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$^{10}_{\Lambda\Lambda}$Be in Faddeev-Yakubovsky formalism

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Abstract. The four-body dynamical equations for two distinct pairs of identical particles derived earlier are applied to investigate the system $^{10}_{\Lambda\Lambda}$Be. The two-body potentials have been taken to be of the Yamaguchi form, and the Bateman approximation has been used for the other amplitudes. From the set of coupled integral equations, the separation energy, $B_{\Lambda\Lambda}$, for the two Λ particles in $^{10}_{\Lambda\Lambda}$Be is obtained as 43·97 MeV.

Keywords. Four-body equations; identical particles; two-body separable potentials; Bateman approximation; separation energy; Lambda particles in $^{10}_{\Lambda\Lambda}$Be; Faddeev-Yakubovsky formalism.

1. Introduction

The interaction of lambda-particle with other hadrons is not yet well understood. There is some breakthrough in the understanding of Λ-N forces by the study of hypernuclei and also by direct Λ-p scattering experiments. The study of Λ-Λ interaction has been possible only through the study of the binding energy of double lambda-hypernuclei. Experimentally two such hypernuclei have so far been discovered. One of them $^{6}_{\Lambda\Lambda}$He (Prowse 1966) and the other is often interpreted as $\mathrm{Be}^{10}_{\Lambda\Lambda}$ (Danysz *et al* 1963). We have already investigated (Roy-Choudhury *et al* 1976) the system $^{6}_{\Lambda\Lambda}$He in a three-body model by using exact dynamical equations but with two-body local potentials. In this paper we explore the properties of $^{10}_{\Lambda\Lambda}$Be as a four-body system by using the set of exact equations derived by us in Faddeev-Yakubovsky formalism (Faddeev 1970; Yakubovsky 1967) for two distinct pairs of identical particles (Roy-Choudhury *et al* 1977, to be henceforth referred to as Ref. I and the equations from this paper to be indicated by Ref. (I.1) etc.). Mitra *et al* (1965), Rosenberg (1965), Takahashi and Mishima (1965) and Alessandrini (1966) have earlier derived four-particle equations as sets of six coupled equations in which the kernels are the connected parts of the amplitudes for three-body subsystems and also for the subsystems consisting of two non-interacting pairs. However, no attempt seems to have been made to study the $^{10}_{\Lambda\Lambda}$Be system with the four-body exact dynamical equations. Calculations for the binding energy in the four-identical-particle case have, however, been carried out (Kharchenko and Kuzmichev 1972; Kharchenko and Shadchin 1974; Kharchenko *et al* 1974; Narodetsky 1974; Narodetsky *et al* 1973; Tjon 1975). The calculations for the four identical particle case are much simpler compared to those for the system of two distinct pairs of identical particles. The number of dynamical equations in our studies is seven as compared to two in the case of four identical particles (Kharchenko and Kuzmichev 1972). This difference is due to the symmetry properties of the $\Lambda\Lambda\alpha\alpha$ system. Before the non-variational exact equations came to be used, four-body systems were theoretically investigated with the variational shell model and resonating group methods in which a variational procedure is also used (Fiarman and Meyerhof 1973). However, it is difficult to find out the error in a variational solution from these methods as the choice is limited to a selective class of trial wave functions.

Binding energy calculations with variational approach have been made for ${}^{10}_{\Lambda\Lambda}$Be by treating it as a three-body system (Λ-Λ-^{8}Be) (Dalitz and Rajasekharan 1964; Tang *et al* 1964; Ali and Bodmer 1965). Subsequently, there have been variational calculations for this $\Lambda\Lambda$-hypernucleus in four-body model (Λ-Λ-α-α) (Deloff 1963; Nakamura 1963; Ali and Bodmer 1967; Tang and Herndon 1965, 1966; Bhamathi *et al* 1973). These calculations clearly show that the four-body model treatment for ${}^{10}_{\Lambda\Lambda}$Be is better as compared to the three-body one.

The seven exact dynamical equations for $\Lambda\Lambda\alpha\alpha$ system can be reduced to a set of seven coupled integral equations in two variables after using separable forms for the two-body interactions and making partial wave analysis (Roy-Choudhury *et al* 1977). Further simplification of these equations have been made by using the Bateman (1922) method which reduces them to a set of coupled one-dimensional integral equations. Bateman method has earlier been used in three-body systems by Kharchenko *et al* (1970) and more recently by Kharchenko *et al* (1974) and Saiwicki and Namyslowuski (1975–76) for the study of four identical particle system. Our set of one-dimensional integral equations has been used to calculate the binding energy of $\Lambda\Lambda\alpha\alpha$ system. The two α's have been labelled as 1 and 2 and the two Λ's as 3 and 4.

2. Theory

For the four-body system under consideration, the total angular momentum is zero, and of all the possible partial components of the three-body scattering amplitudes, only the s-component has been taken into account. Thus the set of equations (I. 8) for s-wave reduces to the form (after dropping the angular momentum subscripts):

$$Q^{1a}(p,q) = \frac{1}{2\pi^2}\tau_{12}\left(z_1 - \frac{p^2}{2\mu_{12,3}}\right)\left[\frac{2m_a+m_b}{m_b}\int_0^\infty dq'\right.$$

$$\times \int_{|q-[m_b/(2m_a+m_b)]q'|}^{q+[m_b/(2m_a+m_b)]q'} dp' \frac{p'q'}{q}\mathscr{K}_{12,31}(p,p_1';z_1)\,Q^{1a}(p',q')$$

$$+\frac{2m_b+m_a}{2m_b}\int_0^\infty dq' \int_{|q-[m_b/(2m_b+m_a)]q'|}^{q+[m_b/(2m_b+m_a)]q'} dp' \frac{p'q'}{q}$$

$$\times \{\mathscr{K}_{12,12}(p, p_2'; z_1) + \mathscr{K}_{12,23}(p, p_2'; z_1)\} Q^{2a}(p', q') + 4 \int_0^\infty ds'$$

$$\times \int_{|q - \frac{1}{2}s'|}^{q + \frac{1}{2}s'} d\kappa' \frac{\kappa' s'}{q} \mathscr{K}_{12,31}(p, p_3'; z_1) R^{1b}(\kappa', s') + \frac{m_a + m_b}{2m_b} \int_0^\infty ds'$$

$$\times \int_{|q-[m_b/(m_a+m_b)]s'|}^{q+[m_b/(m_a+m_b)]s'} d\kappa' \frac{\kappa' s'}{q} \{\mathscr{K}_{12,12}(p, p_4'; z_1)$$

$$+ \mathscr{K}_{12,23}(p.p_4'; z_1)\} R^{3b}(\kappa', s,)\Big], \qquad \textbf{(1a)}$$

$$Q^{2a}(p, q) = \frac{1}{\pi^2} \tau_{23}\left(z_2 - \frac{p^2}{2\mu_{23,4}}\right)\left[\frac{2m_b + m_a}{m_a} \int_0^\infty dq'\right.$$

$$\times \int_{|q-[m_a/(2m_b+m_a)]q'|}^{q+[m_a/(2m_b+m_a)]q'} dp' \frac{p'q'}{q} \mathscr{K}_{23,23}(p, p_5'; z_2) Q^{3a}(p', q')$$

$$+ \frac{2m_a + m_b}{2m_a} \int_0^\infty dq' \int_{|q-[m_a/(2m_a+m_b)]q'|}^{q+[m_a/(2m_a+m_b)]q'} dp' \frac{p'q'}{q}$$

$$\times \{\mathscr{K}_{23,34}(p, p_6'; z_2) + \mathscr{K}_{23,42}(p, p_6'; z_2)\} Q^{4a}(p', q') + 4 \int_0^\infty ds'$$

$$\times \int_{|q-\frac{1}{2}s'|}^{q+\frac{1}{2}s'} d\kappa' \frac{\kappa' s'}{q} \mathscr{K}_{23,23}(p, p_7'; z_2) R^{2b}(\kappa', s') + \frac{m_a + m_b}{2m_a} \int_0^\infty ds'$$

