $\mu_k$ being the viscosity coefficients and $K_{ii}$, $K_2$ the elastic constants of a cholesteric. The constants **a** and **b** are the constant shear stresses in the $zx$ and $zy$ planes respectively. The primary shear stress **a** can bring into existence the secondary shear stress **b** under certain conditions as shall be seen later. The boundary conditions to be imposed are

$$\theta(\pm h/2) = v(\pm h/2) = u(-h/2) = 0;\ u(+h/2) = V \tag{6}$$

For general asymmetric boundary conditions for $\phi$ one can put as per BC1

$$(d\phi/dz)(z = \pm h/2) = q = 2\pi/P = K_2/K_{22}, \tag{7}$$

or,

$$\phi(z = \pm h/2) = \pm qh/2 + \phi_0. \tag{8}$$

as per BC2 for general asymmetric boundary conditions. From equations (2)–(5) it can be shown (Leslie 1969) that if $\phi_0 = 0$ the equations support the mode (Mode 1) in which $\phi$ and $u-\frac{1}{2}u$ are antisymmetric and $\theta$ and $v$ are symmetric. For low shear rates both the boundary conditions lead to identical results for this mode (Kini 1979a). In the case of general asymmetric solutions at low shear rates one can write

$$\phi = (qz + \phi_0) + \phi_1(z),$$

where $\phi_1$ is assumed to be small. Similarly $\theta$, $u$, $v$, **a** and **b** are also assumed to be of first order. Linearising (2)–(5)

$$K_{11}\theta'' - K_{33}q^2\theta + (\lambda_1 + \lambda_2)(\mathbf{a}C + \mathbf{b}S)/(H_1 + H_2) = 0, \tag{9}$$

$$K_{22}\phi_1'' = 0, \tag{10}$$

$$v' = [2\mathbf{b}(H_1 + H_2C^2) - 2\mathbf{a}H_2SC]/[H_1(H_1 + H_2)], \tag{11}$$

$$u' = [2\mathbf{a}(H_1 + H_2S^2) - 2\mathbf{b}SCH_2)]/[H_1(H_1 + H_2)], \tag{12}$$

with $H_1 = \mu_4$, $H_2 = \mu_3 + \mu_6$, $S = \sin(qz + \phi_0)$, $C = \cos(qz + \theta_0)$.
From (10) and (7) or (10) and (8) one finds that $\phi_1 = 0$. Thus at low shear rates, to first order the cholesteric can be assumed to have no change in the twist. From (11) and (6)

$$\mathbf{b} = \mathbf{a}H_2 \sin 2\phi_0 \sin \alpha/[\alpha(2H_1 + H_2) + H_2 \cos 2\phi_0 \sin \alpha], \tag{13}$$

with $\alpha = qh$.
From (6) and (12)

$$H_1(H_1 + H_2)V = \mathbf{a}[h(2H_1 + H_2) - \frac{H_2}{q}\cos 2\theta_0 \sin \alpha] - \frac{\mathbf{b}H_2}{q}\sin 2\phi_0 \sin \alpha.$$

Using (13) the apparent viscosity

$$\eta = \frac{\mathbf{a}h}{V} = \frac{H_1(H_1 + H_2)}{2H_1 + H_2 - H_2 \cos 2\phi_0 \dfrac{\sin \alpha}{\alpha} - \dfrac{(H_2 \sin 2\phi_0 \sin \alpha)^2}{\alpha^2(2H_1 + H_2) + \alpha H_2 \cos 2\phi_0 \sin \alpha}} \tag{14}$$

Thus $\eta$ is independent of shear rate and handedness of twist but is a function of pitch, sample thickness, material constants and $\phi_0$. From (13) for a given $q$, $\mathbf{b} = 0$ if $\phi_0 = 0$ or $\pi/2$. For $\phi_0 = 0$ one gets Mode 1 which was discussed earlier. For $\phi_0 = \pi/2$ Mode 2 can exist for which $\theta$, $u-\frac{1}{2}V$ and $\phi$ are antisymmetric and $v$ is symmetric. One can also observe from (13) that for a given $\phi_0 \neq 0, \pi/2$, $\mathbf{b} = 0$ if $\sin \phi = 0$ or $h = mP/2$ where $m$ is integral. Thus when there are integral or half integral number of pitches in the sample **b** vanishes. When **b** is non zero, in order to get mathematically consistent solutions, all quantities have to be assumed to be asymmetric. This is clear from (11) and (12) where the contributions from stresses **a** and **b** are always opposite in nature. For simple consideration let us start with an antisymmetric $\phi$ and assume that stresses **a** and **b** are applied. From (11) we find that **a** contributes an antisymmetric part whereas **b** contributes a symmetric part. Thus if **a** and **b** act together $v'$ would be asymmetric even if $\phi$ were antisymmetric. One can again consider another simple picture to drive home this point. Consider a cholesteric director aligned at an angle $\phi_0$ to the $x$ axis in the $xy$ plane. If we now apply the stresses **a** and **b** in the $yx$ and $zy$ planes, along any direction, say along the director the contribution of **a** will be **a** $\cos\phi_0$ but that of **b**, **b** $\sin\phi_0$. Since **a** and **b** are perpendicular to one another, if **b** also exists then the total effect of **a** and **b** will be asymmetric.

