Pramāna, Vol. 14, No. 2, February 1980, pp. 103-111. © Printed in India.

Rayleigh-Sommerfeld diffraction theory and Lambert's law

ARVIND S MARATHAY and SHEILA PRASAD*
Optical Sciences Center, University of Arizona, Tucson, Arizona 85721, USA
*Department of Electrical Engineering, Northeastern University, Boston, Mass. 02114, USA

MS received 7 September 1979

Abstract. The Rayleigh-Sommerfeld diffraction theory is used to derive expressions for (1) the spectral irradiance on the surface of a hemisphere covering the aperture and (2) the spectral radiant intensity. For a uniform, noncoherent source-aperture, both calculations predict a cos θ angular variation, as is known to be the case of Lambertian sources. A cosine-fourth dependence of the spectral irradiance on a plane parallel to the aperture plane is also indicated.

Keywords. Diffraction; non-coherence; radiometry; Lambert's law.

1. Introduction

Walther (1968) proposed a tentative definition of the generalised radiance that could be used for any state of coherence of the radiation field. This work was extended by Marchand and Wolf (1974) to include the definition of the generalised radiant intensity. The limiting case of a uniform, noncoherent source field whose cross-power spectral density is delta correlated was examined, and it was concluded that the radiant intensity varied as $\cos^2 \theta$. This same result was also obtained when the cross-power spectral density had the Besinc form $2J_1(kr)/(kr)$. Although the delta function is a mathematical idealisation, the Besinc form is frequently realised in optical experiments. The physical implication of the result obtained by Walther, Marchand, and Wolf is that such 'uniform' sources will look *darker* as one views them at larger angles from the source normal. Their result is particularly perplexing because the spatial frequency spectrum of the cross-power spectral density used in the above discussion is constant over the real plane waves. Such sources deliver the same amount of power in every direction of propagation of the real plane waves. Therefore, it is reasonable to expect that such sources would exhibit the same properties as the so-called Lambertian sources of classical radiometry.

In this paper, we show that a uniform, noncoherent source of finite extent is indeed Lambertian. This is done by providing three independent derivations within the framework of the basic Rayleigh-Sommerfeld diffraction theory without recourse to any kind of special formalism of generalised radiometry.

2. Spectral irradiance

In the Rayleigh-Sommerfeld (RS) formulation of diffraction theory, a Green's function $G(\mathbf{r}-\mathbf{r}')$ is constructed so that it vanishes when the point $\mathbf{r}'$ (or the point $\mathbf{r}$) lies on a plane (say at $z'=0$). The vector $\mathbf{r}$ has components (x, y, z). The (scalar) wave amplitude $\psi(x, y, z)$ in the diffracted field is related to the amplitude $\psi_A(x', y', 0)$ on the aperture by the relation

$$\psi(x, y, z) = \iint \psi_A(x', y', 0)\left[-\frac{\partial G}{\partial n}(x-x', y-y', z)\right] dx'\, dy'. \tag{1}$$

This integral is limited by the Kirchhoff boundary conditions to the area A of the aperture. The symbol $\partial G/\partial n$ is the normal derivative of the Green's function evaluated at the aperture plane $z' = 0$.

The angular spectrum of the plane wave formulation corresponding to the above convolution integral gives

$$\tilde{\psi}(\kappa p, \kappa q, z) = \tilde{\psi}_A(\kappa p, \kappa q, 0) \exp[ikmz], \tag{2}$$

where $\quad m = +[1-p^2-q^2]^{\frac{1}{2}}, \quad p^2+q^2 \leqslant 1,$

$$= +i\,[p^2+q^2-1]^{\frac{1}{2}}, \quad p^2+q^2 > 1,$$

where $\kappa = 1/\lambda$ and $\tilde{\psi}$ is the spatial Fourier transform of ψ given by

$$\psi(x, y, z) = \iint_{-\infty}^{\infty} d(\kappa p)\, d(\kappa q)\, \tilde{\psi}(\kappa p, \kappa q, z) \exp[+i2\pi\kappa(px+qy)]. \tag{3}$$

