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Coefficients of viscosity of a gaseous plasma

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Abstract. We present here an order of magnitude calculation for the coefficients of viscosity with the assumption that the drift velocity introduces asymmetry both in the single-particle distribution function f_1 and the correlation function $P(1, 2)$. These asymmetric parts have been estimated considering the self-relaxation of the system when the cause of drift velocity is suddenly removed. Using these, the kinetic part of the coefficient of electron viscosity has been calculated and the result fairly agrees with similar studies by others. The potential part of shear viscosity coefficient is found to be zero while both parts of the coefficient of bulk viscosity are non-zero.

Keywords. Gaseous plasma; coefficient of viscosity; correlation function; transport coefficient; self-relaxation.

1. Introduction

Due to the long range of Coulomb interaction, the correlational effects on the transport coefficients in a gaseous plasma can sometimes be significant. The central problem in calculating these effects is to determine the binary correlations having a definite type of inhomogeneity, resulting in the transmission process of the dynamical equilibrium conditions.

Following the idea of Balescu and De Gottal (1961), we developed a method (Majumdar *et al* 1973) to calculate potential part of the thermal conductivity. In this paper we calculate the viscosity coefficient by using the same method. The essence of the method is the following.

In the case of transport of momentum, the macroscopic drift velocity $\mathbf{u}$ introduces asymmetry in the single-particle distribution f_1 and the pair correlation function $P(1, 2)$, due to the inhomogeneity of the system. Assuming $\mathbf{u}$ to be very small compared to the thermal velocities of ions and electrons,

$$\mathbf{u} \ll (2k\theta_i/M)^{\frac{1}{2}} \ll (2k\theta_e/m)^{\frac{1}{2}},$$

where θ_e, m and θ_i, M are the temperature and mass of electrons and ions respectively. We expand f_1 and $P(1, 2)$ in powers of grad $\mathbf{u}$

$$f_1^s = f_0^s + f_{ij}^s \frac{\partial u_j}{\partial x_i}, \tag{1}$$

$$P^{s,s'}(1, 2) = P_0^{ss'}(1, 2) + P_{1ij}^{ss'} \frac{\partial u_j}{\partial x_i}, \tag{2}$$

where s, s' denote particle species (electron or ion) and summation over repeated index is implied, and f_0 and $P_0(1, 2)$ are the equilibrium values given by

$$f_0^s(1) = A_s\, n \exp\,[-\beta_s(\mathbf{v}_1 - \mathbf{u})^2],$$

$$P_0^{ss'}(1, 2) = -\frac{e^2}{k\theta_s} f_0^s(1) f_0^{s'}(2) \exp(-k_D\, r)/r. \tag{3}$$

$A_s = m_s/2\pi k\theta_s$, $\beta_s = m_s/2k\theta_s$, $r = |\mathbf{x}_1 - \mathbf{x}_2|$ and $1/k_D$ is the Debye length. The asymmetric parts of f_1^s and $P^{ss'}(1,2)$ are then calculated by using the kinetic equations for f_1^s and $P^{ss'}(1, 2)$ and observing the self-relaxation of the system towards equilibrium. The use of these asymmetric parts in the expression for stress tensor gives directly the coefficient of viscosity.

2. Kinetic part of the viscosity coefficient

The kinetic part of the stress tensor is given by the following expression (Irving and Kirkwood 1950):

$$\sigma_k^s = -m_s \int \mathbf{w}\,\mathbf{w}\, f_1^s\, d^3\mathbf{w}, \tag{4}$$

where $\mathbf{w} = \mathbf{v} - \mathbf{u}$ is the peculiar velocity. Using the expansion (1) and (4), we note that the contribution of f_0^s to σ_k^s is given by

$$\sigma_k^{s(0)} = -m_s \int \mathbf{w}\,\mathbf{w}\, f_0^s\, d\,\mathbf{w} = -n\,k\,\theta_s\,\delta_{ij},$$

which gives the scalar pressure at temperature θ_s. The contribution of the second term in (1) to σ_s^k, which gives the asymmetric part is given by

$$\sigma_k^{s(1)} = -m_s \int \mathbf{w}\,\mathbf{w} \left(f_{ij}^s \frac{\partial u_j}{\partial x_i}\right) d^3\mathbf{w}. \tag{5}$$