$$\times \int_{|q-[m_a/(m_a+m_b)]s'|}^{q+[m_a/(m_a+m_b)]s'} d\kappa' \frac{\kappa' s'}{q} \{\mathscr{K}_{23,34}(p, p_8'; z_2)$$

$$+ \mathscr{K}_{23,42}(p, p_8'; z_2)\} R^{3b}(\kappa' s')\Big], \qquad \text{(1b)}$$

$$Q^{3a}(p, q) = \tfrac{1}{2} \tau_{34}\left(z_2 - \frac{p^2}{2\mu_{34,1}}\right)\left[\tau_{23}\left(z_2 - \frac{p^2}{2\mu_{23,4}}\right)\right]^{-1} Q^{2a}(p, q), \qquad \text{(1c)}$$

$$Q^{4a}(p, q) = 2 \tau_{41}\left(z_1 - \frac{p^2}{2\mu_{41,2}}\right)\left[\tau_{12}\left(z_2 - \frac{p^2}{2\mu_{12,3}}\right)\right]^{-1} Q^{1a}(p, q), \qquad \text{(1d)}$$

$$R^{1b}(\kappa, s) = \frac{1}{2\pi^2} \tau_{12}\left(z_3 - \frac{\kappa^2}{2\mu_{34}}\right)\left[\frac{2m_a + m_b}{4m_a} \int_0^\infty dq'\right.$$

$$\times \int_{|s-[2m_a/(2m_a+m_b)]q'|}^{s+[2m_a/(2m_a+m_b)]q'} dp' \frac{p'q'}{s} \mathscr{V}_{12,34}(\kappa, \kappa_1'; z_3) Q^{1a}(p', q')$$

$$+\frac{2m_b+m_a}{4m_b}\int_0^\infty dq'\int_{|s-[2m_b/(2m_b+m_a)]q'|}^{s+[2m_b/(2m_b+m_a)]q'} dp'\frac{p'q'}{s}$$

$$\times \mathscr{V}_{12,12}(\kappa,\kappa_2';z_3)\,Q^{3a}(p',q')\Big], \tag{1e}$$

$$R^{2b}(\kappa,s)=\tau_{34}\left(z_3-\frac{\kappa^2}{2\mu_{12}}\right)\left[\tau_{12}\left(z_3-\frac{\kappa^2}{2\mu_{34}}\right)\right]^{-1}\mathscr{R}^{1b}(\kappa,s'), \tag{1f}$$

$$R^{3b}(\kappa,s)=\frac{1}{\pi^2}\tau_{23}\left(z_4-\frac{\kappa^2}{2\mu_{41}}\right)\left[\frac{2m_b+m_a}{2(m_a+m_b)}\int_0^\infty dq'\right.$$

$$\times\int_{|s-[(m_a+m_b)/(2m_b+m_a)]q'|}^{s+[(m_a+m_b)/(2m_b+m_a)]q'} dp'\frac{p'q'}{s}\mathscr{V}_{23,23}(\kappa,\kappa_3';z_4)\,Q^{2a}(p',q')$$

$$+\frac{2m_a+m_b}{2(m_a+m_b)}\int_0^\infty dq'\int_{|s-[(m_a+m_b)/(2m_a+m_b)]q'|}^{s+[(m_a+m_b)/(2m_a+m_b)]q'} dp'\frac{p'q'}{s}$$

$$\times\mathscr{V}_{23,23}(\kappa,\kappa_4';z)\,Q^{4a}(p'q')\Big]. \tag{1g}$$

The various notations occurring in the set of equations (1) are defined in Ref. I.

We will convert the set of two-variable integral equations (1) to a set of single variable ones, by using separable representations for the amplitudes $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ of Ref. I.

To get the necessary representations for these amplitudes, we first apply Bateman approximation for the s-wave components of the effective potentials $U_{\alpha,\beta}$ and $W_{\alpha,\beta}$ defined in Ref. I:

$$U_{\alpha,\beta}(p,p';z)\to U^{(N)}_{\alpha,\beta}(p,p';z)=\sum_{n=1}^{N}\frac{U^n_{\alpha,\beta}(p,p_n;z)\,U^n_{\alpha,\beta}(p_n,p';z)}{U^n_{\alpha,\beta}(p_n,p_n;z)}, \tag{2}$$

$$W_{\alpha,\beta}(\kappa,\kappa';z)\to W^{(N)}_{\alpha,\beta}(\kappa,\kappa';z)=\sum_{n=1}^{N'}\frac{W^n_{\alpha,\beta}(\kappa,\kappa_n;z)\,W^n_{\alpha,\beta}(\kappa_n,\kappa';z)}{W^n_{\alpha,\beta}(\kappa_n,\kappa_n;z)}, \tag{3}$$

where the form factors $U^n_{\alpha,\beta}$ and $W^n_{\alpha,\beta}$ are defined by

$$U^n_{\alpha,\beta}(p,p';z)=U^{n-1}_{\alpha,\beta}(p,p';z)-\frac{U^{n-1}_{\alpha,\beta}(p,p_{n-1};z)\,U^{n-1}_{\alpha,\beta}(p_{n-1},p';z)}{U^{n-1}_{\alpha,\beta}(p_{n-1},p_{n-1};z)}$$

$$n=2,3,\ldots N;$$

$$U^1_{\alpha,\beta}(p,p';z)=U_{\alpha,\beta}(p,p';z). \tag{4}$$

$$W^{n}_{\alpha,\beta}(\kappa,\kappa';z) = W^{n-1}_{\alpha,\beta}(\kappa,\kappa';z) - \frac{W^{n-1}_{\alpha,\beta}(\kappa,\kappa_{n-1};z)\,W^{n-1}_{\alpha,\beta}(\kappa_{n-1},\kappa';z)}{W^{n-1}_{\alpha,\beta}(\kappa_{n-1},\kappa_{n-1};z)}$$

$$n = 2, 3, \ldots N';$$

$$W^{1}_{\alpha,\beta}(\kappa,\kappa';z) = W_{\alpha,\beta}(\kappa,\kappa';z). \tag{5}$$

In (4) and (5) p_n and κ_n are corresponding nodal points in the range of the variables p, p' and κ, κ'.

Substituting the expressions for $U_{\alpha,\beta}$ and $W_{\alpha,\beta}$ in equations for the amplitudes $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ of Ref. I, one can get, after doing some algebra, the separable forms for the latter pair. These along with the separable forms of $U_{\alpha,\beta}$, and $W_{\alpha,\beta}$ [equations (4) and (5)] yield the separable representations for $\mathscr{K}_{\alpha,\beta}$ and $\mathscr{V}_{\alpha,\beta}$ occurring in Ref. (I.16) and (I. 17). The representations are given below:

$$\mathscr{K}_{12,12}(p,p';z_1) = \sum_{nn'} U^{n}_{12,23}(p,p^{(1)}_{n};z_1)\,\omega^{nn'}_{12,12}(z_1)\,U^{n'}_{12,23}(p^{(1)}_{n'},p';z_1),$$

$$\mathscr{K}_{12,23}(p,p';z_1) = \sum_{nn'} U^{n}_{12,23}(p,p^{(1)}_{n};z_1)\,\omega^{nn'}_{12,23}(z_1)\,U^{n'}_{23,23}(p^{(1)}_{n'},p';z_1),$$

$$\mathscr{K}_{12,31}(p,p';z_1) = \sum_{nn'} U^{n}_{12,31}(p,p^{(1)}_{n};z_1)\,\omega^{nn'}_{12,31}(z_1)\,U^{n'}_{31,12}(p^{(1)}_{n'},p';z_1), \tag{6a}$$

$$\mathscr{K}_{23,23}(p,p';z_2) = \sum_{nn'} \{U^{n}_{23,34}(p,p^{(2)}_{n};z_2)\,\omega^{(1)nn'}_{23,23}(z_2)\,U^{n'}_{23,24}(p^{(2)}_{n'},p';z_2)$$

$$+ U^{n}_{23,23}(p,p^{(2)}_{n};z_2)\,\omega^{(2)nn'}_{23,23}(z_2)\,U^{n'}_{23,34}(p^{(2)}_{n'},p';z_2)\}$$

