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Charged particle orbits in Kerr geometry with electromagnetic fields as viewed from locally non-rotating frames

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Abstract. The charged particle orbits in electromagnetic fields on Kerr background as viewed from a locally non-rotating frame do not exhibit non-gyrating bound orbits, which was an essential feature in the earlier study of Prasanna and Vishveshwara, thus showing the non gyration to be due to the effect of dragging of inertial frames produced by the rotating black hole.

Keywords. Charged particle orbits; locally non-rotating frame; general relativity.

While studying the orbits of charged particles in an electromagnetic field on the Kerr background geometry Prasanna and Vishveshwara (1978) had found that the particles execute Larmor motion (gyration) in their bound orbits only when they are completely outside the ergosphere of the black hole generating the space-time curvature. This was found to be so because of the inertial frame dragging effect which precludes completely any retrograde motion within the ergosphere. But it is known that this frame dragging arises mainly because of the Boyer-Lindquist coordinates and that if one goes over to a Locally Non-Rotating Frame (LNRF) as defined by Bardeen (1970) there is no frame-dragging. It is essential to see whether as a result of this, the non-gyration of charged particle also is a Boyer-Lindquist effect which may not exist in LNRF. With this in mind we now consider the orbits of charged particles for the same set of parameters as in the earlier case but as viewed from the LNRF.

The Kerr geometry as expressed by Boyer-Lindquist (B-L) coordinates

$$ds^2 = -(1-2\,mr/\Sigma)\,c^2\,dt^2 - (4\,mra/\Sigma)\sin^2\theta\; c\,dt\,d\phi + (\Sigma/\Delta)\,dr^2 + \Sigma\,d\theta^2 + (A/\Sigma)\sin^2\theta\,d\phi^2,$$

with $\Sigma = (r^2+a^2\cos^2\theta)$, $\quad \Delta = (r^2+a^2-2\,mr)$,

$$A = (r^2+a^2)^2 - \Delta\, a^2\sin^2\theta, \quad m = MG/c^2 \tag{1}$$

when expressed in cannonical form is given by (Breuer 1975)

$$\begin{aligned} ds^2 &= -[(\Sigma^{1/2}\Delta^{1/2}/A^{1/2})\,c\,dt]^2 + [(\Sigma^{1/2}/\Delta^{1/2})\,dr]^2 \\ &\quad + [\Sigma^{1/2}\,d\theta]^2 + [(A^{1/2}\sin\theta/\Sigma^{1/2})\{d\phi - (2\,mar/A)\,cdt\}]^2, \\ &= -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2. \end{aligned} \tag{2}$$

If $U^{\hat{i}}$ represents the four-velocity as given by LNRF θ^i it will be related to U^i the four-velocity in the B-L coordinate frame through the transformation

$$U^{\hat{i}} = e^{\hat{i}}_j \, U^j \tag{3}$$

wherein the transformation matrix $e^{\hat{i}}_j$ is given by

$$e^{\hat{i}}_j = \begin{bmatrix} (\Sigma\Delta/A)^{1/2} & 0 & 0 & 0 \\ 0 & (\Sigma/\Delta)^{1/2} & 0 & 0 \\ 0 & 0 & \Sigma^{1/2} & 0 \\ \left(\dfrac{-2\,mar}{\Sigma^{1/2}\,\Delta^{1/2}}\right)\sin\theta & 0 & 0 & (A/\Sigma)^{1/2}\sin\theta \end{bmatrix}. \tag{4}$$

Similarly defining

$$F^{\hat{i}}_{\hat{j}} = e^{\hat{i}}_k \, e^l_{\hat{j}} \, F^k_l \tag{5}$$

and

$$\Gamma^{\hat{i}}_{\hat{j}\hat{k}} = \Gamma^l_{mn} \, e^{\hat{i}}_l \, e^m_{\hat{j}} \, e^n_{\hat{k}} + e^p_{\hat{j},q} \, e^{\hat{i}}_p \, e^q_{\hat{k}} \tag{6}$$

wherein $e^l_{\hat{j}}$ is the inverse matrix of $e^{\hat{i}}_j$, we can obtain the differential equations governing the motion in equatorial plane from those in B-L coordinates as given in the earlier paper (Prasanna and Vishveshwara 1978).

