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Iteration approach to peratisation technique

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Abstract. Peratisation scheme normally used to obtain scattering length $a(\alpha)$ is through Born series. In this paper an iteration approach to this technique has been presented. This approach has also been applied to different potentials successfully.

Keywords. Peratisation technique; first order peratisation; inverse fourth power; logarithmic singular potential; scattering length.

1. Introduction

The study of repulsive singular potentials has been the subject of investigation since the last few years. This study has been applied to a variety of fields such as unrenormalisable field theories (Khuri and Pais 1964; Pais and Wu 1964; Wu 1964; Tiktopoulos and Treiman 1964), high energy behaviour of phase shifts (Paliov and Rosendroff 1967; Calogero 1964, 1967; Spector 1969), low energy behaviour of phase shifts (Stanciu 1967; Handelsman *et al* 1968) and to some physical applications such as molecular physics (Vogt and Wannier 1954; Thaler 1959).

For a singular potential $gV(r)$, the scattering length A considered as a function of the coupling constant g has a singularity at $g=0$ due to the singular nature of the potential. Hence, a power expansion of A in g is frustrated by infinite integrals. To overcome this difficulty the technique of pertisation has been devised. Many workers have applied this technique to singular potentials (Khuri and Pais 1964; Aly *et al* 1965; Cornille 1966; Gale 1967; Frank and Land 1970a,b) since it was first introduced by Feinberg and Pais (1963).

The success of regularisation is presumed to be the precondition for the success of peratisation. In this paper we use θ and $+$ regularisations. The power expansion in g for the scattering length corresponding to regularised potential $gV(r, \alpha)$ can be written as

$$A(\alpha) = \sum_{n=1}^{\infty} A_n(\alpha)\, g^n, \tag{1}$$

where $A_n(\alpha)$ are functions of α which diverge as $\alpha \to 0$. The peratisation consists of summing up the series of most singular terms in $A_n(\alpha)$ in each power of g and to see finally whether the sum is finite, when the limit $\alpha \to 0$ is approached.

The usual procedure of obtaining the series (1) so far is through Born series. In our previous paper (Kulkarni and Sharma 1978), we discussed a different approach to this procedure. In this paper we discuss yet another approach which may be named as an iterative approach. This has been formulated in § 2 and applied to different singular potetials in § 3. In § 4, we establish the relationship between the two approaches developed by us.

2. Formulation of the method

We assume here that the series (1) can be obtained in the limit $r \to \infty$ from

$$A(r, \alpha) = \sum_{n=1}^{\infty} A_n (r, \alpha) g^n, \tag{2}$$

such that

$$\lim_{r \to \infty} A_n (r, \alpha) = A_n (\alpha). \tag{3}$$

Now we obtain series (2) by iteration approach from the standard equation for scattering length which is given as

$$a'(r) = -g\, V(r)\, [r + a(r)]^2. \tag{4}$$

For this we set

$$a'_{m+1}(r) = -gV(r)\, [r + a_m (r)]^2, \tag{5}$$

with $$a_o(r) = 0. \tag{6}$$

Equation (5) for the regularised potential $gV(r, \alpha)$ takes the form

$$a'_{m+1} (r, \alpha) = -g\, V(r, \alpha)\, [r + a_m (r, \alpha)]^2, \tag{7}$$

with $$a_o(r, \alpha) = 0. \tag{8}$$

By repeating the above procedure we obtain number of terms for the expansion $a_m (r, \alpha)$ which ultimately yields series (2). However, it should be noted that mth order iteration $a_m(r, \alpha)$ contains terms of order higher than m in g which need never be computed. As done before the limits of integration of θ regularisation here are from $r = \alpha$ to r and for $+$ regularisation from $r=0$ to r in (7).

3. Applications

3.1. *Inverse fourth power potential*

We now apply the theory developed in § 2 to the potential

$$V(r) = g/r^4. \tag{9}$$

This potential, attractive or repulsive, has been studied most extensively (Spector 1964). Aly and Müller (1966), Challifour and Eden (1963), and Dombey and Jones (1968) have also studied the inverse fourth power potential for Regge behaviour. For θ regularisation of this potential (7) and (8) yield,

$$a_1(r, \alpha) = g\left\{\frac{1}{r} - \frac{1}{\alpha}\right\}. \tag{10a}$$

$$a_2(r, \alpha)_2 = a_1(r, \alpha) + g^2\left\{\frac{1}{3r^3} - \frac{1}{2\alpha r^2} + \frac{1}{6\alpha^3}\right\} + O(g^3), \tag{10b}$$

$$a_3(r, \alpha) = a_2(r, \alpha) + g^3\left\{\frac{7}{15r^5} - \frac{1}{\alpha r^4} + \frac{1}{3\alpha^2 r^3} + \frac{1}{3\alpha^3 r^2} - \frac{2}{15\alpha^5}\right\} + O(g^4) \tag{10c}$$

$$a_4(r, \alpha) = a_3(r, \alpha) + 2g^4\left\{\frac{17}{105r^7} - \frac{4}{9\alpha r^6} - \frac{4}{15\alpha^2 r^5} + \frac{1}{6\alpha^4 r^3} - \frac{1}{15\alpha^5 r^2} + \frac{17}{630\alpha^7}\right\} + O(g^5). \tag{10d}$$

and so on. On increasing the order of iteration, series (2) is automatically obtained.
In the limit $r \to \infty$, this series leads to

$$\lim_{r\to\infty} a(r, \alpha) = a(\alpha) = -g/\alpha + g^2/3\alpha^3 - 2g^3/15\alpha^5 + 17g^4/315\alpha^7 - \dots$$

or
$$a(\alpha) = -g^{1/2} \tan h\, (g^{1/2}/\alpha), \tag{11}$$

which in the limit $\alpha \to 0$ gives $-g^{1/2}$, the exact scattering length for potential (9). Hence peratisation is successful in this case.

