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Some solutions of the Einstein-Maxwell equations in general relativity

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Abstract. A solution of the Einstein-Maxwell equations corresponding to source-free electromagnetic field plus pure radiation is obtained. The solution is algebraically special. A particular case of the solution is considered which encompasses many known solutions. Among them is a radiating Ruban metric.

Keywords. General relativity; algebraically special solution; Einstein-Maxwell equations; electromagnetic field.

1. Introduction

There are two types of solutions of the Einstein-Maxwell equations in general relativity, namely algebraically general solution and algebraically special solution. In spite of the fact that an exact gravitational solution radiating from a finite source must be algebraically general (Sachs 1961) the problem of obtaining algebraically special solutions has received much attention due to several reasons. One reason is that the Schwarzschild solution, the Kerr solution (Kerr 1963) and the NUT solution (Newman *et al* 1963) are familiar members of this class. In the present paper, using the complex vectorial formalism as formulated by Cahen *et al* (1967) we obtain some algebraically special solutions of the Einstein's equations plus pure radiation which are equivalent to (Bhatt and Patel 1978)

$$E_{p\bar{q}} = -2F_p\bar{F}_q - \mu\delta_p^2\delta_{\bar{q}}^{\bar{2}}, \tag{1}$$

$$dF^+ = 0. \tag{2}$$

In (1) and (2), $E_{p\bar{q}}$ is the hermitian tensor corresponding to the trace-free part of the Ricci tensor and μ is the density of flowing radiation. $F^+ = F_pZ^p$ is the self-dual part of the electromagnetic field tensor, Z^p forming a basis for the complex 3-space of self-dual 2-forms. The bar denotes the complex conjugation. A detailed account of this formalism is given by Israel (1970). We shall use his notations. In particular, the Greek and the first half of the Latin indices will range from 1 to 4 and second half of the Latin indices will range from 1 to 3.

2. The metric and the Maxwell equations

We consider the metric (Vaidya *et al* 1976)

$$ds^2=2(du+g\sin\alpha\, d\beta)\, dx-2L(du+g\sin\alpha\, d\beta)^2-M^2\,(d\alpha^2+\sin^2\alpha d\beta^2). \quad (3)$$

Here L and M are functions of u, α and x and g is a function of α only. We use u, α, β and x as the coordinates. Introducing the basic 1-forms

$$\theta^1 = du + g\sin\alpha\, d\beta,\quad \sqrt{2}\theta^2 = M(d\alpha + i\sin\alpha d\beta),$$

$$\theta^4 = dx - L\theta^1,\qquad \theta^3 = \bar{\theta}^2, \quad (4)$$

we can express (3) as

$$ds^2 = 2(\theta^1\theta^4 - \theta^2\theta^3). \quad (5)$$

Using (4) we can obtain $d\theta^a$, which by using the defining expressions

$$Z^1 = \theta^3\wedge\theta^4,\ Z^2 = \theta^1\wedge\theta^2,\ Z^3 = (1/2)\,(\theta^1\wedge\theta^4 - \theta^2\wedge\theta^3),$$

for Z^p, will give us dZ^p. Using these expressions for dZ^p, Cartan's first equations of structure

$$dZ^p = (1/2)\,\epsilon^{pmn}\,\sigma_m\wedge Z_n$$

will then determine the connection 1-forms σ_p, ϵ^{pmn} being the Levi-Civita's permutation symbol. The detailed calculations of dZ^p and σ_p are given in Bhatt and Patel (1978) and we shall reproduce the expressions for σ_p for ready reference:

$$\sigma_1 = 2[-(M_x/M) + i\,(f/M^2)]\theta^2,$$

$$\sigma_2 = -\sqrt{2}[(L_\alpha/M) + ig(L_u/M)]\theta^1 + 2[(1/M)(M_u + LM_x)$$
$$- i(Lf/M^2)]\theta^3, \quad (6)$$

$$\sigma_3 = -2[L_x + i(Lf/M^2)]\theta^1 - \sqrt{2}[F + ig(M_u/M)]\theta^2 +$$
$$\sqrt{2}[F - ig(M_u/M)]\theta^3 + 2i(f/M^2)\theta^4.$$

Here $2f = g_\alpha + g\cot\alpha$ and $M^2F = M_\alpha + M\cot\alpha$, and suffixes denote partial derivatives, *viz.*, $M_\alpha = \partial M/\partial\alpha$, etc.

