Pramāna, Vol. 14, No. 4, April 1980, pp. 315–320. © Printed in India.

Evanescent wave contribution to the diffracted amplitude for spherical geometry

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MS received 7 September 1979; revised 3 March 1980

Abstract. Calculation of the diffracted amplitude for the diffracting aperture on a sphere is briefly reviewed. Explicit formulas are given to compute the diffracted amplitude. In these formulas the real wave and evanescent wave contributions are discussed.

Keywords. Spherical harmonics; Hankel functions; diffraction.

1. Introduction

Spherical harmonics and Hankel functions appear naturally in the study of the (scalar) diffraction of light from apertures on a spherical surface as formulated by Marathay (1975). More recently Collett and Wolf (1977) have employed these functions in their study of image fields. The expression for the calculation of the diffracted amplitude in terms of the above functions does not appear to contain any evanescent wave contributions. Yet it is well known that evanescent waves inevitbly arise in the study of diffraction from apertures of finite spatial extent. In this paper we shall study the expression for the diffracted field to disclose the evanescent wave contributions contained in it.

2. Diffraction of light from an aperture on a spherical surface

It will be helpful to review briefly the formulation of this problem as given by Marathay (1975). In this problem one assumes that the diffracting aperture is on a sphere. By this is meant that the complex values of the amplitude transmittance of the aperture are specified at points on the sphere. Typically in one formulation of the problem, the sources are assumed to be within the sphere, the aperture occupies the area of a spherical cap, and the rest of the surface of the sphere is assumed to be opaque. The diffracted light is studied in the region of space without the sphere. In another formulation the sources are assumed to be in space outside the sphere and the diffracted amplitude is studied inside the sphere. For a direct comparison of the Rayleigh-Sommerfeld diffraction theory for an aperture in a plane with that of the aperture on a sphere the reader may refer to table 1 given by Marathay and Sheila Prasad (1980).

In either case a Greens function $G(\mathbf{r}, \mathbf{r}')$ is constructed so that it vanishes when either of its arguments coincides with any point $\mathbf{s}$ on the sphere of radius s that incorporates the aperture $\mathscr{A}$. This Greens function is the modified form of the primitive Greens function G_0, which like G fulfils

$$(\nabla'^2+k^2)\, G_0(\mathbf{r}, \mathbf{r}') = -\delta\,(\mathbf{r}-\mathbf{r}'),$$

where the prime on the del-operator indicates differentiation with respect to the primed variable. A similar equation holds for the unprimed variable.

We begin with the eigenfunction expansion of $G_0(\mathbf{r}, \mathbf{r}')$ in the form

$$G_0(\mathbf{r}, \mathbf{r}') = \begin{cases} ik \sum_{l=0}^{\infty} j_l\,(kr)\, h_l^{(1)}\,(kr') \sum_{m=-l}^{l} Y_l^m\,(\theta, \phi)\; Y_l^{m*}\,(\theta', \phi'), & r<r' \\ ik \sum_{l=0}^{\infty} j_l\,(kr')\, h_l^{(1)}\,(kr) \sum_{m=-l}^{l} Y_l^m\,(\theta, \phi)\; Y_l^{m*}\,(\theta', \phi'), & r>r'. \end{cases}$$

To this we add a solution of the homogeneous equation, namely,

$$(\nabla'^2+k^2)\, \psi_H\,(\mathbf{r}_{lm}, \mathbf{r}') = 0.$$

This solution ψ_H is chosen so that the modified function G fulfils the boundary condition with respect to the sphere of radius s mentioned above. In this way we find,

$$G(\mathbf{r}, \mathbf{r}') = ik \sum_{l=0}^{\infty} \frac{h_l^{(1)}\,(kr)}{h_l^{(1)}\,(ks)} [j_l\,(kr')\, h_l^{(1)}\,(ks) - j_l\,(ks)\, h_l^{(1)}\,(kr')]$$
$$\times \sum_{m=-l}^{l} Y_l^m\,(\theta, \phi)\; Y_l^{m*}\,(\theta', \phi'),$$

which clearly vanishes when $\mathbf{r}'=\mathbf{s}$. We wish to find the wave amplitude $\psi(\mathbf{r})$ at the point of observation $\mathbf{r}$ in figure 1. It shows a cross-section in the $x-z$ plane of the sphere of radius s containing the aperture $\mathscr{A}$.