$$\begin{aligned} \frac{\rho}{\Delta^{1/2}}\frac{d^2\rho}{d\sigma^2} = \frac{1}{B}\Big\{ & \Delta^{1/2}(1 - a^2/\rho^3)\,(d\phi/d\sigma)^2 + \frac{2a}{\rho}(3 + a^2/\rho^2)(d\phi/d\sigma)(d\tau/d\sigma) \\ & - \frac{1}{\rho^3\Delta^{1/2}}[(\rho^2 + a^2)^2 - 4\,a^2\,\rho]\,(d\tau/d\sigma)^2 \Big\} \\ & + \frac{1}{\rho B^{1/2}}\Big\{\Delta^{1/2}\,\bar{A}_{\phi,\rho}\,(d\phi/d\sigma) + \frac{2a}{\rho}\,\bar{A}_{\phi,\rho}\left(\frac{d\tau}{d\sigma}\right) + B\,A_{\tau,\rho}\,(d\tau/d\sigma)\Big\} \\ & - \frac{1}{\Delta^{1/2}}\Big\{1 - \frac{\rho(\rho-1)}{\Delta}(d\rho/d\sigma)^2\Big\}, \end{aligned} \tag{7}$$

$$d\phi/d\sigma = (L - \bar{A}_\phi)/B^{1/2}, \tag{8}$$

$$d\tau/d\sigma = \frac{1}{(B\,\Delta)^{1/2}}[B(E + A_\tau) - (2\,a/\rho)(L - \bar{A}_\phi)], \tag{9}$$

with $$B = (\rho^2 + \alpha^2 + 2\,\alpha^2/\rho), \quad \Delta = (\rho^2 - 2\,\rho + \alpha^2), \tag{10}$$

whereas ρ, α, σ, L, τ and $\bar{A}_\phi$ are being the same as defined earlier.

From the normalisation condition

$$\eta_{\hat{i}\hat{j}}\, U^{\hat{i}}\, U^{\hat{j}} = -1, \tag{11}$$

we find that putting $U^{\hat{\rho}} = 0$, the effective potential for the ρ-motion is given by

$$V_{\text{eff}} = E_{\text{min}} = -A_\tau + K/R,$$
$$K = 2\alpha\,(L - \bar{A}\phi) + \Delta^{1/2}\,\{\rho^2\,(L - \bar{A}\phi)^2 + \rho R\}^{1/2},$$
$$R = (\rho^3 + \alpha^2\rho + 2\alpha^2), \tag{12}$$

which is the same as what it was for the earlier situation. Thus the turning points of the orbits remain the same and hence fixing the constant physical parameters α, L and E and integrating the system of equations (7) to (9) with the initial condition

$$(d\rho/d\sigma)_0 = \frac{1}{\rho_0}\left\{-\Delta_0 - (1 - 2/\rho_0)\,[L - (\bar{A}_\phi)_0]^2 \right.$$
$$\left. - \frac{4\alpha}{\rho_0}\,[E + (A_\tau)_0]^2\,[L - (\bar{A}_\phi)_0]^2 + B_0\,[E + (A_\tau)_0]^2\right\}^{1/2}, \tag{13}$$

wherein B_0, Δ_0, $(A_\tau)_0$ and $(\bar{A}_\phi)_0$ are the values of these quantities at $\rho = \rho_0$, we can obtain the particle orbits.

Figures 1 to 3 show the plots of the orbits for different parameters α, L, E and λ for the dipole case with the Petterson vector potential:

$$A_\tau = \left(\frac{-3\lambda\alpha}{4\,(1 - \alpha^2)^{3/2}}\right)\left\{\left(1 - \frac{1}{\rho}\right)\ln\left(\frac{\rho - 1 + (1 - \alpha^2)^{1/2}}{\rho - 1 - (1 - \alpha^2)^{1/2}}\right) - \frac{2\,(1 - \alpha^2)^{1/2}}{\rho}\right\}, \tag{14}$$

$$\bar{A}_\phi = \left(\frac{-3\lambda}{8\,(1 - \alpha^2)^{3/2}}\right)\left\{[2\,(1 - \alpha^2)^{1/2}]\,(1 + \rho + 2\alpha^2/\rho)\right.$$
$$\left. - (\rho^2 + \alpha^2 - 2\alpha^2/\rho)\,\ln\left(\frac{\rho - 1 + (1 - \alpha^2)^{1/2}}{\rho - 1 - (1 - \alpha^2)^{1/2}}\right)\right\}, \tag{15}$$