The absence of terms involving θ^3 and θ^4 in σ_1 indicates that the congruence k^a of null tangents is geodesic as well as shear-free.

One can now use σ_p given by (6) and Cartan's second equation of structure

$$\Sigma_p = d\,\sigma_p - (1/2)\,\epsilon_{pmn}\,\sigma^m\wedge\sigma^n$$

to obtain the curvature 2-forms Σ_p. The expressions for Σ_p are recorded in Bhatt and Patel (1978) and are not repeated here. These expressions for Σ_p along with the identity

$$\Sigma_p = C_{pq} Z^q - (1/6) R \gamma_{pq} Z^q + E_{p\bar{q}} \bar{Z}^q,$$

will then determine the hermitian tensor $E_{p\bar{q}}$, the curvature scalar R and the complex-valued trace-free symmetric tensor C_{pq} which correspond to the Weyl tensor. $E_{p\bar{q}}$ and R are given by

$$E_{1\bar{1}} = (2/M) [M_{xx} - (f^2/M^3)],$$

$$E_{1\bar{2}} = \bar{E}_{2\bar{1}} = 0,$$

$$E_{1\bar{3}} = \bar{E}_{3\bar{1}} = \sqrt{2} (g/M) \{(M_x/M)_y - (f/M^2)_u + i \{(M_x/M)_u + (f/M^2)_y\}],$$

$$E_{2\bar{2}} = L^2 E_{1\bar{1}} + (1/M^2) [g^2 (L_{uu} + L_{yy}) + 2f L_y + 2L_u MM_x$$
$$+ 4L MM_{xu} - 2L_x MM_u + 2MM_{uu}],$$

$$E_{3\bar{2}} = \bar{E}_{2\bar{3}} = \sqrt{2} L E_{3\bar{1}} - i\sqrt{2} (g/M) [L_x + (M_u/M) + i (2f L/M^2)]_u$$
$$+ \sqrt{2} (g/M) [L_x + (M_u/M) + i (2f L/M^2)]_y, \tag{7a}$$

$$E_{3\bar{3}} = (2/M^2) \{g^2 [(M_u/M)_u + (M_y/M)_y] - 2M_x M_u - 2L M_x^2$$
$$+ (2f M_y/M) + (6L f^2/M^2)\} + 2L_{xx}.$$

$$R = -E_{3\bar{3}} + 4L E_{1\bar{1}} + 4L_{xx} + (2/M^2) [4MM_{xu} + 4MM_x L_x$$
$$+ (8L f^2/M^2)]. \tag{7b}$$

In the above equations, the variable y replaces the variable α, the defining relation being

$$g\, d\alpha = dy. \tag{8}$$

Since $E_{1\bar{2}} = 0$, it follows from the field equation $E_{1\bar{2}} = -2F_1 \bar{F}_2$ that either (i) $F_1 = 0$ or (ii) $F_2 = 0$ or (iii) $F_1 = 0$ and $F_2 = 0$. We take $F_1 = 0$ and assume the self-dual 2-form F^+ as

$$F^+ = \phi Z^2 + \psi Z^3, \tag{9}$$

where ϕ and ψ are complex-valued functions of u, α and x. Since $F_1 = 0$, it follows from the field equations (1) that $E_{1\bar{1}} = 0$ and $E_{1\bar{3}} = 0$. These equations involve only one unknown function M. They can be solved to get

$$M^2 = (f/Y)(X^2 + Y^2), \tag{10}$$

where $$X_u = -Y_y,\ X_y = Y_u,\ X_x = -1,\ Y_x = 0, \tag{11}$$

With M given by (10) and (11) and Σ_1 given in Bhatt and Patel (1978) we have verified that $C_{11} = 0$, $C_{13} = 0$. Therefore, the space-time given by the metric (3) is algebraically special.