The point of observation $\mathbf{r}$ is enclosed by the surface S. The surface S is made up of three parts S_1, S_2, and S_3, as shown in figure 1. The part S_1 is a sphere of radius s,

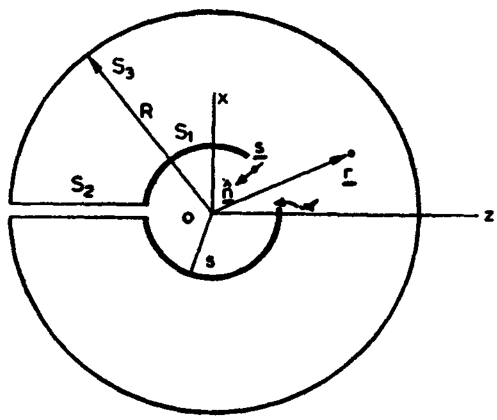

Figure 1. Construction of the closed surface S for the surface integral. It is made up of the sphere S_1 of radius s, the cylinder S_2 of vanishingly small radius, and the sphere S_3 of radius R.

which contains the aperture $\mathscr{A}$. The part S_3 is a sphere of radius R concentric with S_1 about the origin 0. These two spheres are connected by a cylindrical surface S_2 of vanishingly small radius. The contribution to the surface integral from the cylinder S_2 will go to zero when the radius of the cylinder is made vanishingly small. The contribution from the sphere S_3 will be zero when $R \to \infty$, owing to the Sommerfeld radiation condition. Thus we are left with only the contribution of the sphere S_1 of radius s, which leads to

$$\psi(\mathbf{r}) = -\int\int_{\mathscr{A}} \psi(\mathbf{s}) \frac{\partial G(\mathbf{r}, \mathbf{s})}{\partial n} s^2 \, d\Omega_s.$$

In this integral, $s^2 \, d\Omega_s$ is the area element on the sphere with the element of the solid angle $d\Omega_s = \sin\theta_s \, d\theta_s \, d\phi_s$, where θ_s and ϕ_s are the polar angles of the point $\mathbf{s}$ on S_1. In this equation we have made use of the fact that G vanishes on S_1. We will adopt the Kirchhoff-type boundary conditions and ask for the amplitude ψ to be zero on S_1 outside the aperture $\mathscr{A}$, thereby limiting the integral to the area of the aperture $\mathscr{A}$. The unit normal $\hat{n}$ at $\mathbf{s}$ must point out of the volume enclosed by the surface S. That is, as shown in figure 1, $\hat{n}$ points from $\mathbf{s}$ toward the origin 0.

The normal derivative evaluated on this sphere is given by,

$$\partial G(\mathbf{r}, \mathbf{s})/\partial n = -\frac{1}{s^2} \sum_{l=0}^{\infty} [h_l^{(1)}(kr)/h_l^{(1)}(ks)] \times \sum_{m=-l}^{l} Y_l^m(\theta, \phi) \, Y_l^{m*}(\theta_s, \phi_s). \tag{1}$$

This expression has to be used to calculate the diffracted amplitude denoted by $\psi(\mathbf{r})$ or $\psi(r, \theta, \phi)$. The Laplace expansion of this amplitude reads,

$$\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{lm}(r) \, Y_l^m(\theta, \phi). \tag{2}$$

The relationship of the Laplace coefficients, $\psi_{lm}(r)$, of the diffracted amplitude to those of the aperture distribution, $\psi_{lm}^{\mathscr{A}}(s)$, on the sphere of radius s is found by using (1);

$$\psi_{lm}(r) = \psi_{lm}^{\mathscr{A}}(s) \, T_l(r, s),$$

where
$$T_l(r, s) = h_l^{(1)}(kr)/h_l^{(1)}(ks). \tag{3}$$

The function T_l may be referred to as the transfer function of free space for spherical geometry.

The diffracted amplitude on the sphere of radius r is then described by,

$$\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} [h_l^{(1)}(kr)/h_l^{(1)}(ks)] \sum_{m=-l}^{l} \psi_{lm}^{\mathscr{A}}(s) \, Y_l^m(\theta, \phi). \tag{4}$$

In this equation $\psi_{lm}^{\mathscr{A}}(s)$ contains the information about the complex transmittance of the aperture on the sphere and the incident amplitude. We have to study this expression for the evanescent contributions for the case $r > s$.