Observe that equation (2) gives us the transform $\tilde{\psi}$ (of the diffracted field) for constant z values. *These are planes parallel to the plane $z' = 0$ over which the Green's function is made to vanish.* Thus it is seen that the above formalism gives us the diffracted amplitude in any plane $z > 0$ when it is known on the plane at $z' = 0$, over which the Green's function vanishes. In equation (1) also the z co-ordinate is held constant. This fact does not seem to have been appreciated in the earlier literature on the subject. For a further discussion on this point, the reader may refer to the appendix of this paper where the Rayleigh-Sommerfeld theory is compared for the plane and the spherical geometry as formulated by Marathay (1975).

According to the (RS) theory the spectral irradiance is found in a plane labelled by a fixed value of $z > 0$. The spectral irradiance, $\hat{E}$ with units of $[\mathrm{Wm}^{-2}\,(\mathrm{Hz})^{-1}]$ is given by $\{C\,|\psi|^2\}$ where C is a suitable constant. The symbol E is used in conventional radiometry to denote irradiance. The spectral radiant power $[W\,(\mathrm{Hz})^{-1}]$ distribution in the plane $(z > 0)$, is given by $\{C\,|\psi|^2\, d\sigma_p\}$ where $d\sigma_p = dxdy$ is the element of area around the point of interest (figure 1) in the plane at a fixed value of z. To calculate the power distribution on any other surface intersecting the plane z at the point of interest, it is necessary to take the projection of $d\sigma_p$ onto that surface. In particular, let the surface be the hemisphere centered at $(x_0', y_0', 0)$ as shown in figure 1. The elementary area $d\sigma_R$ at the point (x, y, z) on the hemisphere and the surface element $d\sigma_p$ of the plane are related by $d\sigma_R = d\sigma_p \cos\theta$. The spectral radiant power distribution on the hemisphere will be $\{C\,|\psi|^2\, d\sigma_R/\cos\theta\}$. We therefore conclude

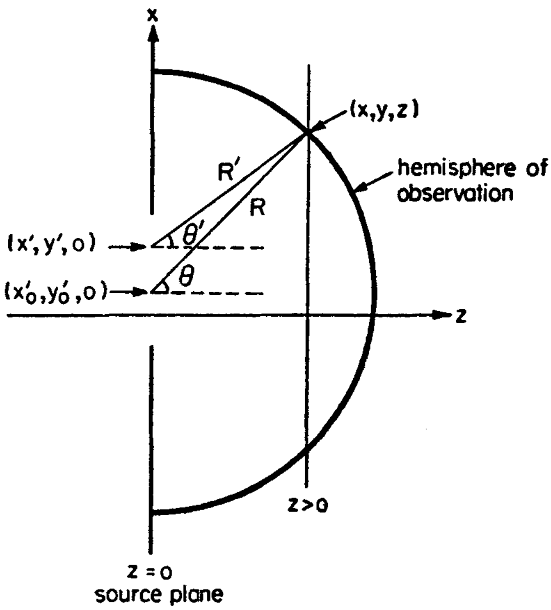

Figure 1. Geometry of the diffraction problem and the hemisphere of observation.

that the spectral irradiance $\hat{E}_R$ at (x, y, z) on the hemisphere may be found in terms of the spectral irradiance $\hat{E}$ on the plane by

$$\hat{E}_R = (C\,|\psi|^2)/\cos\theta = \hat{E}/\cos\theta. \tag{4}$$

Thus starting with the (RS) theory, we will calculate in this manner the spectral irradiance on the hemisphere covering the noncoherently illuminated aperture.