To calculate $\sigma_k^{(s)}$, we compute first the value of $f_{ij}^s\, \partial u_j/\partial x_i$ by observing the relaxation of the system by suddenly withdrawing the condition due to which the drift velocity arises. Since σ_k is due to the momentum transfer processes, and since the momentum transfer in an electron-ion collision is small, we can safely use the one-component description of the plasma:

$$(\partial f_1^{s(1)}/\partial t) = -\mathbf{v}_1 \frac{\partial f_1^s}{\partial \mathbf{x}_1} + \frac{1}{m_s}\int d^3\mathbf{x}_3\, d^3\mathbf{v}_3\, \frac{\partial \phi^{ss}(1,2)}{\partial \mathbf{x}_1} \cdot \frac{\partial P^{ss}(1,2)}{\partial \mathbf{V}_1}, \tag{6}$$

where $\phi^{ss'}(1,2) = e_s\, e_{s'}/r$, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, is the Coulomb potential between the particles. Since the magnitude and the gradient of $\mathbf{u}$ are small by assumption, we neglect higher powers and higher derivatives of $\mathbf{u}$. Using then (1) and (2) in (6), we obtain

$$(\partial f_1^s/\partial t) = -\mathbf{v}_1 \cdot \frac{\partial f_0^s}{\partial \mathbf{x}_1} + \frac{1}{m_s}\int \frac{\partial \phi^{ss}(1,2)}{\partial \mathbf{x}_1} \cdot \frac{\partial P_0^{ss}(1, 2)}{\partial \mathbf{V}_1}\, d^3\mathbf{x}_2\, d^3\mathbf{v}_2 \tag{7}$$

We note that f_0^s and P_0^{ss} are the locally symmetric parts of the quantities f_1^s and P^{ss} in an inhomogeneous system, and their values change from point to point. The inhomogeneity is maximum at $t=0$, and relaxes to almost zero at $t=\tau_s$, the relaxation time for the sth component. Therefore, we can replace the right side of (7) by the averages over these values. Integrating then (7) from t to $t=t+\tau_s$, we obtain

$$\triangle f_1^s \equiv f_{ij}^s \frac{\partial u_j}{\partial x_i} = -\, a_s \tau_s, \tag{8}$$

$$a_s = \tfrac{1}{2}\left[-\mathbf{v}_1 \frac{\partial f_0^s}{\partial \mathbf{x}_1} + \frac{1}{m_s}\int \frac{\partial \phi^{ss}(1,2)}{\partial \mathbf{x}_1}\frac{\partial P_0^{ss}(1,2)}{\partial \mathbf{v}_1}\, d^3\mathbf{x}_2\, d^3\mathbf{v}_2\right]. \tag{9}$$

Now, for the functions whose $\mathbf{x}$ dependence appear only through $\mathbf{u}(\mathbf{x})$ and $n(\mathbf{x})$, we can write

$$\frac{\partial}{\partial x_{1i}} = \frac{\partial u_j}{\partial x_{1i}}\frac{\partial}{\partial u_j} + \frac{\partial n}{\partial x_{1i}}\frac{\partial}{\partial n},$$

$$= \frac{\partial u_j}{\partial x_{1i}}\frac{\partial}{\partial u_j} + g_s n u_j \frac{\partial u_i}{\partial x_{1j}}\frac{\partial}{\partial n}, \tag{10}$$

where $\quad g_s = -\, m_s/k\theta_s = -\, 2\beta_s.$

In deriving the last step we have used the Navier-Stoke's equation.

Using (8), (9) and (10) in (5) we obtain in a straightforward manner

$$\sigma_{kij}^{(1)} = -\tfrac{1}{2}(nk\,\theta_s\,\tau_s)\left(\frac{\partial u_i}{\partial x_{1j}} + \frac{\partial u_j}{\partial x_{1i}} - \frac{2}{3}\delta_{ij}\frac{\partial u_l}{\partial x_{1l}}\right) - \frac{5}{6} nk\,\theta_s\,\tau_s\,\delta_{ij}\frac{\partial u_l}{\partial x_{1l}}. \tag{11}$$

Comparing with the general expression of the stress tensor σ_{ij} for a stream-line flow (Landau and Lifschitz 1959):

$$\sigma_{ij} = -\,p\,\delta_{ij} - \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_l}{\partial x_l}\right) - \zeta\,\delta_{ij}\frac{\partial u_l}{\partial x_l}, \tag{12}$$

we immediately obtain the expressions for the coefficients of shear viscosity η and bulk viscosity ζ (the kinetic part):

$$\eta_k^s = \frac{1}{2} nk\,\theta_s\,\tau_s,$$

$$\zeta_k^s = \frac{5}{6} nk\,\theta_s\,\tau_s. \tag{13}$$

3. Potential part of the coefficient of viscosity

Generalising the method of Irving and Kirkwood (1950), we write down the expression for the potential part of the stress-tensor of a two-component plasma:

$$\sigma_v^s = \frac{1}{2}\sum_{s'}\int \frac{\mathbf{r}\,\mathbf{r}}{r}\,\phi^{ss'}(r)\left(1-\frac{1}{2}\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{x}}+\ldots\right) f_2^{ss'}(1,2)\, d^3\mathbf{r}\, d^3\mathbf{v}_1\, d^3\mathbf{v}_2, \quad (14)$$

where $\mathbf{x}$ and $\mathbf{r}$ are the centre of mass and relative coordinates of particles 1 and 2. We write the two-particle distribution function $f_2^{ss'}$ in terms of the single particle function f_1^s and the correlation function $P^{ss'}(1,2)$ in the form (Dupree 1961)

$$f_2^{ss'}(1,2) = f_1^s(1)\, f_1^{s'}(2) + P^{ss'}(1,2). \quad (15)$$

Substituting (15) in (14) and remembering the form of $\phi^{ss'}(1,2)$ we see that the non-vanishing part of σ_v^s is given by

$$\sigma_v^s = \frac{1}{2}\sum_{s'}\int \frac{\mathbf{r}\,\mathbf{r}}{r}\,\phi^{ss'}(r)\left(1-\mathbf{r}\cdot\frac{\partial}{\partial \mathbf{x}}+\ldots\right) P^{ss'}(1,2)\, d^3\mathbf{r}\, d^3\mathbf{v}_1\, d^3\mathbf{v}_2. \quad (16)$$

If we now use the expansion given in (2) in (16), it is easy to see that the symmetric part $P_0^{ss'}(1,2)$ produces a diagonal tensor giving

$$p_v^s = -\frac{1}{48\pi}\, k\theta k_D^3, \text{ for } \theta_i = \theta_e.$$

This result agrees identically with Balescu (1963).

To calculate the remaining asymmetric part of σ_v^s generated by the second term on the right side of (2), we require the expression for $P_{1ij}^{ss'}\,\partial u_i/\partial x_j$. For this, we study the relaxation of the system in the same way as we have done in the preceding section. Following Dupree (1961), we write down the equation of time evolution of $P^{ss}(1,2)$ for a two-component plasma in the centre of mass and relative coordinates:

$$\frac{\partial P^{ss'}}{\partial t}(1,2) = -\frac{1}{m_s+m_{s'}}(m_s\mathbf{v}_1+m_{s'}\mathbf{v}_2)\cdot\frac{\partial P^{ss'}}{\partial \mathbf{x}} + \frac{1}{m_s+m_{s'}}$$
$$\times\left[\frac{\partial f^s}{\partial \mathbf{V}_1}(1)\, f_1^{s'}(2) + \frac{\partial f^{s'}(2)}{\partial \mathbf{V}_2} f_1^s(1)\right]\cdot\frac{\partial F^{ss'}}{\partial \mathbf{x}}(1,2) + \frac{\mathbf{r}}{r}\cdot\mathbf{G}\cdot(\mathbf{r},\mathbf{v}_1,\mathbf{v}_2). \quad (17)$$

Here, we have written

$$\sum_t n_t\, e_s\, e_t \int V(1,3)\, P^{s't}(3,2)\, d^3\mathbf{x}_3\, d^3\mathbf{v}_3 = f_1^{s'}(2)\, F^{ss'}(1,2),$$

and similarly for $f_1^s(1)\, F^{ss'}(1,2)$. $\phi^{ss'}(r) = e_s\, e_{s'}\, V(r)$, and the particle 3 belongs to the t-th species.

We can formally expand

$$F^{ss'}(1,2) = F_0^{ss'}(1,2) + F_{1kl}^{ss'} \frac{\partial u_k}{\partial x_l}. \tag{18}$$

Using (2) and (18) in (17), we obtain after neglecting higher powers and higher order derivatives of **u**,

$$\frac{\partial P^{ss'}(1,2)}{\partial t} = -\frac{1}{m_s + m_{s'}} (m_s \mathbf{v}_1 + m_{s'} \mathbf{v}_2) \cdot \frac{\partial P_0^{ss'}(1,2)}{\partial \mathbf{x}}$$

$$+ \frac{1}{m_s + m_{s'}} \left[\frac{\partial f_0^s(1)}{\partial \mathbf{v}_1} f_0^{s'}(2) + \frac{\partial f_0^{s'}(2)}{\partial \mathbf{v}_2} f_0^s(1) \right] \cdot \frac{\partial F_0^{ss'}(1,2)}{\partial \mathbf{x}}$$

$$+ \frac{\mathbf{r}}{r} \cdot \mathbf{G}(\mathbf{r}, \mathbf{v}_1, \mathbf{v}_2).$$