Using F^+ given by (9), M^2 given by (10) and (11) and dZ^p listed in Bhatt and Patel (1978) we have verified that the Maxwell equations (2) imply the following equations for ϕ and ψ:

$$\psi_x - 2\psi\,[-(M_x/M) + i\,(f/M^2)] = 0, \tag{12}$$

$$\psi_u + i\,\psi_y = 0, \tag{13}$$

$$2\sqrt{2}\,(\phi M)_x - g\,(i\,\psi_u + \psi_y) = 0, \tag{14}$$

$$\begin{aligned}\sqrt{2}\,(g/M)\,(-i\,\phi_u+\phi_y) + \sqrt{2}\,\phi\,[F-i\,(M_u/M)] - \psi_u - L\psi_x \\ - 2\psi[(M_u/M) - L\,\{-(M_x/M) + i\,(f/M^2)\}] = 0.\end{aligned} \tag{15}$$

We can use (10) to solve (12) for ψ. The function ψ is given by

$$\psi = K(X-iY)^{-2}, \tag{16}$$

where K is a complex function of u and y. With ψ given by (16), equation (13) implies that

$$e_u = h_y,\ e_y = -h_u,\ K = e+ih. \tag{17}$$

Using ψ given by (16) and (17), equation (14) gives us the following form for the function ϕ:

$$\phi = (ig/\sqrt{2}M)\,[K/(X-iY)]_u. \tag{18}$$

Finally, using all the relevant results of this section, equation (15) implies that

$$K = HY, \tag{19}$$

where H is a complex function of y only.

3. The remaining Einstein-Maxwell equations

We set $R = 0$ and use M^2 given by (10) and (11) to determine the following form of function $2L$:

$$2L = -(Y_u/Y)\,X + 2G + (X^2 + Y^2)^{-1}\,(2FX + 2EY), \tag{20}$$

where $2G = (Y/f)\,[(1/2)g^2\,\nabla^2 \log(f/y) + f_y - 1 - 3f\,(Y_y/Y)]$,

$$\nabla^2 = \partial^2/\partial u^2 + \partial^2/\partial y^2. \tag{21}$$

E and F are undetermined functions of u and y. With the form (20) of $2L$ and with M given by (10) and (11), the field equation $E_{3\bar{3}} = -2F_3\bar{F}_3$ then implies that

$$E = -2GY - YY_y - (1/4Y)\,K\bar{K}. \tag{22}$$

It then follows from $E_{3\bar{2}} = -2F_3\bar{F}_2$ that

$$F_u = -[E + (K\bar{K}/4Y)]_y,\; F_y = [E + (K\bar{K}/4Y)]_u. \tag{23}$$

Using $E_{22} = -2F_2\bar{F}_2 - \mu$, a straightforward but lengthy calculation gives the radiation density μ. The expression for μ is recorded in the appendix.

The corresponding electromagnetic field tensor $F_{\alpha\beta}$ can easily be obtained:

$$\begin{aligned} &F_{12} = g(x\psi_1 + y\psi_2)_y, && F_{13} = -g \sin\alpha (x\psi_1 + y\psi_2)_u \\ &F_{14} = \psi_1, && F_{23} = g^2 \sin\alpha (x\psi_1 + y\psi_2)_y - M^2 \sin\alpha\, \psi_2, \\ &F_{24} = 0, && F_{34} = g \sin\alpha\, \psi_1,\; \psi = \psi_1 + i\psi_2. \end{aligned} \tag{24}$$

Here we have named the coordinates

$$x^1 = u,\; x^2 = \alpha,\; x^3 = \beta,\; x^4 = x.$$

We have so far worked with the general line element (3). A case $f = Y$ has been treated in Bhatt and Patel (1978). It has been shown that the radiating Debney-Kerr Schild (1969) metric and the radiating Brill (1964) metric can be incorporated in (3) with $M^2 = X^2 + Y^2$ and $2L$ given by (20). In the next section we shall consider one more case which seems to be of physical interest.