In figure 1 a typical point in the aperture is denoted by $(x', y', 0)$. The field point is denoted by (x, y, z) and R' denotes the distance between the aperture point and the field point. It is seen from the figure that $\cos\theta' = z/R'$, $\cos\theta = z/R$. Beran and Parrent (1964) have considered the radiation from a plane, finite surface and have solved the problem of the propagation of the mutual coherence. Starting from the Rayleigh-Sommerfeld diffraction theory, they obtain a generalisation of the Van Cittert-Zernike theorem:

$$\hat{\Gamma}\,(x_1, y_1; x_2, y_2, z, \nu)$$

$$= \iint_{\Sigma} \iint_{\Sigma} \Big[\hat{\Gamma}_{\Sigma}\,(x_{1s}, y_{1s}; x_{2s}, y_{2s}, 0, \nu)\cos\theta_1\cos\theta_2\,(1 - ik\,\rho_1)\,(1 + ik\,\rho_2)$$

$$\frac{\exp\{ik\,(\rho_1 - \rho_2)\}}{(2\pi\,\rho_1\,\rho_2)^2}\Big]\,dx_{1s}\,dy_{1s}\,dx_{2s}\,dy_{2s}. \tag{5}$$

The geometry is shown in figure 2, and the various symbols are defined by

$$\rho_1 = [(x_1 - x_{1s})^2 + (y_1 - y_{1s})^2 + z^2]^{\frac{1}{2}},$$

$$\rho_2 = [(x_2 - x_{2s})^2 + (y_2 - y_{2s})^2 + z^2]^{\frac{1}{2}},$$

$$\cos\theta_1 = (z/\rho_1), \quad \cos\theta_2 = (z/\rho_2).$$

The spatial and temporal Fourier transform $\mathring{\Gamma}$ of the mutual coherence function Γ is given by

$$\mathring{\Gamma}(\kappa p_1, \kappa q_1, \kappa p_2, \kappa q_2, z, \nu)$$
$$= \mathring{\Gamma}_\Sigma(\kappa p_1, \kappa q_1, \kappa p_2, \kappa q_2, 0, \nu) \exp[+ik(m_1 - m_2) z]. \tag{6}$$

This equation relates the transform of the aperture distribution in the plane at $z = 0$ to the transform of the diffracted field in the plane $z > 0$. This equation is analogous to equation (2) for the amplitude. The symbol m_1 is defined in the same way as m of equation (2), but m_2 stands for

$$m_2 = +[1 - (p_2^2 + q_2^2)]^{\frac{1}{2}}, \quad p_2^2 + q_2^2 \leqslant 1$$

$$= -i[(p_2^2 + q_2^2) - 1]^{\frac{1}{2}}, \quad p_2^2 + q_2^2 > 1.$$

Now we will consider the special case of a noncoherently illuminated aperture. Following Beran and Parrent (1964) the condition of noncoherence will be described by

$$\hat{\Gamma}_\Sigma(x_{1s}, y_{1s}; x_{2s}, y_{2s}, 0, \nu)$$
$$= \frac{\lambda^2}{\pi} \hat{\Gamma}_\Sigma(x_{1s}, y_{1s}; x_{1s}, y_{1s}, 0, \nu)\, \delta(x_{1s} - x_{2s})\, \delta(y_{1s} - y_{2s}). \tag{7}$$

For $x_{1s} = x_{2s}$ and $y_{1s} = y_{2s}$, $\hat{\Gamma}_\Sigma$ is the spectral irradiance of the aperture up to a suitable constant C; thus let us put

$$\hat{E}_\Sigma(x_{1s}, y_{1s}, 0, \nu) \equiv C\,\hat{\Gamma}(x_{1s}, y_{1s}; x_{1s}, y_{1s}, 0, \nu). \tag{8}$$

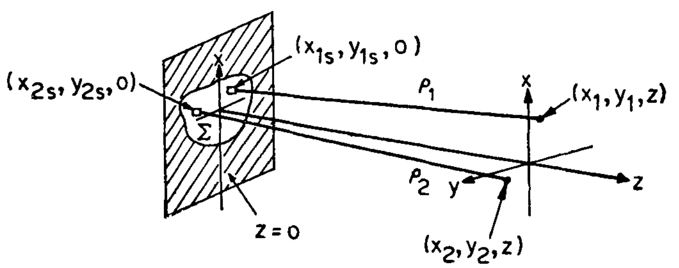

Figure 2. Geometry for calculating the temporal Fourier transform $\hat{\Gamma}$ in the plane z.