$$\mathscr{K}_{23,34}(p,p';z_2) = \sum_{nn'} \{U^{n}_{23,34}(p,p^{(2)}_{n};z_2)\,\omega^{(1)nn'}_{23,34}(z_2)\,U^{n'}_{34,23}(p^{(2)}_{n'},p';z_2)$$

$$+ U^{n}_{23,23}(p,p^{(2)}_{n};z_2)\,\omega^{(2)nn'}_{23,24}(z_2)\,U^{n'}_{34,23}(p^{(2)}_{n'},p';z_2)\}$$

$$\mathscr{K}_{23,42}(p,p';z_2) = \sum_{nn'} \{U^{n}_{23,34}(p,p^{(2)}_{n};z_2)\,\omega^{(1)nn'}_{23,42}(z_2)\,U^{n'}_{42,23}(p^{(2)}_{n'},p';z_2)$$

$$+ U^{n}_{23,23}(p,p^{(2)}_{n};z_2)\,\omega^{(2)nn'}_{23,42}(z_2)\,U^{n'}_{42,23}(p^{(2)}_{n'},p';z_2)\}, \tag{6b}$$

$$\mathscr{K}_{34,34}(p,p';z_2) = \sum_{nn'} U^{n}_{34,41}(p,p^{(2)}_{n};z_2)\,\omega^{nn'}_{34,34}(z_2)\,U^{n'}_{34,41}(p^{(2)}_{n'},p';z_2)$$

$$\mathscr{K}_{34,41}(p,p';z_2) = \sum_{nn'} U^{n}_{34,41}(p,p^{(2)}_{n};z_2)\,\omega^{nn'}_{34,41}(z_2)\,U^{n'}_{41,41}(p^{(2)}_{n'},p';z_2)$$

$$\mathscr{K}_{34,13}(p,p';z_2) = \sum_{nn'} U^{n}_{34,13}(p,p^{(2)}_{n};z_2)\,\omega^{nn'}_{34,13};z_2)\,U^{n'}_{13,34}(p^{(2)}_{n'},p';z_2), \tag{6c}$$

$$\mathscr{K}_{41,41}(p,p';z_1)=\sum_{nn'}\{U^n_{41,12}(p,p^{(1)}_n;z_1)\,\omega^{(1)nn'}_{41,41}(z_1)\,U^{n'}_{41,12}(p^{(1)}_{n'},p';z_1)$$

$$+U^n_{41,41}(p,p^{(1)}_n;z_1)\,\omega^{(2)nn'}_{41,41}(z_1)\,U^{n'}_{41,12}(p^{(1)}_{n'},p';z_1)\}$$

$$\mathscr{K}_{41,12}(p,p';z_1)=\sum_{nn'}\{U^n_{41,12}(p,p^{(1)}_n;z_1)\,\omega^{(1)nn'}_{41,12}(z_1)\,U^{n'}_{12,41}(p^{(1)}_{n'},p';z_1)$$

$$+U^n_{41,41}(p,p^{(1)}_n;z_1)\,\omega^{(2)nn'}_{41,12}(z_1)\,U^{n'}_{12,41}(p^{(1)}_{n'},p';z_1)\}$$

$$\mathscr{K}_{41,24}(p,p';z_1)=\sum_{nn'}\{U^n_{41,12}(p,p^{(1)}_n;z_1)\,\omega^{(1)nn'}_{41,24}(z_1)\,U^{n'}_{24,41}(p^{(1)}_{n'},p';z_1)$$

$$+U^n_{41,41}(p,p^{(1)}_n;z_1)\,\omega^{(2)nn'}_{41,24}(z_1)\,U^{n'}_{24,41}(p^{(1)}_{n'},p';z_1)\},\tag{6d}$$

$$\mathscr{V}_{12,12}(\kappa,\kappa';z_3)=\sum_{nn'}W^n_{12,34}(\kappa,\kappa^{(1)}_n;z_3)\,\pi^{nn'}_{12,12}(z_3)\,W^{n'}_{12,34}(\kappa^{(1)}_{n'},\kappa';z_3)$$

$$\mathscr{V}_{12,34}(\kappa,\kappa';z_3)=\sum_{nn'}W^n_{12,34}(\kappa,\kappa^{(1)}_n;z_3)\,\pi^{nn'}_{12,34}(z_3)\,W^{n'}_{34,12}(\kappa^{(1)}_{n'},\kappa';z_3),\tag{7a}$$

$$\mathscr{V}_{34,34}(\kappa,\kappa';z_3)=\sum_{nn'}W^n_{34,12}(\kappa,\kappa^{(1)}_n;z_3)\,\pi^{nn'}_{34,34}(z_3)\,W^{n'}_{34,12}(\kappa^{(1)}_{n'},\kappa';z_3)$$

$$\mathscr{V}_{34,12}(\kappa,\kappa';z_3)=\sum_{nn'}W^n_{34,12}(\kappa,\kappa^{(1)}_n;z_3)\,\pi^{nn'}_{34,12}(z_3)\,W^{n'}_{12,34}(\kappa^{(1)}_{n'},\kappa';z_3)\tag{7b}$$

$$\mathscr{V}_{23,23}(\kappa,\kappa';z_4)=\sum_{nn'}W^n_{23,23}(\kappa,\kappa^{(2)}_n;z_4)\,\pi^{nn'}_{23,23}(z_4)\,W^{n'}_{23,23}(\kappa^{(2)}_{n'},\kappa';z_4).\tag{7c}$$

The expressions for $\omega_{\alpha,\beta}$'s and $\pi_{\alpha,\beta}$'s appearing in the set of equations (6) and (7), respectively are quite lengthy. For want of space, only a few of them are given in appendix 1. The rest are available with the authors. The nodal points $p^{(1)}_n$, $p^{(2)}_n$, $\kappa^{(1)}_n$ and $\kappa^{(2)}_n$ correspond respectively to the systems (123, 4), (234, 1), (12, 34) and (23, 14).

We now substitute the expressions for $\mathscr{K}_{\alpha,\beta}$'s and $\mathscr{V}_{\alpha,\beta}$'s into the set of integral equations (1) for the functions Q and R. Introducing the functions h, g and r defined by

$$Q^{1a}(p,q)=\frac{1}{2\pi^2}\tau_{12}\left(z_1-\frac{p^2}{2\mu_{12,3}}\right)\sum_{nn'}U^n_{12,23}(p,p^{(1)}_n;z_1)\,h^{1a}_{n'}(q),\tag{8a}$$

$$Q^{2a}(p,q)=\frac{1}{\pi^2}\tau_{23}\left(z_2-\frac{p^2}{2\mu_{23,4}}\right)\sum_{nn'}\{U^n_{23,34}(p,p^{(2)}_n;z_2)\,h^{2a}_{n'}(q)$$

$$+U^n_{23,23}(p,p^{(2)}_n;z_2)\,g^{2a}_{n'}(q)\},\tag{8b}$$

$$Q^{3a}(p,q)=\frac{1}{2\pi^2}\tau_{34}\left(z_2-\frac{p^2}{2\mu_{34,1}}\right)\sum_{nn'}U^n_{34,13}(p,p^{(2)}_n;z_2)\,h^{3a}_{n'}(q),\tag{8c}$$

$$Q^{4a}(p, q) = \frac{1}{\pi^2}\tau_{41}\left(z_1 - \frac{p^2}{2\mu_{41,2}}\right)\sum_{nn'} \{U^n_{23,12}(p, p^{(1)}_n; z_1)\, h^{4a}_{n'}(q)$$

$$+ U^n_{23,23}(p, p^{(1)}_n; z_1)\, g^{4a}_{n'}(q)\}, \tag{8d}$$

$$R^{1b}(\kappa, s) = \frac{1}{2\pi^2}\tau_{12}\left(z_3 - \frac{\kappa^2}{2\mu_{34}}\right)\sum_{nn'} W^n_{12,34}(\kappa, \kappa^{(1)}_n; z_3)\, r^{2b}_{n'}(s), \tag{9a}$$