From (14) if $\alpha\to\infty$ (highly twisted cholesteric) then $\mathbf{b}\to 0$ and $\eta\to H_1(H_1+H_2)/(2H_1+H_2)$ which is independent of $\phi_0$. Thus for a highly twisted cholesteric at low shear rates there will at best be a weak dependence of $\eta$ on $\phi_0$. Equivalently a measurement of $\eta$ at low shear rates will not yield any information about $\phi_0$. On the other hand if $\alpha\to 0$ (untwisted cholesteric or nematic) $\eta\to\eta(0,\phi_0) = (H_1+H_2\cos^2\phi_0)/2$. For $\phi_0=0$ and $\pi/2$ two of the Miesowicsz coefficients are recovered. At this stage it is instructive to study the case $\alpha\to 0$ in greater detail. For $\alpha\to 0$, $\mathbf{b}\to\mathbf{a}\,H_2\sin 2\phi_0/[(2H_1+H_2)+H_2\cos 2\phi_0]$. Thus as long as $\sin 2\phi_0\neq 0$, even in the absence of a permanent twist a stress **b** will exist when a stress **a** is applied in the $zx$ plane. Equations (11) and (12) now become

$$v' = [2b(H_1 + H_2C^2) - 2aH_2SC]/[H_1(H_1 + H_2)], \tag{11$'$}$$

$$u' = [2a(H_1 + H_2S^2) - 2bH_2SC]/[H_1(H_1 + H_2)]. \tag{12$'$}$$

From (6) and (11)$'$ $v=0$. Thus there is no secondary flow in the sample even though there is a stress **b** in the $yz$ plane. Using (6) and (12),

$$V = 2\mathbf{a}h/(H_1 + H_2C^2),$$

$$\eta_S = \mathbf{a}h/V = \tfrac{1}{2}(H_1 + H_2C^2).$$

Figure 1 contains plots of some relevant quantities for different values of $\phi_0$ with MBBA parameters [$K_{11} = 6\times 10^{-7}$ dynes, $K_{33} = 7\times 10^{-7}$ dynes, $H_1 = 0{\cdot}832$ poise, $H_2 = -0{\cdot}336$ poise] for a sample thickness of 200 $\mu$m. The effect of $\phi_0$ on the ratio $\eta(\alpha,\phi_0)/\eta(0,\phi_0)$ is quite striking especially for pitches which are of the order of the sample thickness. From (14), $\eta(\alpha,\phi_0) = \eta(\alpha,\phi_0+\pi)$ in accordance with the fundamental property of a nonpolar director. However $\eta(\alpha,\phi_0)\neq\eta(\alpha,\phi_0+\pi/2)$ in general. Thus if $\phi$ is increased beyond $\pi/2$ the plot for $\phi_0'$ will not in general coincide with that for $\phi_0'+\pi/2$. Still the extrema become higher as $\phi_0$ is increased until for $\phi_0=\pi$ one gets back the plot corresponding to $\phi_0=0$.

For a given $\phi_0$, the position of the extrema of $\eta(\alpha,\phi_0)$ with $\alpha$ can be shown to satisfy the condition

$$(\sin\alpha_M - \alpha_M\cos\alpha_M)\,[\cos 2\phi_0\,\{(H_2\sin\alpha_M)^2 + \alpha_M^2(2H_1 + H_2)^2\} + 2\alpha_M(2H_1 + H_2)H_2\sin\alpha_M] = 0.$$