The spectral irradiance $\hat{E}$ at a point (x, y) in the plane at $z > 0$ is obtained by using equations (7) and (8) in equation (5) and setting $x_1 = x_2 = x$ and $y_1 = y_2 = y$,

$$\hat{E}(x, y, z, \nu) = \frac{\lambda^2}{4\pi^3} \iint_{\Sigma} \hat{E}_{\Sigma}(x', y', 0, \nu) \cos^2\theta' \left[\frac{1+k^2 R'^2}{R'^4}\right] dx'\, dy'. \tag{9}$$

For the geometry considered in the above equation, $\cos\theta' = z/R'$ and ρ_1 and ρ_2 of equation (5) are both equal to R'.

We will now examine the spectral irradiance $\hat{E}_R(x, y, \nu)$ on the surface of the hemisphere centered on $(x'_0, y'_0, 0)$ in figure 1. The z value is fixed by the condition $z = +[R^2 - x^2 - y^2]^{\frac{1}{2}}$. The combination of equations (9) and (4) yields

$$\hat{E}_R(x, y, \nu) = \frac{\lambda^2}{4\pi^3} \cdot \frac{1}{\cos\theta} \cdot \iint_{\Sigma} \hat{E}_{\Sigma}(x', y', 0, \nu) \cos^2\theta' \left[\frac{1+k^2 R'^2}{R'^4}\right] dx'\, dy'. \tag{10}$$

This is the expression for the irradiance on the hemisphere. The contribution $d\hat{E}_R$ to the spectral irradiance on the hemisphere by an area element $(dx'dy')$ in the aperture is given by

$$d\hat{E}_R = \frac{\lambda^2}{4\pi^3} \left[\frac{1+k^2 R'^2}{R'^4}\right] \left[\frac{\cos^2\theta'}{\cos\theta}\right] \hat{E}_{\Sigma}(x', y', 0, \nu)\, dx'\, dy'. \tag{11}$$

For a uniformly illuminated aperture, $\hat{E}_{\Sigma}$ is a constant. If the element of area is chosen to be at the location $(x'_0, y'_0, 0)$ then $\theta' = \theta$ and $d\hat{E}_R$ is found to be proportional to $\cos\theta$, characteristic of a Lambertian source. So the uniformly illuminated noncoherent (source) aperture gives an irradiance distribution on the hemisphere like a Lambertian source. With respect to a finite size aperture the irradiance distribution on the hemisphere is not exactly $\cos\theta$. For an experimental verification it will be necessary to choose the radius R very much greater than the dimensions of the aperture. In this situation the variation of $[\cos\theta']$ over the aperture may be neglected. In this approximation the small aperture will also exhibit a $\cos\theta$ power distribution on the hemisphere.

3. Spectral radiant intensity

We propose to show that the radiant intensity has a $\cos\theta$ behaviour for a spatially homogeneous case as well as in the limit of a delta-correlated cross-spectral density function. The calculation is based on the basic Rayleigh-Sommerfeld diffraction theory. We begin with the spectral radiant power $\hat{\Phi}$ in some plane $z > 0$,

$$\hat{\Phi} = C \iint_{-\infty}^{\infty} dx\, dy\, \hat{\Gamma}(x, y; x, y, z, \nu). \tag{12}$$