$$R^{2b}(\kappa, s) = \frac{1}{2\pi^2}\tau_{34}\left(z_3 - \frac{\kappa^2}{2\mu_{12}}\right)\sum_{nn'} W^n_{34,12}(\kappa, \kappa^{(1)}_n; z_3)\, r^{2b}_{n'}(s), \tag{9b}$$

$$R^{3b}(\kappa, s) = \frac{1}{\pi^2}\tau_{23}\left(z_4 - \frac{\kappa^2}{2\mu_{14}}\right)\sum_{nn'} W^n_{23,23}(\kappa, \kappa^{(2)}_n; z_4)\, r^{3b}_{n'}(s), \tag{9c}$$

one can easily see that h, g and r satisfy the following set of single variable integral equations:

$$h^{1a}_n(q) = \sum_{n'n''}\Big[\int_0^\infty \{u^{(23,12)(12,23)}_{nn'}(q, q'; Z)\, \omega^{n'n''}_{12,31}(z_1)\, h^{1a}_{n''}(q')$$

$$+ [u^{(12,23)(23,34)}_{nn'}(q, q'; Z)\, \omega^{n'n''}_{12,12}(z_1) + u^{(23,23)(23,34)}_{nn'}(q, q'; Z)$$

$$\times \omega^{n'n''}_{12,23}(z_1)]\, h^{2a}_{n''}(q') + [u^{(12,23)(23,23)}_{nn''}(q, q'; Z)\, \omega^{n'n''}_{12,12}(z_1)$$

$$+ u^{(23,23)(23,23)}_{nn'}(q, q'; Z)\, \omega^{n'n''}_{12,23}(z_1)]\, g^{2a}_{n''}(q')\}\, q'^2\, [dq'/(2\pi^2)]$$

$$+ \int_0^\infty \{v^{(23,12)(12,34)}_{nn'}(q, s'; Z)\, \omega^{n'n''}_{12,31}(z_1)\, r^{1b}_{n''}(s') + [v^{(12,23)(23,23)}_{nn'}$$

$$(q, s'; Z)\, \omega^{n'n''}_{12,12}(z_1) + v^{(23,23)(23,23)}_{nn'}(q, s'; Z)\, \omega^{n'n''}_{12,23}(z_1)]$$

$$\times r^{3b}_{n''}(s')\}\, s'^2\, [ds'/(2\pi^2)]\Big], \tag{10a}$$

$$h^{3a}_n(q) = \sum_{n'n''}\Big[\int_0^\infty \{u^{(23,34)(34,23)}_{nn'}(q, q'; Z)\, \omega^{(1)n'n''}_{23,23}(z_2)\, h^{3a}_{n''}(q')$$

$$+ [u^{(34,23)(23,12)}_{nn'}(q, q'; Z)\, \omega^{(1)n'n''}_{23,34}(z_2) + u^{(23,23)(23,12)}_{nn'}(q, q'; Z)$$

$$\times \omega^{(1)n'n''}_{23,42}(z_2)]\, h^{4a}_{n''}(q') + [u^{(34,23)(23,23)}_{nn'}(q, q'; Z)\, \omega^{(1)n'n''}_{23,34}(z_2)$$

$$+ u^{(23,23)(23,23)}_{nn'}(q, q'; Z)\, \omega^{(1)n'n''}_{23,42}(z_2)]\, g^{4a}_{n''}(q')\}\, q'^2\, [dq'/(2\pi^2)]$$

$$+ \int_0^\infty \{v_{un'}^{(23,34)(34,12)} (q, s'; Z)\, \omega_{23,23}^{(1)n'n''} (z_2)\, r_{n''}^{2b} (s') + [v_{nn'}^{(34,23)(23,23)}$$

$$(q, s'; Z)\, \omega_{23,34}^{(1)n'n''} (z_2) + v_{nn'}^{(23,23)(23,23)} (q, s'; Z)\, \omega_{23,42}^{(1)n'n''} (z_2)]$$

$$\times\, r_{n''}^{3b} (s')\}\, s'^2\, [ds'/(2\pi^2)]\Big], \tag{10b}$$

$$g_n^{2a} (q) = \sum_{n'n''} \Big[\int_0^\infty \{u_{nn'}^{(23,34)(34,23)} (q, q'; Z)\, \omega_{23,23}^{(2)n'n''} (z_2)\, h_{n''}^{3a} (q')$$

$$+ [u_{nn'}^{(23,23)(23,12)} (q, q'; Z)\, \omega_{23,42}^{(2)n'n''} (z_2) + u_{nn'}^{(34,23)(23,12)} (q, q'; Z)$$

$$\times\, \omega_{23,34}^{(2)n'n''} (z_2)]\, h_{n''}^{4a} (q') + [u_{nn'}^{(23,23)(23,23)} (q, q'; Z)\, \omega_{23,42}^{(2)n'n''} (z_2)$$

$$+ u_{nn'}^{(34,23)(23,23)} (q, q'; Z)\, \omega_{23,34}^{(2)n'n''} (z_2)]\, g_{n''}^{4a} (q')\}\, q'^2\, [dq'/(2\pi^2)]$$

$$+ \int_0^\infty \{v_{nn'}^{(23,34)(34,12)} (q, s'; Z)\, \omega_{23,23}^{(2)n'n''} (z_2)\, r_{n''}^{2b} (s') + [v_{nn'}^{(23,23)(23,23)}$$

$$(q, s'; Z)\, \omega_{23,42}^{(2)n'n''} (z_2) + v_{nn'}^{(34,23)(23,23)} (q, s'; z)\, \omega_{23,34}^{(2)n'n''} (z_2)]$$

$$\times\, r_{n''}^{3b} (s')\}\, s'^2\, [ds'/(2\pi^2)]\Big], \tag{10c}$$

$$h_n^{3a} (q) = \sum_{n'n''} \Big[\int_0^\infty \{u_{nn'}^{(23,34)(34,23)} (q, q'; Z)\, \omega_{34,13}^{n'n''} (z_2)\, h_{n''}^{3a} (q')$$

$$+ [u_{nn'}^{(34,23)(23,12)} (q, q'; Z)\, \omega_{34,34}^{n'n''} (z_2) + u_{nn'}^{(23,23)(23,12)} (q, q'; Z)$$

$$\times\, \omega_{34,41}^{n'n''} (z_2)]\, h_{n''}^{4a} (q') + [u_{nn'}^{(34,23)(23,23)} (q, q'; Z)\, \omega_{34,34}^{n'n''} (z_2)$$

$$+ u_{nn'}^{(23,23)(23,23)} (q, q'; Z)\, \omega_{34,41}^{n'n''} (z_2)]\, g_{n''}^{4a} (q')\}\, q'^2\, [dq'/(2\pi^2)]$$

$$+ \int_0^\infty \{v_{nn'}^{(23,34)(34,12)} (q, s'; Z)\, \omega_{34,13}^{n'n''} (z_2)\, r_{n''}^{2b} (s') + [v_{nn'}^{(34,23)(23,23)}$$

$$(q, s'; Z)\, \omega_{34,34}^{n'n''} (z_2) + v_{nn'}^{(34,23)(23,23)} (q, s'; Z)\, \omega_{34,41}^{n'n''} (z_2)]$$

$$\times\, r_{n''}^{3b} (s')\}\, s'^2\, [ds'/(2\pi^2)]\Big], \tag{10d}$$

$$h_n^{4a} (q) = \sum_{n'n''} \Big[\int_0^\infty \{u_{nn'}^{(23,12)(12,23)} (q, q'; Z)\, \omega_{41,41}^{(1)n'n''} (z_1)\, h_{n''}^{1a} (q')$$

$$+ [u_{nn'}^{(12,23)(23,34)} (q, q'; Z)\, \omega_{41,12}^{(1)n'n''} (z_1) + u_{nn'}^{(23,23)(23,34)} (q, q'; Z)$$

$$\times\, \omega^{(1)n'n''}_{41,24}(z_1)]\, h^{2a}_{n''}(q') + (u^{(12,23)\,(23,23)}_{nn'}(q, q'; Z)\, \omega^{(1)n'n''}_{41,12}(z_1)$$