4. The case f ≠ Y, Y = Y(y)

We consider a case in which $Y = Y(y)$. It then follows from (11) that

$$Y = -ay + b,\; X = au - x, \tag{25}$$

a and b being constants of integration. No additional constant is added in X because such a constant can always be incorporated in x coordinate.

Since Y is a function of y only, equations (21) and (19) imply that G and K are also so. Consequently (17) would imply that K is a complex constant and E given by (22) will be a function of y only. It then follows from (23) that

$$E + (K\bar{K}/4Y) = ky + l, \quad F = -ku - m, \tag{26}$$

k, l, m being constants of integration.

We now introduce a variable and a function p as follows:

$$(f/Y)^{1/2}\, d\alpha = d\theta, \quad (f/Y)^{1/2} \sin\alpha = p(\theta). \tag{27}$$

Then we find from (21) that

$$2G = (p_{\theta\theta}/p) + 2a.$$

This expression for $2G$ along with equations (22) and (26), then implies that

$$p_{\theta\theta} + [a + (1/Y)(ky + l)]\, p = 0. \tag{28}$$

We now consider a case in which $a + (1/Y)(ky + l)$ is a constant, say ϵ, where $\epsilon = 1, 0, -1$. It then follows that

$$k = a(a - \epsilon), \quad 1 = -b(a - \epsilon). \tag{29}$$

Equation (28) can then be integrated. We get

$$p = \begin{cases} \sin\theta \\ \theta \\ \sinh\theta, \cosh\theta, e^{\theta} \end{cases} \quad \text{when } \epsilon = \begin{cases} 1 \\ 0 \\ -1, \end{cases} \tag{30}$$

The expressions for $2G$, $E + (K\bar{K}/4Y)$ and F then become

$$2G = 2a - \epsilon, \quad E + (K\bar{K}/4Y) = -(a - \epsilon)Y, \quad F = -a(a - \epsilon)u - m. \tag{31}$$

The foregoing expressions for $2G$, E and F determine $2L$ as

$$2L = \epsilon - (X^2 + Y^2)^{-1}\{2\,[(a - \epsilon)x + m]X + (1/2)(e^2 + h^2)\}. \tag{32}$$

The density of flowing radiation μ and the electromagnetic field $F_{\alpha\beta}$ in this case are given by

$$\mu = -2a(a - \epsilon)/(X^2 + Y^2). \tag{33}$$

$$\begin{aligned} F_{12} &= ag\psi_2, & F_{13} &= ag \sin\alpha\psi_1, \\ F_{14} &= \psi_1, & F_{23} &= -\sin\alpha(ag^2 + M^2)\psi_2, \\ F_{24} &= 0, & F_{34} &= g\sin\alpha\,\psi_1, \end{aligned} \tag{34}$$

where $\psi = \psi_1 + i\psi_2$ and ψ is given by (16). In view of the relations (8), (25) and (27), we have the following relations between Y and $g \sin\alpha$:

$$2pY = (g\sin\alpha)_{\theta}, \qquad pY_{\theta} = -ag\sin\alpha, \tag{35}$$

where p is given by (30). These relations together then give the following differential equation:

$$Y_{\theta\theta} + (p_{\theta}/p)\, Y_{\theta} + 2aY = 0. \tag{36}$$

With p given by (30), (36) is a differential equation for Y. One can solve it and use Y in (35) to get $g \sin \alpha$. M^2 and $2L$ are given by (10), (11) and (32). Thus all the metric potentials are determined. The case $\epsilon = 1$ and $a \neq 0$ has been discussed earlier by us (Bhatt and Patel 1978). In the next section we shall give some other particular cases.

5. Particular cases

We cite several particular cases here. In all the cases we shall give the forms of the line elements along with the corresponding values of the density of flowing radiation μ and the electromagnetic field tensor $F_{\alpha\beta}$.