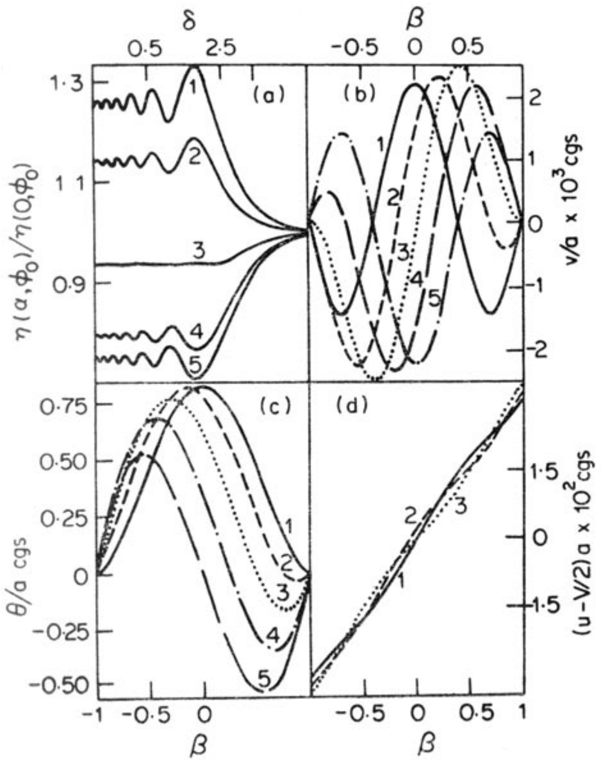


**Figure 1.** Shear flow (a) $\eta(\alpha,\phi_0)/\eta(0,\phi_0)$ vs $\delta = P/h$ for $\phi_0 =$ (1) 0 (2) $\pi/8$ (3) $\pi/4$ (4) $3\pi/8$ (5) $\pi/2$ radians. (b) $v/a$ vs $\beta = 2z/h$ and (c) $\theta/a$ vs $\beta$ with P=280 $\mu$m. $\phi_0 =$ (1) 0 (2) $\pi/8$ (3) $\pi/4$ (4) $3\pi/8$ (5) $\pi/2$. (d) $(u - V/2)/a$ vs $\beta$ with P=280 $\mu$m and $\phi_0 =$ (1) 0 (2) $\pi/4$ (3) $\pi/2$. h=200 $\mu$m in all cases.

There are thus two possibilities:

(i) $\tan \alpha_M = \alpha_M$, (15)

(ii) $\sin^2 \alpha_M (H_2^2 \cos 2\phi_0) + 2\alpha_M \sin \alpha_M \{(2H_1 + H_2) H_2\}$

$$+ \alpha^2 (2H_1 + H_2)^2 \cos 2\phi_0 = 0. \qquad (16)$$

Equation (15) is independent of $\phi_0$ and the material parameters, but the smallest root other than the trivial root zero is $\alpha_M = 4{\cdot}49$. The roots of (16) are restricted by the fact that

$$\left|\frac{\sin \alpha_M}{\alpha_M}\right| = \left|\frac{(2H_1 + H_2)}{H_2} \frac{(\pm \sin 2\phi_0 - 1)}{\cos 2\phi_0}\right| \leqslant 1. \qquad (17)$$

For MBBA parameters only the root

$$\frac{\sin \alpha_M}{\alpha_M} = \frac{(2H_1 + H_2)(\sin 2\phi_0 - 1)}{H_2 \cos 2\phi_0}, \qquad (18)$$

can be found, that too in the restricted interval $31° < \phi_0 < 59°$. (For $\phi_0 = \pi/4$ the function is undefined. But one can take the limit as $\phi_0 \rightarrow \pi/4$.) In this interval of $\phi_0$ one can find extrema with $\alpha$ but for $\alpha<4{\cdot}49$. These extrema are quite shallow as can be seen from figure 2a. Thus in the region of $\alpha>4{\cdot}49$ for all the values of $\phi_0$, the extrema of $\eta$ with $\alpha$ are determined by (15) and are independent of $\phi_0$. For a restricted range of $\phi_0$, (16) can determine shallow extrema in the long pitch region. The ratio (figure 1a) at the primary extremum varies by as much as 30% as $\phi_0$ increases from 0 to $\pi/2$. However the value of $\eta$ varies by about 10%.

The effect of changing $\phi_0$ is seen in the profiles of $u$, $\theta$ and $v$. From (12)

$$H_1\,(H_1+H_2)\left(u-\frac{V}{2}\right)=\mathbf{a}\left[(2H_1+H_2)\,z+\frac{H_2}{2q}\{\sin 2\phi_0\cos\alpha-\sin\zeta\}\right]$$

$$+\frac{\mathbf{b}H_2}{2q}[\cos\zeta-\cos 2\phi_0\cos\alpha]\ \text{ where } \zeta=2qz+2\phi_0.$$

For $\phi_0=0$ and $\pi/2$, $u-V/2$ is antisymmetric, though numerically different (figure 1*d*). For other values of $\phi_0$, $u-V/2$ is asymmetric which is clearly a consequence of the symmetric contribution toward $u'$ arising from **b**. The profiles of $v$ and $\theta$ are more interesting

$$H_1\,(H_1+H_2)\,v=\frac{\mathbf{a}\,H_2}{2q}[\cos\zeta-\cos\epsilon]$$

$$+\mathbf{b}\left[(2H_1+H_2)\left(z+\frac{h}{2}\right)+\frac{H_2}{2q}(\sin\zeta-\sin\epsilon)\right], \tag{19}$$