$$+\, u^{(23,23)\,(23,23)}_{nn'}(q, q'; Z)\, \omega^{(1)n'n''}_{41,24}(z_1))\, g^{2a}_{n''}(q')\}\, q'^2\, (dq'/(2\pi^2))$$

$$+ \int_0^\infty \{ v^{(23,12)\,(12,34)}_{nn'}(q, s'; Z)\, \omega^{(1)n'n''}_{41,41}(z_1)\;\; r^{1b}_{n''}(s')$$

$$+\, (v^{(12,23)\,(23,23)}_{nn'}(q, s'; Z)\, \omega^{(1)n'n''}_{41,12}(z_1) + v^{(23,23)\,(23,23)}_{nn'}(q, s'; Z)$$

$$\times\, \omega^{(1)n'n''}_{41,24}(z_1))\, r^{3b}_{n''}(s')\}\, (ds')/(2\pi^2))\Big], \tag{10e}$$

$$g^{4a}_n(q) = \sum_{n'n''} \Big[\int_0^\infty \{ u^{(23,12)\,(12,23)}_{nn'}(q, q'; Z)\, \omega^{(2)n'n''}_{41,41}(z_1)\;\; h^{1a}_{n''}(q')$$

$$+\, (u^{(12,23)\,(23,34)}_{nn'}(q, q'; Z)\, \omega^{(2)n'n''}_{41,12}(z_1) + u^{(23,23)\,(23,34)}_{nn'}(q, q'; Z)$$

$$\times\, \omega^{(2)n'n''}_{41,24}(z_1))\, h^{2a}_{n''}(q') + (u^{(12,23)\,(23,23)}_{nn'}(q, q'; Z)\, \omega^{(2)n'n''}_{41,12}(z_1)$$

$$+\, u^{(23,23)\,(23,23)}_{nn''}(q, q'; Z)\, \omega^{(2)n'n''}_{41,24}(z_1))\, g^{2a}_{n''}(q')\}\, q'^2\, (dq'/(2\pi^2))$$

$$+ \int_0^\infty \{ v^{(23,12)\,(12,34)}_{nn'}(q, s'; Z)\, \omega^{(2)n'n''}_{41,41}(z_1)\; r^{1b}_{n''}(s')$$

$$+\, (v^{(12,23)\,(23,33)}_{nn'}(q, s'; Z)\, \omega^{(2)n'n''}_{41,12}(z_1) + v^{(23,23)\,(23,23)}_{nn'}(q, s'; Z)$$

$$\times\, \omega^{(2)n'n''}_{41,24}(z_1))\, r^{3b}_{n''}(s')\}\, s'^2\, (ds'/(2\pi^2))\Big], \tag{10f}$$

$$r^{1b}_n(s) = \sum_{n'n''} \Big[\int_0^\infty \{ w^{(34,12)\,(12,23)}_{nn'}(s, q'; Z)\, \pi^{n'n''}_{12,34}(z_3)\, h^{1a}_{n''}(q')$$

$$+\, w^{(12,34)\,(34,23)}_{nn'}(s, q'; Z)\, \pi^{n'n''}_{12,12}(z_3)\, h^{3a}_{n''}(q')\}\, q'^2\, (dq'/(2\pi^2))\Big], \tag{11a}$$

$$r^{2b}_n(s) = \sum_{n'n''} \Big[\int_0^\infty \{ w^{(34,12)\,(12,23)}_{nn'}(s, q'; Z)\, \pi^{n'n''}_{34,34}(z_3)\, h^{1a}_{n''}(q')$$

$$+\, w^{(12,34)\,(34,23)}_{n'n}(s, q'; Z)\, \pi^{n'n''}_{34,12}(z_3)\, h^{3a}_{n''}(q')\}\, q'^2\, (dq'/(2\pi^2))\Big], \tag{11b}$$

$$r^{3b}_n(s) = \sum_{n'n''} \Big[\int_0^\infty \{ w^{(23,23)\,(23,34)}_{nn'}(s, q'; Z)\, \pi^{n'n''}_{23,23}(z_4)\, h^{2a}_{n''}(q')$$

$$+\, w^{(1)\,(23,23)\,(23,23)}_{nn''}(s, q'; Z)\, \pi^{n'n''}_{(23,23)}(z_4)\, g^{2a}_{n''}(q')$$

$$+ w_{nn'}^{(23,23)(23,12)} (s, q'; Z) \pi_{23,23}^{n'n''} (z_4) h_{n''}^{4a} (q')$$

$$+ w_{nn'}^{(2)\,(23,23)\,(23,23)} (s, q'; Z) \pi_{23,23}^{n'n''} (z_4) g_{n'}^{4a} (q')\} q'^2 (dq'/(2\pi^2)) \Big]. \quad (11c)$$

The notations introduced in the set of equations (10) and (11) are defined as follows:

$$u_{nn'}^{(23,12)\,(12,23)} (q, q'; Z) = \int_{-1}^{1} U_{23,12}^{n} \Bigg(p_n^{(1)}, \Bigg[\frac{m_b^2}{(2m_a+m_b)^2} q^2 + q'^2$$

$$+ \frac{2m_b}{2m_a+m_b} qq' x_1 \Bigg]^{1/2}; z_1 \Bigg)$$

$$\times \tau_{12} \Bigg[z_1 - \frac{1}{2\mu_{12,3}} \Bigg(q^2 + \frac{m_b^2}{(2m_a+m_b)^2} q'^2 + \frac{2m_b}{2m_a+m_b} qq' x_1 \Bigg) \Bigg]$$

$$\times U_{12,23}^{n'} \Bigg(\Bigg[q^2 + \frac{m_b^2}{(2m_a+m_b)^2} q'^2 + \frac{2m_b}{2m_a+m_b} qq' x_1 \Bigg]^{1/2}, p_{n'}^{(1)}; z_1 \Bigg) dx_1, \quad (12a)$$

$$u_{nn'}^{(12,23)\,(23,34)} (q, q'; Z) = \tfrac{1}{2} \int_{-1}^{1} U_{12,23}^{n} \Bigg(p_n^{(1)}, \Bigg[\frac{m_a^2}{(2m_a+m_b)^2} q^2 + q'^2$$

$$+ \frac{2m_a}{2m_a+m_b} qq' x_2 \Bigg]^{1/2}; z_1 \Bigg)$$

$$\times \tau_{23} \Bigg[z_2 - \frac{1}{2\mu_{23,4}} \Bigg(q^2 + \frac{m_a^2}{(2m_a+m_b)^2} q'^2 + \frac{2m_a}{2m_a+m_b} qq' x_2 \Bigg) \Bigg]$$

$$\times U_{23,34}^{n'} \Bigg(\Bigg[q^2 + \frac{m_a^2}{(2m_a+m_b)^2} q'^2 + \frac{2m_a}{2m_a+m_b} qq' x_2 \Bigg]^{1/2}, p_{n'}^{(2)}; z_2 \Bigg) dx_2. \quad (12b)$$

The expressions for $u_{nn'}^{(23,\,23)\,(23,\,34)}$, $u_{nn'}^{(12,\,23)\,(23,\,23)}$ and $u_{nn'}^{(23,\,23)\,(23,\,23)}$ can be found from (12b) by replacing the corresponding subscripts of the functions U^n appearing within the integral on the right-hand side. To cite an example the expression for $u_{nn'}^{(12,\,23)\,(23,\,23)}$ can be obtained by making the changes $U_{12,\,23}^{n} \to U_{12,\,23}^{n}$ and $U_{23,\,34}^{n'} \to U_{23,\,23}^{n'}$ in the right side of (12b).