The product $C\hat{\Gamma}$ has the dimensions of [$\mathrm{Wm^{-2}\,(Hz)^{-1}}$], and those of $\hat{\Phi}$ are [$\mathrm{W(Hz)^{-1}}$]. In this expression we substitute for $\hat{\Gamma}$ in terms of the space time Fourier transform $\overset{\circ}{\Gamma}$ to get

$$\hat{\Phi} = C\iint_{-\infty}^{\infty} dx\, dy \iint_{-\infty}^{\infty} dp_1\, dq_1 \iint_{-\infty}^{\infty} dp_2\, dq_2\, \kappa^4\, \overset{\circ}{\Gamma}(\kappa p_1, \kappa q_1, \kappa p_2, \kappa q_2, z, \nu)$$

$$\exp\left[+i\, 2\pi\kappa \left\{(p_1-p_2)\, x + (q_1-q_2)\, y\right\}\right].$$

Since the integrals on x and y give δ-functions involving the p and q variables, we find that

$$\hat{\Phi} = C\kappa^2 \iint_{-\infty}^{\infty} dp\, dq\, \overset{\circ}{\Gamma}(\kappa p, \kappa q, \kappa p, \kappa q, z, \nu). \tag{13}$$

At this point we neglect the evanescent waves, that is restrict the integrals to $p^2+q^2 \leqslant 1$ and replace the elements $(dp\,dq)$ by $(d\Omega m)$. The symbol $d\Omega$ stands for the element of solid angle, $d\Omega = \sin\theta\; d\theta\; d\phi$, $m = \cos\theta$, $p = \sin\theta\, \cos\phi$, and $q = \sin\theta\, \sin\phi$. In this way equation (13) may be rewritten as

$$\hat{\Phi} = C\kappa^2 \iint_{(\frac{1}{2})} \overset{\circ}{\Gamma}(\kappa p, \kappa q, \kappa p, \kappa q, z, \nu)\, m d\Omega, \tag{14}$$

where the symbol $(\frac{1}{2})$ on the integrals is introduced to remind us that it is restricted to half of the total solid angle, that is over 2π steradians. In this case $0 \leqslant \theta \leqslant \pi/2$ and $0 \leqslant \phi \leqslant 2\pi$.

The situation is now ripe for calculating the spectral radiant intensity $\hat{I}$ as the spectral radiant power per unit solid angle. Thus, from equation (14) we find that

$$\hat{I} = \partial\hat{\Phi}/\partial\Omega = C\kappa^2\, \overset{\circ}{\Gamma}(\kappa p, \kappa q, \kappa p, \kappa q, z, \nu)\, m. \tag{15}$$

The units of I are [$\mathrm{W(sr)^{-1}\,(Hz)^{-1}}$].

It now remains to calculate $\overset{\circ}{\Gamma}$ for the special case of a uniform noncoherent (source) aperture Σ. For this purpose we could make use of the expression of $\hat{\Gamma}_\Sigma$ given in equation (7). Instead, we use a slightly more general expression, namely

$$\hat{\Gamma}_\Sigma(x_1, y_1; x_2, y_2, 0, \nu) = \hat{\Gamma}_\Sigma(x_1, y_1; x_1, y_1, 0, \nu)\, h\,(x_1-x_2, y_1-y_2), \tag{16}$$

where $h(x_1-x_2, y_1-y_2)$ is some sufficiently sharply peaked function to describe the condition of noncoherence. What we mean by 'sufficiently sharply peaked' will be given later. But for now we ask that it should have the property

$$h(0, 0) = 1. \tag{17}$$

The uniform nature of the source aperture may be described by

$$\hat{\Gamma}_\Sigma(x_1, y_1; x_1, y_1, 0, \nu) = \hat{\Gamma}_0(0, \nu)\, \mathscr{A}(x_1, y_1), \tag{18}$$