5.1. *Case (i)*: $a = 0$

The metric in this case is

$$ds^2 = 2\,(du + g \sin \alpha d\beta)\, dx - (x^2 + b^2)\,(d\theta^2 + p^2\,(\theta)\, d\beta^2)$$
$$+ \{\epsilon - (x^2 + b^2)^{-1}\, [2mx + 2\epsilon b^2 - (1/2)\,(e^2 + h^2)]\}\,(du + g \sin \alpha d\beta)^2, \quad (37)$$

where $p\,(\theta)$ is given by (30) and $g \sin \alpha$ is given by

$$g \sin \alpha = \begin{cases} -2b \cos \theta \\ b\theta^2 \\ 2b \cos h\theta,\ 2b \sin h\theta,\ 2be^{\theta} \end{cases} \quad \text{when } \epsilon = \begin{cases} 1 \\ 0 \\ -1 \end{cases}, \quad (38)$$

$$\mu = 0. \quad (39)$$

$$F_{14} = \psi_1,\ F_{23} = -(x^2 + b^2)\, p(\theta)\, \psi_2,\ F_{34} = g \sin \alpha\, \psi_1. \quad (40)$$

where $g \sin \alpha$ and $\psi = \psi_1 + i\, \psi_2$ are given by (38) and (16), respectively. The metric (37) along with (38), (39) and (40) is the same as that obtained by Ruban (1972) with $\lambda = 0$ there.

5.2. *Case (ii)*: $a = \epsilon$

The subcase $\epsilon = 0$ is discussed in case (i). Therefore, we shall give the results for $\epsilon = 1$ and $\epsilon = -1$. The metric in this case is

$$ds^2 = 2\,(du + g \sin \alpha\, d\beta)\, dx - (X^2 + Y^2)\,(d\theta^2 + p^2(\theta)\, d\beta^2)$$
$$- \{\epsilon - (X^2 + Y^2)^{-1}\, [2mX + (1/2)\,(e^2 + h^2)]\}\,(du + g \sin \alpha d\beta)^2, \quad (41)$$

where $X = \epsilon\, u - x$ and $p\,(\theta)$ is given by (30). The functions Y and $g \sin \alpha$ are given by

$$Y = \begin{cases} k \cos \theta \\ k \cos h\theta,\ k \sin h\theta,\ ke^{\theta} \end{cases} \quad \text{when} \quad \epsilon = \begin{cases} 1 \\ -1 \end{cases}, \quad (42)$$

and $$g \sin \alpha = \begin{cases} k \sin^2 \theta \\ k \sin h^2 \theta,\ k \cos h^2 \theta,\ ke^{2\theta} \end{cases} \quad \text{when} \quad \epsilon = \begin{cases} 1 \\ -1 \end{cases}, \tag{43}$$

$$\mu = 0, \tag{44}$$

$$\begin{aligned} F_{12} &= \epsilon\, g\, \psi_2 \quad , & F_{13} &= \epsilon\, g \sin \alpha\, \psi_1, \\ F_{14} &= \psi_1 \quad , & F_{23} &= -\sin \alpha\, (\epsilon\, g^2 + X^2 + Y^2)\, \psi_2, \\ F_{24} &= 0 \quad , & F_{34} &= g \sin \alpha\, \psi_1, \end{aligned} \tag{45}$$

where $g \sin \alpha$ and $\psi = \psi_1 + i\, \psi_2$ are given by (43) and (16), respectively. Here it should be noted that α and θ are related by

$$d\alpha/\sin \alpha = d\theta/p\,(\theta).$$

The metric (41) is an electromagnetic generalisation of Kinnersley's (1969) vacuum metrics (cases IIA, IIB, IIC, IID with $l = 0$ there).

5.3. *Case (iii)*: $\epsilon = -1$, $p = e^{\theta}$, $a \neq 0$.