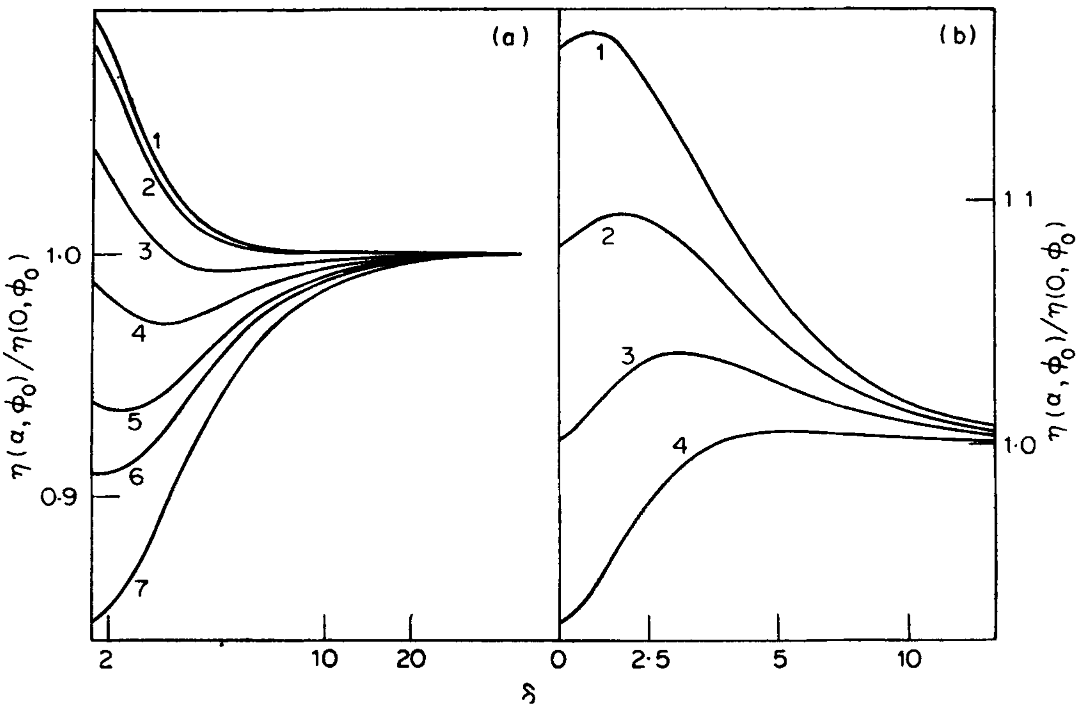


**Figure 2.** Plot of $\eta(\alpha,\phi_0)/\eta(0,\phi_0)$ vs $\delta=P/h$; $h=200\ \mu$m (**a**) shear flow for $\phi_0=$(1) 30 (2) 31 (3) 35 (4) 40 (5) 45 (6) 48 (7) 55 degrees. (**b**) Plane Poiseuille flow for $\phi_0=$ (1) 35 (2) 40 (3) 45 (4) 50 degrees.

$$\theta = \Delta_1 \left[ \cos \nu \sinh \omega + \cos \frac{\epsilon}{2} \sinh \psi - \cos \frac{\zeta}{2} \sinh kh \right] + \Delta_2 \left[ \sin \nu \sinh \omega + \sin \frac{\epsilon}{2} \sinh \psi - \sin \frac{\zeta}{2} \sinh kh \right], \tag{20}$$

with $\Delta_1 = \Delta\, a/\sinh kh$, $\Delta_2 = \Delta\, b/\sinh kh$, $k^2 = K_{33}\, q^2/K_{11}$,

$$\Delta = -(\lambda_1 + \lambda_2)/[(H_1 + H_2)\, K_{11}\, (k^2 + q^2)],\ \epsilon = 2\phi_0 - \alpha,$$

$$\nu = \phi_0 + a/2,\ \omega = kz + kh/2,\ \psi = \frac{kh}{2} - kz.$$

From (19) $v$ is found to be symmetric for $\phi_0 = 0$ and $\phi_0 = \pi/2$ (figure 1b) and further $v(z, \phi_0 = 0) = -v\,(z, \phi_0 = \pi/2)$ at all points $z$. Thus the $v$ profiles for $\phi_0 = 0$ and $\pi/2$ are exactly opposite to one another, which is also true for the net secondary flow. However for any general $\phi_0$, $v$ is asymmetric. The net secondary flow per unit width of the flow cell

$$F_S = \int_{-h/2}^{h/2} v\, dz = \frac{\mathbf{a} H_2[(2H_1 + H_2)\alpha \cos 2\phi_0 + H_2 \sin \alpha][(\sin \alpha - \alpha \cos \alpha)]}{2q^2[\alpha(2H_1 + H_2) + H_2 \cos 2\phi_0 \sin \alpha]}.$$

Observe that on reversing the sign of twist, $F_S$ changes sign in the sample.

$$F_S = 0 \text{ for any } \phi_0 \text{ if } \sin \alpha = \alpha \cos \alpha.$$

Thus at the extrema given by (15) there will be no net secondary flow. Alternatively for a given $\alpha$ (or pitch)

$$F_S = 0 \text{ if } \cos 2\phi_0^{(M)} = -H_2 \sin \alpha/[(2H_1 + H_2)\alpha].$$

$\phi_0^{(M)} = 46°$ for MBBA parameters for $P = 280\ \mu$m which is the reason why for $\phi_0 = 45°$, $v$ is almost antisymmetric. Using (20) the $\theta$ profile is found to be symmetric for $\phi_0 = 0$ (figure 1c) corresponding to Mode 1 and antisymmetric for $\phi_0 = \pi/2$ in Mode 2. For intermediate values of $\phi_0$, $\theta$ is asymmetric. Figure 5a is a plot of the ratio $b/a$ as a function of pitch for different values of $\phi_0$. As can be expected from (13), **b** exhibits oscillations with pitch similar to those of $\eta$.