where up to a constant the function $\hat{\Gamma}_0$ expresses the constant spectral radiant existence of the source and $\mathscr{A}$ specifies the geometrical shape of the aperture in the x, y plane at $z = 0$. Admittedly, equation (16) is an approximation because it neglects to specify the shape restriction on the variables x_2, y_2. But the approximation is good for our present discussion if the function $h\,(x_1 - x_2, y_1 - y_2)$ is sufficiently sharply peaked. A straightforward calculation of the spatial Fourier transform of $\hat{\Gamma}_\Sigma$ of equation (16) gives us $\mathring{\Gamma}_\Sigma$ at $z = 0$. We find that

$$\mathring{\Gamma}_\Sigma(\kappa p_1, \kappa q_1, \kappa p_2, \kappa q_2, 0, \nu)$$
$$= \hat{\Gamma}_0(0, \nu)\, \tilde{\mathscr{A}}\, [\kappa(p_1 - p_2), \kappa(q_1 - q_2)]\, \tilde{h}\,(\kappa p_2, \kappa q_2), \tag{19}$$

where $\tilde{\mathscr{A}}$ and $\tilde{h}$ are the respective spatial Fourier transforms of the aperture shape $\mathscr{A}$ and the sharp function h. For example

$$\tilde{h}\,(\kappa p_2, \kappa q_2) = \iint\limits_{-\infty}^{\infty} dx'\, dy'\, h(x', y') \exp\,[-i2\pi\kappa\,(p_2 x' + q_2 y')]. \tag{20}$$

It is a trivial matter to calculate $\mathring{\Gamma}$ at $z > 0$, owing to the product relationship of equation (6). Furthermore, for the calculation of the spectral radiant intensity $\hat{I}$ of equation (15), we only need $\mathring{\Gamma}$ with $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Thus, by using equation (19) in equation (6) and then putting the result in equation (15) we get

$$\hat{I} = \partial\hat{\Phi}/\partial\Omega = C\kappa^2\, \hat{\Gamma}_0\,(0, \nu)\, \tilde{\mathscr{A}}\,(0, 0)\, \tilde{h}\,(\kappa p, \kappa q)\, m. \tag{21}$$

At this point we introduce the refinement of what we mean by a ' sufficiently sharply peaked ' function h to describe noncoherence. By that we mean that the spatial Fourier transform $\tilde{h}$ be a constant or very nearly a constant over the spectrum of real plane waves,

$$\tilde{h}(\kappa p, \kappa q) \simeq \text{constant}, \; p^2 + q^2 \leqslant 1. \tag{22}$$

It encompasses a rather large class of sharply peaked functions. With the proper adjustment of the parameters in the familiar functions such as sinc, Besinc, Gaussian, etc., it is possible to make them sharp enough to satisfy equation (22). Any one of them is just as good to describe noncoherence regardless of its functional form. They are all such that the coherence interval is on the order of a wavelength, and they attain negligible values over distances on the order of several wavelengths. The δ-function used in equation (7) also belongs to this class as a limiting case.

Owing to the condition already imposed in equation (17), the constant in equation (22) is $[1/(\pi\kappa^2)]$. Consistent with the above discussion, equation (21) gives us

$$\hat{I} = \partial\hat{\Phi}/\partial\Omega = \frac{1}{\pi}\, C\, \hat{\Gamma}_0(0, \nu)\, \tilde{\mathscr{A}}\,(0, 0)\, m. \tag{23}$$

This is the spectral radiant intensity for a uniform, noncoherent source. We observe that its angular dependence is described by $m = \cos\theta$, a characteristic of a Lambertian source.

It is important to note that radiant intensity is defined as something pertaining to the entire source. It is not the source variation but the angular dependence of the radiation that is examined. As a practical matter, this measurement should be made far enough away from the source or appropriate optical elements should be used to measure the radiant power in different directions.