3. Calculation of the diffracted amplitude

The spherical Hankel functions that depend on the radius vector r or s are related to the Hankel functions of half integer order by,

$$h_l^{(1)}(kr) = (\pi/2kr)^{1/2} H_{l+(1/2)}^{(1)}(kr). \tag{5}$$

In optics, $kr \sim 10^7$, when the radial distance r is of the order of one meter. For large arguments, the behaviour of the Hankel functions changes according to whether the index l is less than or greater than the argument. For large l we may neglect the $(\frac{1}{2})$ in (5). The behaviour of the Hankel functions for large arguments and large l has been studied by Debye and Watson. A lucid description of the mathematics for deriving these formulas is given by Sommerfeld (1964). The Debye formula for $l < \rho$ is

$$H_l^{(1)}(\rho) = (2/\pi\rho \sin \alpha)^{1/2} \exp [+ i\rho (\sin \alpha - \alpha \cos \alpha) - i(\pi/4)], \tag{6}$$

where the parameter α is defined by $l = \rho \cos \alpha$. For the index $l > \rho$ the Debye formula is

$$H_l^{(1)}(\rho) = (2/\pi\rho \sinh \alpha)^{1/2} \exp [+ \rho (\alpha \cosh \alpha - \sinh \alpha) - i(\pi/2)], \tag{7}$$

where we have put $l = \rho \cosh \alpha$. It is easy to establish that the exponent is positive. The Hankel function increases without bounds for increasing l.

In the diffraction problem we encounter the ratio of the spherical Hankel functions, namely the transfer function,

$$T_l(r, s) = h_l^{(1)}(kr)/h_l^{(1)}(ks). \tag{8}$$

This ratio may be studied by using the Debye formulas for the following cases:

for $l < ks < kr$,

$$T_l(r, s) = (s/r)(\sin \alpha_s/\sin \alpha_r)^{1/2} \exp \{+ ik [r(\sin \alpha_r - \alpha_r \cos \alpha_r) - s(\sin \alpha_s - \alpha_s \cos \alpha_s)]\}. \tag{9}$$

for $ks < l < kr$,

$$T_l(r, s) = (s/r)(\sinh \alpha_s/\sin \alpha_r)^{1/2} \exp [ikr(\sin \alpha_r - \alpha_r \cos \alpha_r) - ks(\alpha_s \cosh \alpha_s - \sinh \alpha_s) + i(\pi/4)], \tag{10}$$

and for $ks < kr < l$,

$$T_l(r, s) = (s/r)(\sinh \alpha_s/\sinh \alpha_r)^{1/2} \exp [+ kr(\alpha_r \cosh \alpha_r - \sinh \alpha_r) - ks(\alpha_s \cosh \alpha_s - \sinh \alpha_s)]. \tag{11}$$

In the first case, $l < ks < kr$, the ratio T_l is purely oscillatory. By studying the exponents of the other two cases with their respective conditions it is easy to establish that they all exhibit a decaying exponential behaviour; that is, they are evanescent.

Owing to the denominators $(\sin \alpha)^{1/2}$ and $(\sinh \alpha)^{1/2}$ the Debye formulas fail when α is close to zero; that is, when the index l is on the same order of magnitude as the argument ρ. In this situation the Watson formulas are useful. Without going into the details we note that for $l \lesssim \rho$ the Hankel function is oscillatory. For $l > \rho$ but not too large compared to ρ, that is when α is small, the behaviour of the Hankel function is dominated by $\exp\,[+ l(\alpha - \tanh \alpha - \frac{1}{3}\tanh^3 \alpha)]$. This exponent is positive. As before the ratio T_l will be oscillatory for $l \lesssim \rho$ and will decay exponentially for $l > \rho$.

Finally for the special case of the index l equal to the argument, $l=\rho$, Watson's formula yields,

$$H_\rho^{(1)}(\rho) = \frac{2}{\Gamma(2/3)}\,(2/9\rho)^{1/3} \exp\,(- i\pi/3). \tag{12}$$

By using this expression the following two special cases are obtained:
for, $l=ks < kr$,

$$T_l\,(r, s) = \frac{\Gamma(2/3)}{r}\,(s/2\pi k \sin \alpha_r)^{1/2}\,(9ks/2)^{1/3}$$
$$\exp\left[+ ikr\,(\sin \alpha_r - \alpha_r \cos \alpha_r) + i\frac{\pi}{12}\right], \tag{13}$$

which is oscillatory and for $ks < kr = l$,

$$T_l\,(r, s) = \frac{s}{\Gamma(2/3)}\,(2\pi k \sin \alpha_s/r)^{1/2}\,(2/9kr)^{1/3}$$
$$\exp\left[- ks\,(\alpha_s \cosh \alpha_s - \sinh \alpha_s) + i\frac{\pi}{6}\right], \tag{14}$$

which decays exponentially.

4. Conclusions

We conclude that for $s < r$, the expression for the diffracted amplitude of (4) and the normal derivative of the Greens function of (1) contain evanescent terms. These terms have the index l greater than the argument ks, which is itself very large for optical diffraction. The formulas given in this paper for different situations are useful in the calculation of the diffracted amplitude for the case of spherical geometry.

Acknowledgements

One of the authors (ASM) was supported in part by the Air Force Office of Scientific Research and the Army Research Office of the United States.

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