$$v_{nn'}^{(23,\,12)\,(12,\,34)} (q, s'; Z) = 2 \int_{-1}^{1} U_{23,\,12}^{n} \Bigg(p_1^{(1)}, \Bigg[\frac{4m_a^2}{(2m_a+m_b)^2} q^2 + s'^2$$

$$- \frac{4m_a}{2m_a+m_b} qs' y_1 \Bigg]^{1/2}; z_1 \Bigg)$$

$$\times\ \tau_{12}\left[z_3 - \frac{1}{2\mu_{34}}\ (q^2+\tfrac{1}{4}\ s'^2\ -q\ s'^2\ y_1)\right]$$

$$\times\ W^{n}_{12,34}\ [(q^2+\tfrac{1}{4}s'^2-qs'y_1]^{1/2},\ \kappa^{(1)}_{n'};\ z_3)\ dy_1, \tag{13a}$$

$$v^{(12,23)(23,23)}_{nn'}\ (q,\ s';\ Z) = \tfrac{1}{2}\int_{-1}^{1}\ U^{n}_{12,23}\left[p^{(1)}_{n},\left(\frac{(m_a+m_b)^2}{(2m_a+m_b)^2}q^2+s'^2\right.\right.$$

$$\left.\left.-\frac{2(m_a+m_b)}{2m_a+m_b}\ qs'\ y_2\right)^{1/2};\ z_1\right] \times \tau_{23}\left[z_1 - \frac{1}{2\mu_{23}}\left(q^2+\frac{m_b^2}{(m_a+m_b)^2}s'^2\right.\right.$$

$$\left.\left.-\frac{2m_b}{m_a+m_b}\ qs'\ y_2\right)\right] \times W^{n'}_{23,23}\left[\left(q^2+\frac{m^2{}_b}{(m_a+m_b)^2}s'^2\right.\right.$$

$$\left.\left.-\frac{2m_b}{m_a+m_b}\ qs'y_2\right)^{1/2},\ \kappa^{(2)}_{n'};\ z_4\right]\ dy_2 \tag{13b}$$

The expression for $v^{(23,23)(23,23)}_{nn'}$ is found from (13b) by replacing $U^{n}_{12,23}$ by $U^{n}_{23,23}$.

$$u^{(23,34)(34,23)}_{nn'}\ (q,\ q';\ Z) = \int_{-1}^{1}\ U^{n}_{23,31}\left[p^{(2)}_{n},\left(\frac{m_a^2}{(2m_b+m_a)^2}q^2+q'^2\right.\right.$$

$$\left.\left.+\frac{2m_a}{2m_b+m_a}\ qq'x_3\right)^{1/2};\ z_2\right] \times \tau_{31}\left[z_2 - \frac{1}{2\mu_{34,1}}\left(q^2+\frac{m_a^2}{(2m_b+m_a)^2}q'^2\right.\right.$$

$$\left.\left.+\frac{2m_a}{2m_b+m_a}\ qq'x_3\right)\right] \times U^{n'}_{34,23}\left[\left(q^2+\frac{m_a^2}{(2m_b+m_a)^2}\ q'^2\right.\right.$$

$$\left.\left.+\frac{2m_a}{2m_b+m_a}\ qq'x_3\right)^{1/2},\ p^{(2)}_{n'};\ z_3\right]\ dx_3, \tag{14a}$$

$$u^{(34,23)(23,12)}_{nn'}\ (q,\ q';\ Z) = \tfrac{1}{2}\int_{-1}^{1}\ U^{n}_{34,23}\left[p^{(2)}_{n},\left(\frac{m_b^2}{(2m_b+m_a)^2}q^2+q'^2\right.\right.$$

$$\left.\left.+\frac{2m_b}{2m_b+m_a}\ qq'x_4\right)^{1/2};\ z_2\right] \times \tau_{23}\left[z_1 - \frac{1}{2\mu_{23,1}}\left(q^2+\frac{m_b^2}{(2m_b+m_a)^2}\ q'^2\right.\right.$$

$$\left.\left.+\frac{2m_b}{2m_b+m_a}\ qq'x_4\right)\right] \times U^{n'}_{23,12}\left[\left(q^2+\frac{m_b^2}{(2m_b+m_a)^2}\ q'^2\right.\right.$$

$$\left.\left.+\frac{2m_b}{2m_b+m_a}\ qq'x_4\right)^{1/2},\ p^{(1)}_{n'};\ z_1\right]\ dx_4. \tag{14b}$$

$u_{nn'}^{(23,23)(23,12)}$, $u_{nn'}^{(34,23)(23,23)}$ and $u_{nn'}^{(23,23)(23,23)}$ have expressions similar to that for $u_{nn'}^{(34,23)(23,12)}$ [equation (14b)] but with appropriate replacements of U's.

$$v_{nn'}^{(23,34)(34,12)}(q,s';Z)=2\int_{-1}^{1}U^n_{23,34}\left[p_n^{(2)},\left(\frac{4m_b^2}{(2m_b+m_a)^2}q^2+s'^2\right.\right.$$

$$\left.\left.-\frac{4m_b}{2m_b+m_a}qs'y_3\right)^{1/2};z_2\right]\times\tau_{34}\left[z_3-\frac{1}{2\mu_{12}}\left(q^2+\frac{1}{4}s'^2-qs'y_3\right)\right]$$

$$\times W^{n'}_{34,12}\ [(q^2+\tfrac{1}{4}s'^2-qs'y_3)^{1/2},\kappa_{n'}^{(1)};z_3]\ dy_3, \tag{15a}$$

$$v_{nn'}^{(34,23)(23,23)}(q,s';Z)=\tfrac{1}{2}\int_{-1}^{1}U^n_{34,23}\left[p_n^{(2)},\left(\frac{(m_a+m_b)^2}{(2m_b+m_a)^2}q^2+s'^2\right.\right.$$

$$\left.\left.-\frac{2(m_a+m_b)}{2m_b+m_a}qs'y_4\right)^{1/2};z_2\right]\times\tau_{23}\left[z_4-\frac{1}{2\mu_{23}}\left(q^2+\frac{m_b^2}{(m_a+m_b)^2}s'^2\right.\right.$$

$$\left.\left.-\frac{2m_a}{m_a+m_b}qs'y_4\right)\right]\times W^{n'}_{23,23}\left[\left(q^2+\frac{m_b^2}{(m_a+m_b)^2}s'^2\right.\right.$$

$$\left.\left.-\frac{2m_a}{m_a+m_b}qs'y_4\right)^{1/2},\kappa_{n'}^{(2)};z_4\right]dy_4. \tag{15b}$$

Replacing $U^n_{34,23}$ by $U^n_{23,23}$ in (15b) one can get the corresponding expression for $v_{nn'}^{(23,23)(23,23)}$.

$$w_{nn'}^{(34,12)(12,23)}(s,q';Z)=\tfrac{1}{2}\int_{-1}^{1}W^n_{34,12}\ [\kappa_n^{(1)},\ (\tfrac{1}{4}s^2+q'^2+sq'z_1')^{1/2},z_3]$$

$$\times\tau_{12}\left[z_1-\frac{1}{2\mu_{12,3}}\left(s^2+\frac{4m_a^2}{(2m_a+m_b)^2}q'^2+\frac{4m_a}{2m_a+m_b}sq'z_1'\right)\right]$$

$$\times U^{n'}_{12,23}\left[\left(s^2+\frac{4m_a^2}{(2m_a+m_b)^2}q'^2+\frac{4m_a}{2m_a+m_b}sq'z_1\right)^{1/2},p_{n'}^{(1)};z_1\right]dz_1' \tag{16a}$$

$$w_{nn'}^{(12,34)(34,23)}(s,q';Z)=\tfrac{1}{2}\int_{-1}^{1}W^n_{12,34}\ [\kappa_n^{(1)},\ (\tfrac{1}{4}s^2+q'^2+sq'z_2')^{1/2};z_3]$$

$$\times\tau_{34}\left[z_2-\frac{1}{2\mu_{34,1}}\left(s^2+\frac{4m_b^2}{(2m_b+m_a)^2}q'^2+\frac{4m_b}{2m_b+m_a}sq'z_2'\right)\right]$$

$$\times U^{n'}_{34,23}\left[\left(s^2+\frac{4m_b^2}{(2m_b+m_a)^2}q'^2+\frac{4m_b}{2m_b+m_a}sq'z_2\right)^{1/2},p_{n'}^{(2)};z_2\right]dz_2, \tag{16b}$$

$$w_{nn'}^{(23,23)\,(23,34)}(s, q'; Z) = \int_{-1}^{1} W_{23,23}^{n}\left[\kappa_n^{(2)}, \left(\frac{m_a^2}{(m_a+m_b)^2}s^2+q'^2\right.\right.$$

$$\left.\left.+\frac{2m_a}{m_a+m_b}sq'z_3\right)^{1/2}; z_4\right)\Big] \times \tau_{23}\left[z_2 - \frac{1}{2\mu_{23'4}}\left(s^2 + \frac{(m_a+m_b)^2}{(2m_b+m_a)^2}q'^2\right.\right.$$

$$\left.\left.+\frac{2(m_a+m_b)}{2m_b+m_a}sq'z_3'\right)\right] \times U_{23,34}^{n'}\left[\left(s^2 + \frac{(m_a+m_b)^2}{(2m_b+m_a)^2}q'^2\right.\right.$$

$$\left.\left.+\frac{2(m_a+m_b)}{2m_b+m_a}sq'z_3'\right)^{1/2}, p_{n'}^{(2)}; z_2\right]dz_3', \tag{16c}$$