4. Spectral irradiance on a plane parallel to the source aperture

We start with a noncoherent source aperture in the plane $z = 0$ and ask for the spectral irradiance on the plane $z =$ constant. Since the calculation proceeds on the lines discussed in the previous sections, we shall simply state the result (Marathay *et al* 1977)

$$d\hat{E} = \frac{1}{\pi}\frac{\cos^4\theta'}{z}\,\hat{E}_\Sigma\,(x', y', 0, \nu)\, dx'dy'.$$

In this equation $d\hat{E}$ is the differential spectral irradiance contributed by an elementary area $dx'dy'$ in the source plane. The noncoherent source aperture also yields a cosine-fourth law as is the case for a Lambertian source.

5. Conclusions

In this paper we presented three independent calculations by using the basic Rayleigh-Sommerfeld diffraction theory as applied to a *uniform, noncoherent* source. In the first we calculated the spectral irradiance on a hemisphere covering the above source, and in the second we studied the spectral radiant intensity. A $\cos\theta$ angular dependence is obtained in each of the first two cases. Furthermore, it was pointed out that the spectral irradiance distribution on a plane parallel to the aperture has a $\cos^4\theta$ dependence. In addition, any formalism of generalised radiometry must relate to conventional radiometry of partially coherent fields.

Acknowledgements

This work was supported in part by the Air Force Office of Scientific Research (AFSC), United States Air Force, and the Army Research Office, United States Army.

Appendix

It is proposed here to compare the Rayleigh-Sommerfeld theory for the plane and spherical geometry to show that the surface over which the Green's function $G = 0$

Table 1. Comparison of the Rayleigh-Sommerfeld theory for plane and spherical geometry.

Aperture A on the plane defined by $z = z_1$.	Aperture A on the sphere of radius r_1.
Green's function $G = 0$ on $z = z_1$.	$G = 0$ on $r = r_1$.
Diffracted field: $\phi(x_2, y_2, z_2) = \iint dx_1\, dy_1 \left[\phi_A(x_1, y_1, z_1) \times \left\{-\frac{\partial G}{\partial n}(x_2-x_1, y_2-y_1, z_2-z_1)\right\}\right].$	Diffracted field: $\phi(r_2, \theta_2, \phi_2) = \iint_A r_1^2\, d\Omega_1 \left[\phi_A(r_1, \theta_1, \phi_1) \times \left\{-\frac{\partial G}{\partial n}(\mathbf{r}_2, \mathbf{r}_1)\right\}\right].$
The eigenfunctions for this geometry are plane waves, hence use (spatial) Fourier transform:	The eigenfunctions for this geometry are spherical harmonics, hence use Laplaces series
$\tilde{\phi}(\kappa p, \kappa q, z_2) = \tilde{\phi}_A(\kappa p, \kappa q, z_1) \frac{\exp[ikmz_2]}{\exp[ikmz_1]}$	$\phi_{lm}(r_2) = \phi_{lm}^A(r_1) \frac{h_l^{(1)}(kr_2)}{h_l^{(1)}(kr_1)}$
$\exp[ikmz]$ = z-dependent solution of the Helmholtz equation.	$h_l^{(1)}(kr)$ = radial solution of the Helmholtz equation.
Amplitude ϕ or intensity $\lvert\phi\rvert^2$ is defined over *planes* of constant z which are parallel to the $z = z_1$ plane over which $G = 0$.	ϕ or $\lvert\phi\rvert^2$ is defined over *spheres* of radius r_2 concentric to the sphere of radius r_1, on which $G = 0$.

is important. The results exhibited in the table above are according to the formulation given by Marathay (1975).

References

Beran M J and Parrent Jr G B 1964 *Theory of partial coherence* (Englewood Cliffs, N J: Prentice Hall) pp 42 and 57

Marathay A S 1975 *J. Opt. Soc. Am.* **65** 909

Marathay A S, Sheila Prasad and Shack R V 1977 Talk presented at Optical Society of America Meeting, Toranto, Canada

Marchand E W and Wolf E 1974 *J. Opt. Soc. Am.* **64** 1219

Walther A 1968 *J. Opt. Soc. Am.* **58** 1256