The difficulty in solving the shear flow problem at general shear rates for asymmetric boundary conditions should be quite clear. When Mode 1 or Mode 2 is treated **b** $=0$ and **a** can be uniquely related to $V$ as follows: Starting with **a**, equations (2) and (3) are solved and the $\theta$, $\phi$ profiles obtained. Now (4) and (5) are solved to obtain $v$ and $u$ as functions of $z$, with $u(z{=}h/2){=}V$. On the other hand when we have an asymmetric case, both **a** and **b** figure as independent constants in the begining and determine the $\theta$ and $\phi$ profiles, using which the $u$ and $v$ profiiles have to be obtained. An iterative procedure has to be used, keeping **a** fixed and varying **b** till $v(\pm h/2) = 0$ and using **a** and **b** to calculate $u(+h/2) = V$. When this is done there may be more than one set of (**a**, **b**) values which give rise to the same $V$ in which case **b** may have to be studied as a function of **a** and that set of (**a**, **b**) values taken which satisfy some extremum condition. Thus a solution of the shear flow problem for asymmetric boundary conditions is not a simple task even by numerical means.

## 3. Plane Poiseuille flow

If flow is taking place under the action of a constant pressure gradient $p_{,x} = \mathbf{a}_1$ applied along the $x$ axis, equations (4) and (5) have to be modified to read

$$[H_1+H_2]\,[u'C_\phi + v'S_\phi] = 2[(\mathbf{a}C_\phi + \mathbf{b}S_\phi) + z(\mathbf{a}_1 C_\phi + \mathbf{b}_1 S_\phi)], \tag{21}$$

$$H_1[u'S_\phi - v'C_\phi) = 2[(\mathbf{a}S_\phi - \mathbf{b}C_\phi) + z(\mathbf{a}_1 S_\phi - \mathbf{b}_1 C_\phi]), \tag{22}$$

where $\mathbf{b}_1 = p_{,y}$ a constant pressure gradient along $y$. Since $\mathbf{a}_1$ is the motive force behind the flow, as in the case of shear flow the other constants like **a**, **b** or $\mathbf{b}_1$ which result either due to secondary flow or due to lack of symmetry in director orientation or both, have to be related to $\mathbf{a}_1$. The boundary conditions to be imposed are

$$u(\pm h/2) = v(\pm h/2) = \theta(\pm h/2) = 0. \tag{23}$$

For a unique solution of (21) and (22) in terms of $\mathbf{a}_1$, one of the constants **a**, **b** or $\mathbf{b}_1$ has to be equated to zero, the other two getting determined by (23) in terms of $\mathbf{a}_1$. Thus we have three cases

(i) $\mathbf{a}_1$, **a**, **b**; $\mathbf{b}_1 = 0$,

(ii) $\mathbf{a}_1$, **b**, $\mathbf{b}_1$; $\mathbf{a} = 0$,

(iii) $\mathbf{a}_1$, **a**, $\mathbf{b}_1$; $\mathbf{b} = 0$.

Apart from the difficulty in understanding physically how a pressure gradient $p_{,x} = \mathbf{a}_1$ along $x$ can give rise to a pressure gradient $p_{,y} = \mathbf{b}_1$ along $y$, one finds that cases (ii) and (iii) lead to unphysical results for $\eta$, making $\eta$ zero and even negative for certain ranges of pitches and $\phi_0$, a situation that is totally divorced from reality so far as liquid crystals are concerned. Because of this reason only case (i) is studied which corresponds to a situation in which the pressure gradient $\mathbf{a}_1$ along $x$ gives rise to constant shear stresses in the $xz$ and $yz$ planes. Assuming low shear rates ($\mathbf{a}_1$, **a**, **b** are small) and putting

$\phi = qz + \phi_0 + \phi_1$ one gets by linearisation the following set of equations:

$$K_{11}\theta'' - K_{33}q^2\theta + (\lambda_1 + \lambda_2)\,[(\mathbf{a}C + \mathbf{b}S) + \mathbf{a}_1 zC]/(H_1 + H_2) = 0, \tag{24}$$

$$K_{22}\phi_1'' = 0, \tag{25}$$

$$u' = [2(\mathbf{a} + \mathbf{a}_1 z)(H_1 + H_2S^2) - 2\mathbf{b}H_2SC]\,/\,[H_1(H_1 + H_2)], \tag{26}$$

$$v' = [2\mathbf{b}(H_1 + H_2C^2) - 2H_2SC(\mathbf{a} + \mathbf{a}_1 z)]\,/\,[H_1(H_1 + H_2)]. \tag{27}$$