Similarly for $+$ regularisation of the potential (9), the application of the theory developed in § 2 gives the following final expression for $a(\alpha)$

$$a(\alpha) = \lim_{r\to\infty} a(r, \alpha) = \frac{-g}{3\alpha} + \frac{g^2}{45\alpha^3} - \frac{2g^3}{945\alpha^5} + \dots,$$

or
$$a(\alpha) = -g^{1/2} \cot h\, (g^{1/2}/\alpha) + \alpha, \tag{12}$$

and
$$\lim_{\alpha\to 0} a(\alpha) = -g^{1/2}. \tag{13}$$

Thus we observe that even for $+$ regularisation, iteration approach to peratisation succeeds.

3.2. *Logarithmic singular potential*

We next apply the theory to the following singular potential which is logarithmic in nature and has been used by Aly *et al* (1964) in studying the peratisation technique

$$V(r) = g\frac{\ln^2 r}{r^4}. \tag{14}$$

For θ regularisation of this potential different iterations a_1 (r, α), $a_2(r, \alpha)$, etc are obtained with the help of (5) and (6). After applying the limit $r\to\infty$ to a (r, α) and isolating the leading singularities in α, one gets for the first order peratisation

$$a(\alpha) = -g^{1/2} (\ln \alpha) \left\{ g^{1/2}\frac{\ln \alpha}{\alpha} - \frac{g^{3/2}}{3}\frac{\ln^3 \alpha}{\alpha^3} + \frac{g^{5/2}}{15}\frac{\ln^5 \alpha}{\alpha^5} - \ldots \right\},$$

or
$$a(\alpha) = -g^{1/2} (\ln \alpha) \tan h\left(g^{1/2}\frac{\ln \alpha}{\alpha}\right). \tag{15}$$

It may be interesting to note that the above result can also be obtained directly through the application of the theorem given by Spector (1966).

Now $a(\alpha)$ in the limit $\alpha\to 0$ yields

$$a = -g^{1/2} (\ln \alpha) \ldots\ldots\ldots\ldots \tag{16}$$

This result is identical to that obtained by Aly *et al* (1964).

4. Concluding remarks

Following the notations for A's from equations (1), (2), (3) (which have also been used in our previous paper), a relationship between iterations a_n (r, α) and A's can easily be established. In fact, we observe that

$$a_1(r, \alpha) = gA_1 (r, \alpha), \tag{17a}$$

$$a_2(r, \alpha) = gA_1 (r, \alpha)+g^2 A_2 (r, \alpha), \tag{17b}$$

$$a_3(r, \alpha) = gA_1 (r, \alpha)+g^2A_2 (r, \alpha)+g^3A_3 (r, \alpha), \tag{17c}$$

and so on, where $A_1(r, \alpha)$, $A_2(r, \alpha)$, $A_3(r, \alpha)$, etc have been defined in our previous paper. Thus it can easily be seen that

$$a(r, \alpha) \equiv A(r, \alpha), \tag{18}$$

and hence, in the limit $r\to\infty$

$$a(\alpha) \equiv A(\alpha). \tag{19}$$

The results obtained in this paper completely agree with the results obtained so far by the usual peratisation technique, which shows the correctness of our approach. Another distinct advantage of the present approach is that the calculations involved are comparatively easier.

References

Aly H H, Riazuddin and Zimmerman A H 1964 *Phys. Rev.* **B136** 1174
Aly H H, Riazuddin and Zimmerman A H 1965 *Nuovo Cimento* **35** 324
Aly H H and Müller H J W 1966 *J. Math. Phys.* **7** 1
Challifour J and Eden R J 1963 *J. Math. Phys.* **4** 359
Calogero F 1964 *Phys. Rev.* **135** B693
Calogero F 1967 *Variable phase approach to potential scattering* (New York: Academic Press).
Cornille H 1966 *Nuovo Cimento* **A43** 786
Dombey N and Jones R 1968 *J. Math. Phys.* **9** 986
Feinberg G and Pais A 1963 *Phys. Rev.* **131** 2724
Frank W M and Land D J 1970a *J. Math. Phys.* **11** 2041
Frank W M and Land D J 1970b *J. Math. Phys.* **11** 2058
Gale W A 1967 *J. Math. Phys.* **8** 1050
Handelaman R A, Pao Y and Lew J S 1968 *Nuovo Cimento* **A55** 453
Khuri N N and Pais A 1964 *Rev. Mod. Phys.* **36** 590
Kulkarni S V and Sharma L K 1978 *Pramāṇa* **11** 733
Pais A and Wu T T 1964 *J. Math. Phys.* **5** 799
Paliov A and Rosendroff S 1967 *J. Math. Phys.* **8** 1829
Spector R 1964 *J. Math. Phys.* **5** 1185
Spector R 1966 *J. Math. Phys.* **7** 2103
Spector R 1969 *J. Math. Phys.* **10** 503
Stanciu G N 1967 *Nuovo Cimento* **A50** 293
Thaler R M 1959 *Phys. Rev.* **114** 827
Tiktopoulos G and Treiman S B 1964 *Phys. Rev.* **134** B844
Vogt E and Wannier G 1954 *Phys. Rev.* **95** 1190
Wu T T 1964 *Phys. Rev.* **136** B1176