Equation (16c) gives the expression for $w_{nn'}^{(1)\,(23,23)\,(23,23)}$ after replacing $U_{23,34}^{n'}$ by $U_{23,23}^{n'}$.

$$w_{nn'}^{(23,23)\,(23,12)}(s,q'; Z) = \int_{-1}^{1} W_{23,23}^{n}\left[\kappa_n^{(2)}, \left(\frac{m_b^2}{(m_a+m_b)^2}s^2+q'^2\right.\right.$$

$$\left.\left.+\frac{2m_b}{m_a+m_b}sq'z_4'\right)^{1/2}; z_4\right] \times \tau_{23}\left[z_1 - \frac{1}{2\mu_{41,2}}\left(s^2 + \frac{(m_a+m_b)^2}{(2m_a+m_b)^2}q'^2\right.\right.$$

$$\left.\left.+\frac{2(m_a+m_b)}{2m_a+m_b}sq'z_4'\right)\right] \times U_{23,12}^{n'}\left[\left(s^2 + \frac{(m_a+m_b)^2}{(2m_a+m_b)^2}q'^2\right.\right.$$

$$\left.\left.+\frac{2(m_a+m_b)}{2m_a+m_b}sq'z_4'\right)^{1/2}, p_{n'}^{(1)}; z_1\right]dz_4. \tag{17}$$

With the replacement $U_{23,12}^{n'} \to U_{23,23}^{n'}$, equation (17) gives the corresponding expression for $w_{nn'}^{(2)\,(23,23)\,(23,23)}$.

For the binding energy calculation the integrals in (10) and (11) have been replaced by sums using the Gauss-Legendre quadrature method. By this process, these equations are converted to a set of linear algebraic equations, the coefficients of which depend upon $u_{nn'}^{(\alpha,\beta)\,(\gamma,\delta)}$, $v_{nn'}^{(\alpha,\beta)\,(\gamma,\delta)}$ and $w_{nn'}^{(\alpha,\beta)\,(\gamma,\delta)}$ [equations (12) to (17)]. Those u's, v's and w's have been evaluated numerically by a similar quadrature method. For finding a solution to (10) and (11), 16-point quadrature has been used. Thus from (10) and (11), we arrive at a Fredholm determinant of order 144×144, which is evaluated for different values of the four-body energy Z. We have found out that particular value of Z say Z_0 for which this Fredholm determinant vanishes. Then Z_0 is the binding energy of the system $^{10}_{\Lambda\Lambda}Be$ in our model.

3. Results and discussion

The form factors appearing in the two-body potentials [equation (I.10)] for $\Lambda\Lambda$, $\alpha\Lambda$ and $\alpha\alpha$ interactions have been chosen to be of the Yamaguchi form.

$$g_{ij} = 1/(k^2 + \beta_{ij}^2).$$

The values of the potential parameters λ_{ij} and β_{ij} have been taken to be the same as those used by Monga and Mitra (1966) and Monga (1967) in their three-body calculations of the systems $\Lambda\Lambda\alpha$ and $\alpha\alpha\Lambda$ respectively (the reason for this choice will be explained later). These parameters are listed in table 1.

For solving the four-body equations a simplification can be made by expressing $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ in a suitable form. As the number of coupled integral equations in our case is quite large, we have decided to work with only one-term Bateman approximation for $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$. This approximation amounts to $n' = n'' = 1$ in equations (10) and (11). Consequently the effective potentials $U_{\alpha,\beta}$ and $W_{\alpha,\beta}$ reduce to one-term separable form for $N = N' = 1$. It may be noted that Kharchenko *et al* (1974) have used a two-term Bateman approximation in their calculations on four identical particle systems. They have chosen the points for the Bateman representation so as to give reasonable values for the energies and the wave functions for three identical particle subsystems. As explained below, our choice for Bateman points is optimal in the sense that the correct binding energies for the subsystems involving non-identical particles are obtained. We believe that, because of this optimal choice, our one-term Bateman approximation is similar in accuracy to the two-term approximation used by Kharchenko *et al.*

We now describe how we have fixed up the Bateman points $p_1^{(1)}$, $p_1^{(2)}$, $\kappa_1^{(1)}$ and $\kappa_1^{(2)}$. As defined earlier $p_1^{(1)}$ and $p_1^{(2)}$ correspond respectively to the subsystems $\alpha\alpha\Lambda$ and $\Lambda\Lambda\alpha$. We have chosen the Bateman points $p_1^{(1)}$ and $p_1^{(2)}$ so as to reproduce the binding energies of these subsystems. Thus in our four-body calculations we need as input the two-body off-shell t-matrix for $\Lambda\Lambda$, $\alpha\Lambda$ and $\alpha\alpha$ channels, as well as the binding energies for the three-body systems, $\alpha\alpha\Lambda$ and $\Lambda\Lambda\alpha$. So one has first to do the three-body calculations and then proceed to the four-body calculations. But in the present paper, one of the primary aims has been to provide the required machinery for dynamical calculation of a system of two distinct pairs of identical particles. As an application, we have calculated the binding energy $B_{\Lambda\Lambda}$ in ${}_{\Lambda\Lambda}^{10}$Be. We find in the literature that there exist the calculations of Monga and Mitra (1966) and Monga

Table 1. The two-body potential parameters and the corresponding three-body binding energies for the systems $\alpha\alpha\Lambda$ and $\Lambda\Lambda\alpha$.

Two-body potential parameters			Three-body binding energies (Mev)	
	λ(fm^{-1})^3	β(fm^{-1})	$B_{\Lambda\Lambda}$ (${}_{\Lambda\Lambda}{}^6$He)	B_Λ (${}_\Lambda{}^9$Be)
$\Lambda\Lambda$	−0·3942	1·8061	10·0 (from $\Lambda\Lambda$, $\alpha\Lambda$)	
$\alpha\Lambda$	−0·3809	1·3198		6.5 (from $\alpha\Lambda$, $\alpha\alpha$)
$\alpha\alpha$	−0·1191	0·7641		

(1967), who have determined the binding energies of the $\Lambda\Lambda\alpha$ and $\alpha\alpha\Lambda$ systems, respectively, corresponding to suitable two-body one term nonlocal separable potentials for $\Lambda\Lambda$, $\alpha\Lambda$ and $\alpha\alpha$ channels. These parameters are, of course, not the best possible choice. Better parameters for the two-body potentials have recently become available. In particular for the $\alpha\alpha$ interaction, Buck *et al* (1977) have presented a set of two-body local potential parameters. The corresponding nonlocal potential will be more complicated than the rank-1 Yamaguchi form used by Monga (1967). One can use such improved potentials within the framework of our theory, but the number of coupled equations will be large and unwieldy. Moreover, the three-body calculations with these potentials need to be performed. Instead of doing this, we have used the readily available three-body binding energies of $\alpha\alpha\Lambda$ and $\Lambda\Lambda\alpha$ systems consistent with the two-body potential parameters as employed by Monga (1967) and Monga and Mitra (1966). These are listed in table 1. The values obtained for $p_1^{(1)}$ and $p_1^{(2)}$ are 1·81 fm^{-1} and 1·17 fm^{-1} respectively. The Bateman point $\kappa_1^{(2)}$ has been chosen to be 0·64 fm^{-1} which is consistent with the $\alpha\Lambda$ binding energy 3·1 MeV. The Bateman point $\kappa_1^{(1)}$ is taken to be zero.