Arguing out as before $\phi$ is found to be zero. Using the boundary conditions for $u$ and $v$

$$\mathbf{a} = \frac{\mathbf{a}_1 h H_2 \sin 2\phi_0 \,(2H_1 + H_2)\,(\alpha \cos \alpha - \sin \alpha)}{2[\alpha^2(2H_1 + H_2)^2 - H^2{}_2 \sin^2 \alpha]}, \tag{28}$$

$$\mathbf{b} = -\frac{\mathbf{a}_1 h H_2(\alpha \cos \alpha - \sin \alpha)\,[\alpha(2H_1+H_2) \cos 2\phi_0 - H_2 \sin \alpha]}{2\alpha[\alpha^2(2H_1+H_2)^2 - H_2^2 \sin^2 \alpha]}. \tag{29}$$

The profiles for $u$, $v$ and $\theta$ are:

$$H_1(H_1 + H_2)u = \mathbf{a}[(2H_1 + H_2)\left(z + \frac{h}{2}\right) + \frac{H_2}{2q}(\sin \varepsilon - \sin \zeta)]$$
$$+ \mathbf{a}_1\left[\left(\frac{2H_1+H_2}{2}\right)\left(z^2 - \frac{h^2}{4}\right) + \frac{H_2}{4q^2}(\cos \varepsilon - \alpha \sin \varepsilon - 2qz \sin \zeta - \cos \zeta)\right].$$
$$+ \frac{\mathbf{b}H_2}{2q}[\cos \zeta - \cos \varepsilon], \tag{30}$$

$$H_1(H_1 + H_2)v = \frac{\mathbf{a}H_2}{2q}[\cos \zeta - \cos \varepsilon]$$
$$+ \frac{\mathbf{a}_1 H_2}{4q^2}[2qz \cos \zeta - \sin \zeta + \alpha \cos \varepsilon + \sin \varepsilon]$$
$$+ \mathbf{b}\left[(2H_1+H_2)\left(z + \frac{h}{2}\right) + \frac{H_2}{2q}(\sin \zeta - \sin \varepsilon)\right], \tag{31}$$

$$\theta / \mathbf{a}_1 = \left[\left(B_2\frac{h}{2} - B_1\right)\cos\frac{\varepsilon}{2} - B_3 \sin\frac{\varepsilon}{2}\right] \sinh \psi / \sinh kh$$
$$-\left[\left(\frac{B_2 h}{2} + B_1\right)\cos \nu + B_3 \sin \nu\right] \sinh \omega / \sinh kh$$
$$+ (B_1 + B_2 z)\cos\frac{\zeta}{2} + B_3 \sin\frac{\zeta}{2}, \tag{32}$$

with $\Delta' = -\Delta\,(k^2 + q^2)$, $B_1 = \Delta'\mathbf{a} / [\mathbf{a}_1\,(k^2 + q^2)]$,

$$B_2 = \Delta'/(k^2 + q^2),\; B_3 = \left[\frac{b}{\mathbf{a}_1} - \frac{2q}{(k^2 + q^2)}\right]B_2.$$

To find $\eta$,

$$A_L = \int_{h/2}^{h/2} u(z)\, dz$$

the volume of fluid flowing per unit width of the sample is calculated whence $\eta$ is given by

$$\eta = -\frac{\mathbf{a}_1 h^3}{12 A_L} = \frac{H_1 (H_1 + H_2)}{\left[ (2H_1 + H_2) + \frac{3H_2 \cos 2\phi_0}{\alpha^3} (2 \sin \alpha - \alpha \cos \alpha) \right.}$$

$$\left. - \frac{3H_2}{\alpha^2} (\cos \epsilon - \alpha \sin \epsilon) + \frac{6a}{\mathbf{a}_1 h} \left\{ \frac{H_2 \sin 2\phi_0}{\alpha^2} \sin \alpha - \frac{H_2}{\alpha} \sin \epsilon - (2H_1 + H_2) \right\} - \frac{6H_2 \mathbf{b}}{\alpha^2 \mathbf{a}_1 h} (\cos 2\phi_0 \sin \alpha - \alpha \cos \epsilon) \right].$$

Thus $\eta$ is independent of $\mathbf{a}_1$ and handedness of twist but is a funtion of $\alpha$ and $\phi_0$ alone. For $\alpha \to \infty$ (highly twisted cholesterics) $\mathbf{a} \to 0$, $\mathbf{b} \to 0$ and $\eta \to H_1 (H_1 + H_2)/(2H_1 + H_2)$ the same limit as in shear flow, which is independent of $\phi_0$. Thus for small pitches the effect of $\phi_0$ will not be significant. However for $\alpha \to 0$ (untwisted cholesteric or nematic with the director aligned in the $xy$ plane making an angle $\phi_0$ with $x$) $\mathbf{a} \to 0$, $\mathbf{b} \to 0$ and $\eta \to \eta_P = H_1 (H_1 + H_2)/2(H_1 + H_2 S^2)$ which is different from $\eta_S$. Thus for general values of $\phi_0$, $\eta$ for the two flows will not coincide for nematics. But for $\phi_0 = 0$ or $\pi/2$, $\eta_S$ and $\eta_P$ have the same value. The reason for this discrepancy appears to be due to the vanishing of both $\mathbf{a}$ and $\mathbf{b}$ with the twist in the structure. The driving force (viz., $\mathbf{a}_1$) is responsible for both the primary velocity $u = \mathbf{a}_1 (H_1 + H_2 S^2)(z^2 - h^2/4)/[(H_1 + H_2) H_1]$ and the secondary velocity $v = \mathbf{a}_1 H_2 SC (z^2 - h^2/4)/[(H_1 + H_2) H_1]$ which are fully decoupled. There will be no secondary flow, for $\phi_0 = 0$ or $\pi/2$. Thus the flow situations in the two cases of shear and plane