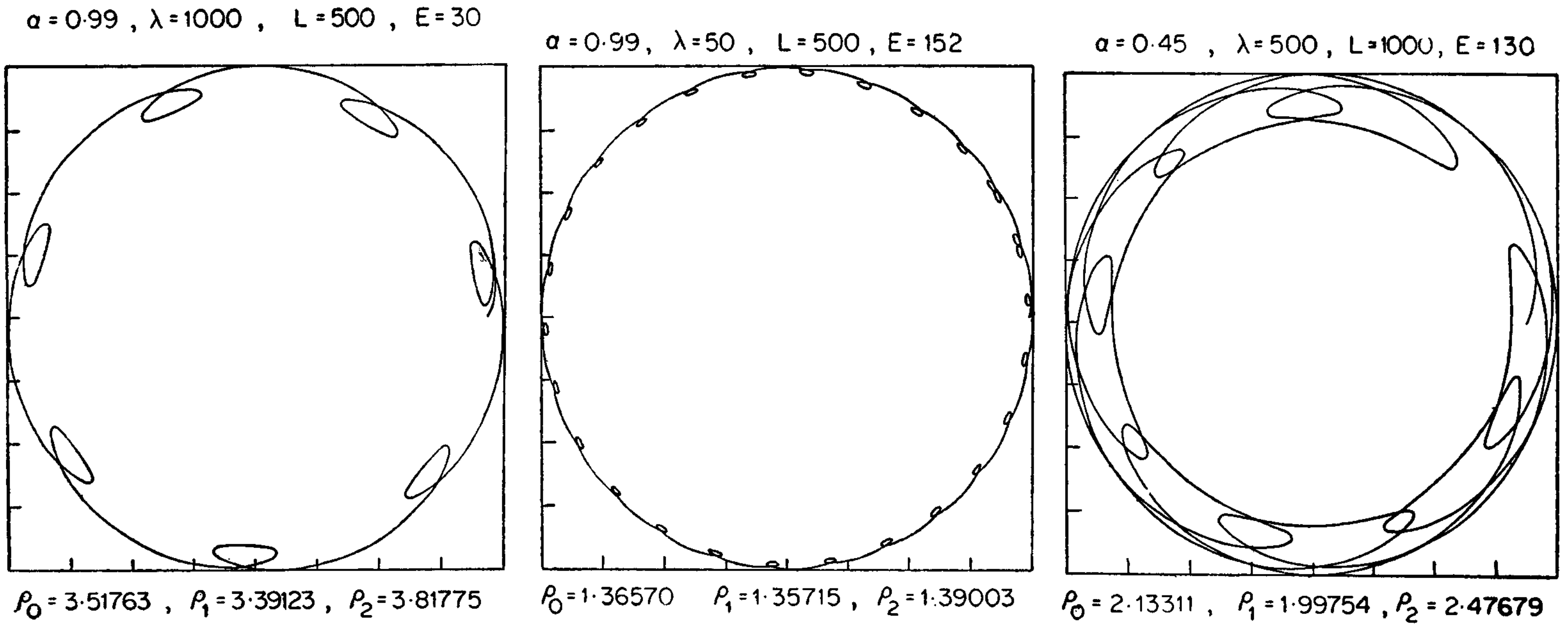
$$\lambda = e\mu/m^2c^2,$$

whereas figures 4 to 6 show the plots for the case of uniform magnetic field with the Wald vector potential