The metric in this case is

$$\begin{aligned} ds^2 = {} & 2\,(du - (1/2)\, e^{\theta} Y_{\theta}\, d\beta)\, dx - (X^2 + Y^2)\,(d\theta^2 + e^{2\theta}\, d\beta^2) \\ & + \{1 + (X^2 + Y^2)^{-1}\, [2\bar{m}x + (1/2)\,(e^2 + h^2)]\}\,(du - (1/a)\, e^{\theta} Y_{\theta}\, d\beta)^2, \end{aligned} \tag{46}$$

where $$Y = e^{-\theta/2}\,[Ae^{q\theta/2} + Be^{-q\theta/2}],\ q = (1 - 8a)^{1/2}$$

$$X = au - x, \tag{47}$$

$$\bar{m} = (a+1)\, x + m,$$

$$\mu = -2a\,(a+1)/(X^2 + Y^2), \tag{48}$$

$$\begin{aligned} F_{12} &= -Y_{\theta}\, \psi_2, & F_{13} &= -e^{\theta} Y_{\theta}\, \psi_1, \\ F_{14} &= \psi_1, & F_{23} &= -e^{\theta}\, [(1/a)\,(Y_{\theta}^2 + X^2 + Y^2)]\, \psi_2, \\ F_{24} &= 0, & F_{34} &= -(1/a)\, e^{-\theta}\, Y_{\theta}\, \psi_1. \end{aligned} \tag{49}$$

5.4. *Case (iv)*: $\epsilon = 0$, $p(\theta) = \theta$, $a \neq 0$.

The space-time geometry of this case is described by the line element

$$\begin{aligned} ds^2 = {} & 2\,(du - (1/a)\, \theta\, Y_{\theta}\, d\beta)\, dx - (X^2 + Y^2)\,(d\theta^2 + \theta^2\, d\beta^2) \\ & (X^2 + Y^2)^{-1}\, [2\,(ax+m)\, X + (1/2)\,(e^2+h^2)]\,(du - (1/a)\, \theta\, Y_{\theta}\, d\beta)^2, \end{aligned} \tag{50}$$

where $$Y = bJ_0(\sqrt{2a\theta}) + cY_0(\sqrt{2a\theta}), \tag{51}$$

J_0 and Y_0 being Bessel functions of zero order of the first and second kind, respectively. The radiation density μ and the electromagnetic field tensor $F_{\alpha\beta}$ in this case are given by

$$\mu = -2a^2/(X^2 + Y^2), \tag{52}$$

$$F_{12} = -Y_\theta\psi_2, \qquad F_{13} = -\theta Y_\theta\psi_1,$$

$$F_{14} = \psi_1, \qquad F_{23} = -\theta[(1/a)Y_\theta^2 + X^2 + Y^2]\psi_2, \tag{53}$$

$$F_{24} = 0, \qquad F_{34} = -(1/a)\theta Y_\theta\psi_1,$$

where $\psi = \psi_1 + i\psi_2$ is given by (16). When $c = 0$, it can be verified that as $a \to 0$, $(1/a)\theta Y_\theta \to -b\theta^2$, and so the metric (50) will go over to the metric (37) with $\epsilon = 0$. Since Y is singular for $a = 0$, it follows that the metric (50) is also singular for $a = 0$. Thus, the metric (50) is a radiating Ruban metric for the case when the cosmological constant λ vanishes.

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Appendix

The expression for density of flowing radiation μ is given as follows:

$$\begin{aligned} -\mu = {} & (g^2/M^2)\{\nabla^2 G - (f/M^2)\nabla^2(K\bar{K}/4Y) - 2(M_x/M)_y(K\bar{K}/4Y)_u \\ & + 2(M_x/M)_u(K\bar{K}/4Y)_y + (X^2 + Y^2)^{-1}[(e_u^2 + h_u^2) + 2K\bar{K}((1/2) \\ & (X^2 + Y^2)^{-1}(X_u^2 + Y_u^2) - (\log\sqrt{X^2 + Y^2})_u(\log\sqrt{e^2 + h^2})_u \\ & + (\tan^{-1}(h/e)_u(\tan^{-1}(Y/X))_u)]\} + \\ & (X^2 + Y^2)^{-1}[-Y^2(Y_u/Y)_u - 2F_u + 3F(Y_u/Y) + 2YG_y - 3Y_u^2 \\ & + 2YY_{uu}] + (X^2 + Y^2)^{-2}[-2Y^2(K\bar{K}/4Y)_y \\ & + 2XY(K\bar{K}/4Y)_u - (K\bar{K}/4Y)(XY_u - YX_u)]. \end{aligned}$$

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