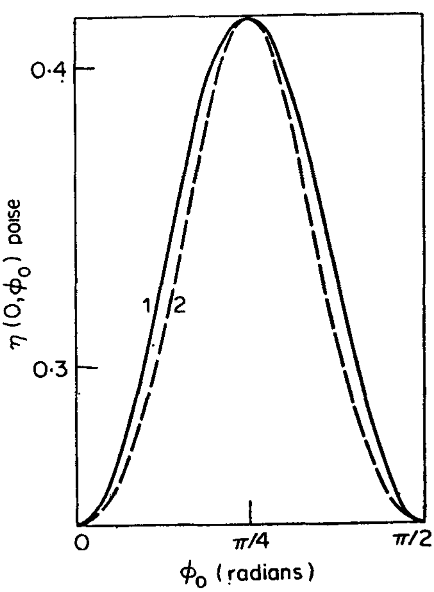


**Figure 3.** $\eta(0, \phi_0)$ as a function of $\phi_0$ for (1) Plane Poiseuille flow ($\eta_P$) (2) shear flow ($\eta_S$.)

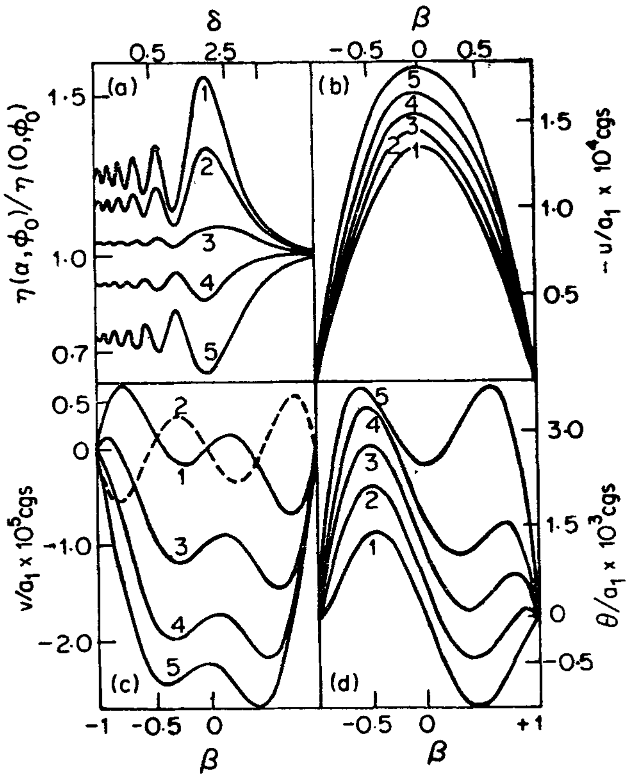


**Figure 4.** Plane Poiseuille flow
(a) $\eta(a, \phi_0)/\eta(0, \phi_0)$ vs $\delta$ for $\phi_0$=(1) 0 (2) 25 (3) 40 (4) 55 (5) 90 degrees. (b) $-u/a_1$ vs $\beta$ for $\phi_0$=(1) 0 (2) 25 (3) 40 (4) 55 (5) 90 degrees. (c) $v/a_1$ vs $\beta$ for $\phi_0$=(1) 0 (2) 90 (3) 12·5 (4) 25 (5) 40 degrees. (d) $\theta/a_1$ vs $\beta$ for $\phi_0$=(1) 0 (2) 12·5 (3) 25 (4) 40 (5) 90 degrees. P=339 $\mu$m in (b), (c) and (d). h=200 $\mu$m in all cases.