$$A_\tau = -\lambda\,\alpha\left(1 - \frac{1}{\rho}\right), \tag{16}$$

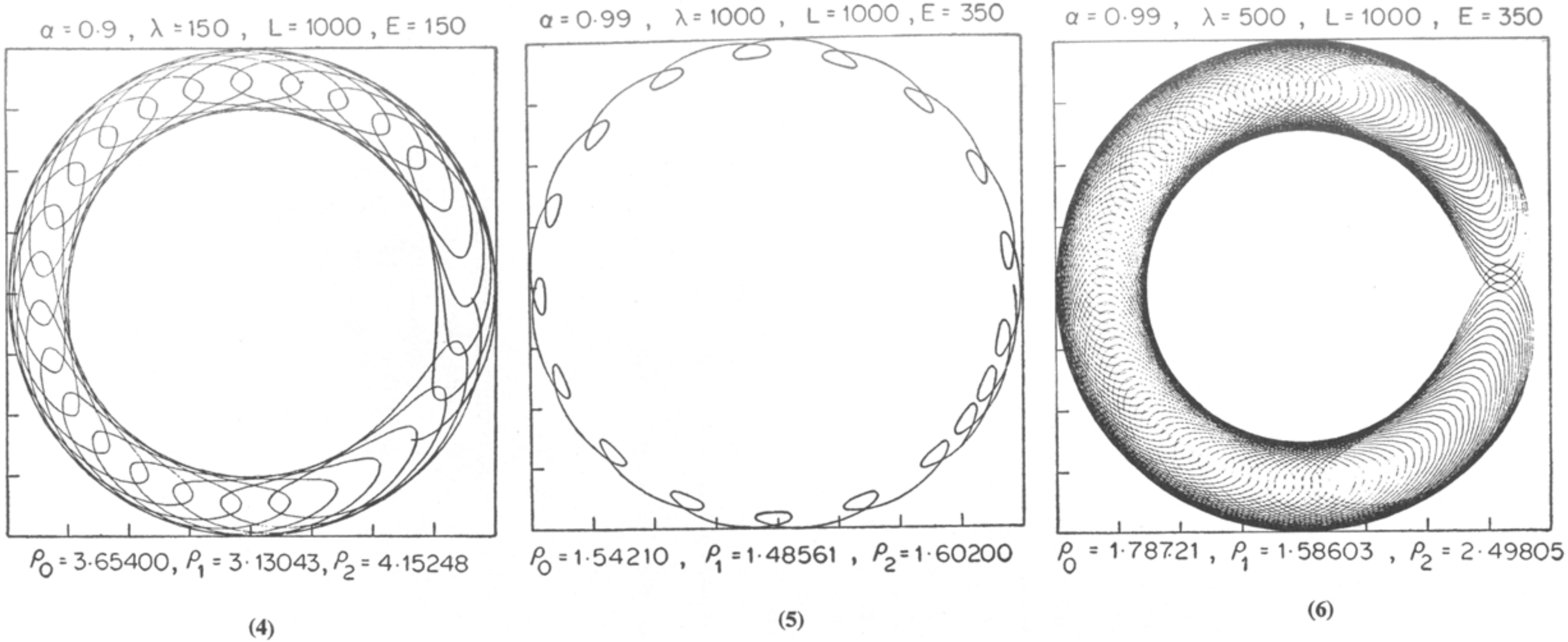
$$\bar{A}_\phi = \frac{\lambda}{2}\left[\rho^2 + \alpha^2\left(1 - \frac{2}{\rho}\right)\right]$$

$$\lambda = e\,B_0\,m/c^2 \tag{17}$$



(1) (2) (3)

Figures 1 to 3. Orbits in the equatorial plane as viewed from a LNRF for the case of dipole magnetic field on Kerr background, parameters being same as those in figures 8, 12 and 14 of Prasanna and Vishveshwara (1978).



Figures 4 to 6. Orbits in the equatorial plane as viewed from a LNRF for the case of uniform magnetic field on Kerr background, parameters being same as those in figures 26, 22 and 21 of Prasanna and Vishveshwara (1978).

As may be seen from the plots the gyration exists in all cases irrespective of the fact whether the particle is completely inside (figures 2 and 5), or completely outside (figures 1 and 4) or moves in and out of the ergosurface (figures 3 and 6).

In fact if we examine this analytically through the condition $(d\phi/d\sigma)_{\rho_g} = 0$, i.e. $L = (\bar{A}_\phi)_{\rho_g}$, we find that

$$(d\rho/d\sigma)_{\rho_g} = \left[\frac{\Delta^{1/2}}{\rho}\left\{-1 + \frac{B}{\Delta}(E + A_\tau)^2\right\}^{1/2}\right]_{\rho_g}$$

which can be made real irrespective of what ρ_g is. Thus it is clearly borne out that the inertial frame dragging of Kerr geometry is a coordinate effect which does not appear for a locally Lorentz observer.

The analysis made above clearly shows that an LNRF observer will continue to see the charged particle gyrating in its orbit whether inside or outside the ergosurface. But for a distant stationary observer (following the global time-like Killing vector) the charged particle does not gyrate when it is inside the ergosphere because of the frame dragging effect which in turn is due to the angular momentum of the black hole. This implies that in the LNRF the effect of frame dragging has been cancelled out completely at least in regard to the motion of charged particle around Kerr black hole. The analysis as seen from an LNRF may be considered as similar to the case of the analysis of a co-moving observer in the surface of a collapsing sphere who would cross the event horizon in finite time whereas for the distant observer this will happen only asympotically as $t \to \infty$.

Even though the emission characteristics as seen by the local observer with LNRF may not drastically change at the ergosurface, for a distant observer, there would be a change analogous to the changes that one encounters between rotating frames while studying the electromagnetic phenomena, and therefore the possible spectral cut off as pointed out earlier (Prasanna and Vishveshwara 1978) for emissions from non-gyrating orbits may be worth looking for.

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