Then following the method described in the previous section we have evaluated from (10) and (11), $B_{\Lambda\Lambda}$, the separation energy for the two Λ particles in $^{10}_{\Lambda\Lambda}$Be. The result obtained is 43·97 MeV, which is larger than the experimental finding 17·5$\pm$0·4 MeV (Danysz *et al* 1963). This is not unusual because with the approach followed here, the binding energy in the case of four identical particles (Kharchenko *et al* 1974) has also been found to be higher than the experimental value. It may be mentioned that a better representation of the two-body potentials in different channels is expected to give improved four-body results. An interesting calculation for four nucleon system has been carried out by Tjon (1975), who used more than one term separable approximation at the two-body and three-body levels and found good agreement with the experimental data. Keeping in view that the number of coupled integral equations (10) and (11) is nine against only two for the four identical particle case, we feel that an attempt to work out $\Lambda\Lambda\alpha\alpha$ problem with Tjon's approach is not feasible at present.

Appendix 1

As mentioned in the text, the expressions for the $\omega_{\alpha,\beta}$'s and $\pi_{\alpha,\beta}$'s occurring in equations (6), (7), (10) and (11) are lengthy. Therefore only a few of them are given below and the rest can be made available on request.

$$\omega^{nn'}_{12,12}(z_1) = \frac{\delta nn'}{U^{n}_{12,23}(p_n^{(1)}, p_n^{(1)}; z_1)} + \sum_{m=1}^{N} \omega^{nm}_{12,12}(z_1)\, d^{(23,12)\,(12,23)}_{mn'}(z_1, p_{n'}^{(1)})$$

with
$$\bar{\omega}^{nm}_{12,12}(z_1) = \frac{1}{\triangle(z_1)\, U^{n}_{12,23}(p_n^{(1)}, p_n^{(1)}; z_1)}\,[(\triangle^{nm}_{22}(z_1) + \triangle^{nm}_{24}(z_1) + \triangle^{nm}_{42}(z_1)$$

$$+ \triangle_{44}^{nm}(z_1))\, d_{nm}^{(12,23)\,(23,12)}(z_1, p_m^{(1)}) + (\triangle_{23}^{nm}(z_1) + \triangle_{25}^{nm}(z_1)$$

$$+ \triangle_{43}^{nm}(z_1) + \triangle_{45}^{nm}(z_1))\, d_{nm}^{(23,23)\,(23,12)}(z_1, p_m^{(1)})]. \tag{A1}$$

$$\pi_{12,12}^{nn'}(z_3) = \frac{\delta_{nn'}}{W_{12,34}^{n}(\kappa_n^{(1)}, \kappa_n^{(1)}; z_3)} + \sum_{m=1}^{N'} \bar{\pi}_{12,12}^{nm}(z_3)\, a_{mn'}^{(34,12)\,(12,34)}(z_3, \kappa_n^{(1)})$$

with $\bar{\pi}_{12,12}^{nm}(z_3) = \dfrac{1}{\Lambda(z_3)\, W_{12,34}^{n}(\kappa_n^{(1)}, \kappa_n^{(1)}; z_3)}$.

$$\Lambda_{22}^{nm}(z_3)\, a_{nm}^{(12,34)\,(34,12)}(z_3, \kappa_m^{(1)}). \tag{A2}$$

In the above equations $\triangle_{ij}^{nm}(z_1)$ and $\Lambda_{ij}^{nm}(z_3)$ are respectively the co-factors of the ijth elements of the determinants

$$\triangle = |\delta_{mn} - A_{mn}^{\alpha,\beta}(z_1)| \text{ and } \Lambda = |\delta_{mn} - C_{mn}^{\alpha,\beta}(z_3)|, \tag{A3}$$

where

$$|A_{mn}^{\alpha,\beta}(z_1)| = \begin{vmatrix} 0 & d_{mn}^{\binom{23,12}{12,23}}(z_1, p_n^{(1)}) & 0 & d_{mn}^{\binom{23,12}{12,23}}(z_1, p_n^{(1)}) & 0 \\ d_{mn}^{\binom{12,23}{23,12}}(z_1, p_n^{(1)}) & 0 & 0 & 0 & d_{mn}^{\binom{12,23}{23,23}}(z_1, p_n^{(1)}) \\ d_{mn}^{\binom{23,23}{23,12}}(z_1, p_n^{(1)}) & 0 & 0 & 0 & d_{mn}^{\binom{23,23}{23,23}}(z_1, p_n^{(1)}) \\ d_{mn}^{\binom{12,23}{23,12}}(z_1, p_n^{(1)}) & 0 & d_{mn}^{\binom{12,23}{23,23}}(z_1, p_n^{(1)}) & 0 & 0 \\ d_{mn}^{\binom{23,23}{23,12}}(z_1, p_n^{(1)}) & 0 & d_{mn}^{\binom{23,23}{23,23}}(z_1, p_n^{(1)}) & 0 & 0 \end{vmatrix} \tag{A4}$$

$$
| C_{mn}^{\alpha,\beta}(z_3) | = \begin{vmatrix} & \begin{pmatrix} 34, 12 \\ 12, 34 \end{pmatrix} \\ 0 & a_{mn}(z_3, \kappa_n^{(1)}) \\ \begin{pmatrix} 12, 34 \\ 34, 12 \end{pmatrix} & \\ a_{mn}(z_3, \kappa_n^{(1)}) & 0 \end{vmatrix} \tag{A5}
$$

and

$$
\begin{aligned}
&\overset{\begin{pmatrix} \alpha, \beta \\ \beta, \gamma \end{pmatrix}}{d_{mn}}(z_3, p_n^{(\nu)}) = d_{mn}^{(\alpha,\beta)(\beta,\gamma)}(z_\nu, p_n^{(\nu)}) \\
&= \frac{1}{U_{\beta,\gamma}^n(p_n^{(\nu)}, p_n^{(\nu)}; z_\nu)} \int_0^\infty U_{\alpha,\beta}^m(p_m^{(\nu)}, p; z_\nu)\, \tau_\beta \left[z_\nu - \frac{M}{2m_\beta M_\beta} p^2 \right] \\
&\times U_{\beta,\gamma}^n(p, p_n^{(\nu)}; z_\nu)\, p^2\, [dp/(2\pi^2)]\,;\ \nu = 1, 2,
\end{aligned} \tag{A6}
$$

$$
\begin{aligned}
&\overset{\begin{pmatrix} \alpha, \beta \\ \beta, \alpha \end{pmatrix}}{a_{mn}}(z_\mu, \kappa_n^{(\nu)}) = a_{mn}^{(\nu,\beta)(\beta,\alpha)}(z_\mu, \kappa_n^{(\nu)}) \\
&= \frac{1}{W_{\beta,\alpha}^n(\kappa_n^{(\nu)}, \kappa_n^{(\nu)}; z_\mu)} \int_0^\infty W_{\alpha,\beta}^m(\kappa_m^{(\nu)}, \kappa; z_\mu)\, \tau_\beta \left[z_\mu - \frac{\kappa^2}{2\mu_\alpha} \right] \\
&\times W_{\beta,\alpha}^n(\kappa, \kappa_n^{(\nu)}; z_\mu)\, \kappa^2\, [d\kappa/(2\pi^2)]\,;\ \nu = 1, 2;\ \mu = 3, 4.
\end{aligned} \tag{A7}
$$

In equations (A6) and (A7)

$$
m_\beta = m_{ij} = M - m_i - m_j, \quad M_\beta = M - m_\beta
$$

and

$$
\mu_\alpha = \mu_{ij} = \frac{m_i m_j}{m_i + m_j},
$$

where m_i is the mass of the ith particle and M is the sum of the masses of the three particles i, j and k.

The zeros of the determinant $\triangle$ determine the binding energies of the three-particle systems (123). The zeros of the determinant Λ are connected to the sum of the binding energies of the two-particle systems (12) and (34).

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