Poiseuille flow are entirely different. Figure 3 contains plots of $\eta_S$ and $\eta_P$ with $\phi_0$. For MBBA parameters $\eta_P$ is greater than $\eta_S$ except at $\phi_0=0$, $\pi/4$ and $\pi/2$. Figure 4 contains the plots of a few relevant quantities for different value of $\phi_0$. The oscillations of $\eta$ are more pronounced (figure 4a) as compared to those in shear flow. The primary extremum of the ratio can vary by nearly 70% though $\eta$ varies by 30% as $\phi_0$ goes from 0 to $\pi/2$. From $d\eta/da=0$ the extrema of $\eta$ with $a$ are found to depend on material parameters and $\phi_0$ in general, there being a marked change in the primary extremum for some values of $\phi_0$ (figure 2b). The $u$ profile is symmetric for $\phi_0=0$ and $\pi/2$ and is asymmetric for general $\phi_0$ (figure 4b). But $v$ is antisymmetric for $\phi_0=0$ and $\pi/2$ though the profiles in the two cases have opposite signs (figure 4c). For general $\phi_0$, $v$ is asymmetric. The $\theta$ profile is antisymmetric for $\phi_0=0$ (figure 4d) but is symmetric for $\phi_0=\pi/2$ and asymmetric for general values of $\phi_0$. Thus in plane Poiseuille flow there are two distinct modes for $\mathbf{a}=0$: For $\phi_0=0$, Mode 1 can exist with $\theta$, $\phi$ and $v$ antisymmetric and $u$ symmetric; for $\phi_0=\pi/2$ there is Mode 2 with $\phi$, $v$ antisymmetric and $\theta$, $u$ symmetric. Figure 5 contains plots of $\mathbf{a}/\mathbf{a}_1$ and $\mathbf{b}/\mathbf{a}_1$ functions of pitch for different values of $\phi_0$ (figures 5b, 5c). As can be seen from (28) and (29), $\mathbf{a}$ and $\mathbf{b}$ also exhibit oscillations with pitch. The net secondary flow $F_s$ is given by

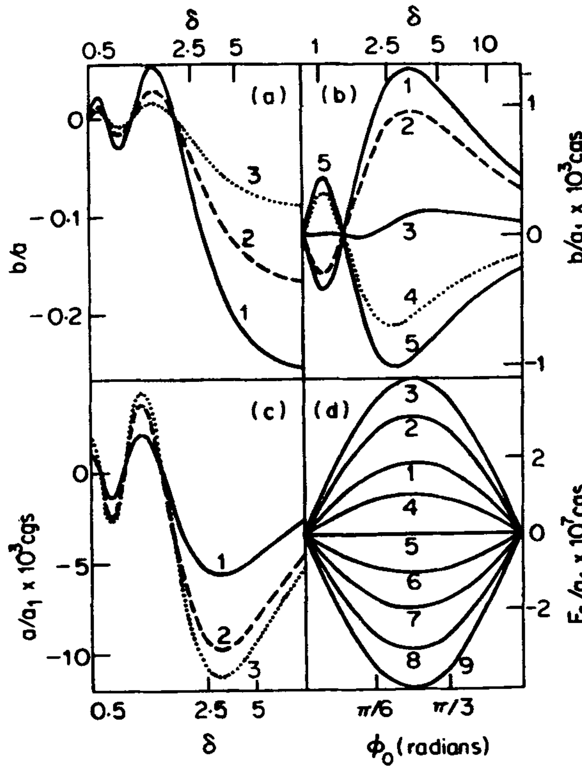


**Figure 5.** Shear flow (a) $b/a$ vs $\delta$ for $\phi_0$=(1) 38 (2) 16 (3) 8 degrees.
Plane Poiseuille flow. (b) $b/a_1$ vs $\delta$ for $\phi_0$=(1) 0 (2) $\pi/8$ (3) $\pi/4$ (4) $3\pi/8$ (5) $\pi/2$ radian, (c) $a/a_1$ vs $\delta$ for $\phi_0$=(1) $\pi/12$ (2) $\pi/6$ (3) $\pi/4$ radian. (d) $F_s$ (net secondary flow)/$a$ vs $\phi_0$ for P=(1) 240 (2) 265 (3) 320 (4) 536 (5) 610 (6) 700 (7) 810 (8) 1010 (9) 1360 $\mu$m.

$$\frac{H_1(H_1+H_2)F_S}{\mathbf{a}_1} = \left(\frac{\mathbf{b}}{\mathbf{a}_1}\right)\frac{[H_2\sin 2\phi_0 \sin\alpha + \alpha\{\alpha(2H_1+H_2) - H_2\sin\epsilon\}]}{2q^2}$$

$$+\left(\frac{\mathbf{a}}{\mathbf{a}_1}\right)H_2\frac{[\cos 2\phi_0\sin\alpha - \alpha\cos\epsilon]}{2q^2}$$

$$+\frac{H_2}{4q^3}[\alpha\sin\epsilon + \alpha^2\cos\epsilon + \sin 2\phi_0(\alpha\cos\alpha - 2\sin\alpha)].$$

For $\phi_0 = n\pi$ or $(2n+1)\pi/2$ ($n$ = integer), $F_S = 0$, the secondary velocity becoming antisymmetric. Figure 5d is a plot of $F_S$ with $\phi_0$ for different pitches. For a given $\phi_0$, $F_S$ oscillates with pitch. But for a given pitch $F_S$ is maximum for $\phi_0 = \pi/4$ (or odd integral multiples of $\pi/4$).

For reasons similar to those discussed in § 2 a numerical treatment of plane Poiseuille flow is difficult at general shear rates. For Modes 1 and 2 there are two constants $\mathbf{a}_1$ and $\mathbf{b}$ which have to be determined uniquely, while for general asymmetric cases there are three constants $\mathbf{a}_1$, $\mathbf{a}$ and $\mathbf{b}$ to be determined uniquely using equation (23).

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