We now apply similar arguments as we have used in deriving (8), and integrate the last equation from t to $t+\tau_{ss'}$, where $\tau_{ss'}$ is the relaxation time between the s and s' particles. We thus obtain

$$\Delta P^{ss'} \equiv -P_{1lj}^{ss'} \frac{\partial u_l}{\partial x_j} = B_{ss'} \tau_{ss'} + \int_0^{\tau_{ss'}} \frac{\mathbf{r}}{r} \cdot \mathbf{G}(\mathbf{r}, \mathbf{v}_1, \mathbf{v}_2)\, dt \tag{19}$$

with

$$B_{ss'} = \frac{1}{2(m_s + m_{s'})} \left[-(m_s \mathbf{v}_1 + m_{s'} \mathbf{v}_2) \cdot \frac{\partial P_0^{ss'}}{\partial \mathbf{x}_1} \right.$$

$$\left. + \left(\frac{\partial}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_2} \right) f_0^s(1) f_0^{s'}(2) \cdot \frac{\partial F_0^{ss'}(1,2)}{\partial \mathbf{x}} \right]. \tag{20}$$

Now, it is easy to see that for many-component plasma, the transformation (10) should be replaced by

$$\frac{\partial}{\partial x_k} = \frac{\partial u_l}{\partial x_k} \frac{\partial}{\partial u_l} + (g_s + g_{s'})\, n u_l \frac{\partial u_k}{\partial x_l} \frac{\partial}{\partial n}. \tag{21}$$

Using (21) and (3), and noting that $F_0^{ss'}(1,2)$ is independent of **u** and **v**, we obtain in a straightforward manner,

$$B_{ss'} = -\frac{1}{2k\theta(m_s + m_{s'})} \{ m_s^2 w_{1k} w_{1l} + m_{s'}^2 w_{2k} w_{2l} + m_s m_{s'} (w_{1k} w_{2l} + w_{2k} w_{1l})$$

$$+ m_s m_{s'} u_k (w_{1l} + w_{2l}) + m_s^2 u_k w_{1l} + m_{s'}^2 u_k w_{2l} \} P_0^{ss'}(1,2) \frac{\partial u_k}{\partial x_l}$$

$$- \frac{1}{2k\theta} (m_s + m_{s'}) u_k u_l \left(2 - \frac{k_D r}{2} \right) P_0^{ss'}(1,2) \frac{\partial u_k}{\partial x_l}$$

$$-\frac{1}{2k\theta}\left\{\left(2-\frac{k_D r}{2}\right)P_0^{ss'}(1,2)+\frac{1}{k\theta}f_0^s(1)f_0^{s'}(2)\frac{\partial F_0^{ss'}(1,2)}{\partial n}\right\}$$

$$\times\ (m_s w_{1k}+m_{s'} w_{2k})\, u_l \frac{\partial u_k}{\partial x_e}. \tag{22}$$

With this expression (22) for $B_{ss'}$, we calculate the asymmetric part $\triangle P_{1ij}^{ss'}$ by comparing (2) and (19). Noting that the terms containing $\mathbf{G}\cdot(\mathbf{r}/r)$ vanish on $\mathbf{r}$ integration, we obtain the contribution of the second term of (2) to σ_v^s given by (16):

$$\sum_{s'}\tau_{ss'}\frac{k\,\theta\,k_D^3}{48\pi}\delta_{ij}\,\delta_{kl}\frac{\partial u_k}{\partial x_l}+\sum_{s'}\tau_{ss'}\frac{n\,e^2\,k_D}{8}(\beta_s+\beta_{s'})\,u_k\,u_l\,\delta_{ij}\frac{\partial u_k}{\partial x_l}. \tag{23}$$

The second term in (23) is small compared to the first term and can be neglected. Thus, from (16), we obtain finally,

$$\sigma_{vij}^s=\frac{1}{48\pi}k\,\theta\,k_D^3\,\delta_{ij}+\sum_{s'}\frac{\tau_{ss'}}{48\pi}k\,\theta\,k_D^3\,\delta_{ij}\,\delta_{kl}\frac{\partial u_{kl}}{\partial x_l}. \tag{24}$$

Comparing (24) with (11), we obtain the potential part of the viscosity coefficients:

$$\eta_v^s=0 \tag{25}$$

and

$$\zeta_v^s=-\frac{1}{48\pi}k\,\theta\,k_D^3\sum_{s'}\tau_{ss'}.$$

4. Conclusions

The following points could be noted. First, the value η_k^l computed with the value of τ_l given by Balescu and De Gottal (1961) is fairly in agreement with that of Hochstim and Massel (1969).

Secondly, the potential part of shear viscosity η_v^s is zero in the stream-line flow approximation, for a plasma with central interaction among the particles. This fact itself is interesting.

Thirdly, the method developed here is quite simple and strightforward, and can easily be extended to study other transport processes in a plasma.

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