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6j, 9j symbols and 3jm factors for the group chain $D_5 \supset C_5$

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Abstract. A complete set of nontrivial inequivalent 6j and 9j symbols for the double group D_5 and the complete set of 3jm factors (equivalently isoscalar factors) associated with the group chain $D_5 \supset C_5$ are calculated using the most general phase choices specified by Butler.

Keywords. Recoupling coefficients; quasi-ortinogonal; quasi-symplectic; group theory; Wigner-Eckart theorem; double group.

1. Introduction

Elementary applications of group theory to a physical system yield qualitative information in the form of enumeration of the symmetries, degeneracies of states of the system and selection rules. To produce quantitative results one must first calculate the coupling coefficients and then use the celebrated Wigner-Eckart theorem. In many practical applications of group theoretical methods, it is frequently necessary to couple products of the basis states of three or more irreducible representations of a group. For example, the unitary transformation from *LS* to *jj* coupling in atomic spectroscopy requires a quadruple coupling of angular momenta. The coupling of the products may be performed in various possible sequences. The various resultant coupled states are related by a unitary transformation. The elements of these unitary matrices are known as ‘ generalised recoupling coefficients ’. A more symmetrical form of the generalised recoupling coefficient, when *n* irreducible representations (irreps) are coupled, is known as 3*nj* symbol. But it is the 6*j* and 9*j* symbols that have been found to be most useful in physical applications.

Racah’s irreducible tensorial method was extended to molecular symmetry groups by Griffith (1962), who dealt with the true (single-valued) representations of the octahedral group and the dihedral groups. Extensions to the spinor (double-valued) representations of the octahedral group and the icosahedral group were carried out by Harnung (1973) and Golding (1973) respectively. Coupling coefficients associated with the 32 crystallographic double point groups were derived by Koster *et al* (1963). Derome and Sharp (1965) treated the problem for an arbitrary group. Butler (1975) extended the work of Derome and Sharp (1965) for arbitrary compact Lie groups (finite or continuous) and their sub-group chains. As pointed out by König and Kremer (1977), any formulation of the irreducible tensorial method and the computation of *j* and *jm* symbols for an arbitrary compact Lie group (finite or continuous) must be consistent with the general formulation of Derome and Sharp (1965) and the phase choices specified by Butler (1975). In a series of papers Butler and Wybourne established a systematic methodology for computing 6*j* symbols and 3*jm* factors in a group-subgroup chain (Butler and Wybourne 1976a) and applied their method to SO_3 (Butler 1976) and $SO_3 \supset T \supset C_3 \supset C_1$ (Butler and Wybourne 1976b).

Consideration of a group-subgroup chain throws light on the structural significance of the system under consideration and leads to a separation of the multiplicity, thus solving the problem of labelling of the basis states.

The 6*j*, 9*j* and 3*jm* symbols (equivalently *W*, *X* and *V* coefficients) are known only for a few non-crystallographic point groups. While discussing the aromatic hydrocarbons, Griffith (1962) proposed a scheme for naphthalene in which the π-electron system has symmetry group D_{10h} ($\equiv D_{10} \times C_i$ and $D_{10} = D_5 \times C_2$). The group D_{5d} is the same as $D_5 \times C_i$ and ferrocene ($Fe(C_5H_5)_2$) is an example of a physical system having D_{5d} as symmetry group. To calculate the 6*j*, 9*j* and 3*jm* symbols of D_{10} and D_{5d}, it would suffice if we calculate the 6*j*, 9*j* and 3*jm* symbols of D_5. We calculate these symbols taking the group independently whereas Golding and Newmarch (1977)† calculated $\bar{V}$ coupling coefficients for D_n^*, C_n^* and T^* using the fact that they are subgroups of the special unitary group SU(2). In this paper, the non-crystallographic double point group D_5 is considered and its 6*j*, 9*j* symbols and 3*jm* factors for the chain $D_5 \supset C_5$ are completely evaluated following the systematic method developed by Butler and Wybourne and the phase choices of Butler (1975). In this method, the calculation of 6*j*, 9*j* symbols and 3*jm* factors do not require any specific choice of bases for the irreps of the group considered. Their calculation depends entirely on the characters of the irreps. For any physical application which use coupling coefficients, one has to choose suitable bases for only the irreps of the lowest group in the chain (C_5 in the case considered in this paper) and then using the resulting coupling coefficients and the 3*jm* factors for the chains, one can calculate the coupling coefficients of the largest group in the chain (D_5 in the case considered in this paper).

In § 2 we calculate the 2*j* and 3*j* symbols for the double group D_5. A set of fundamental 6*j* symbols are calculated and then a complete set of primitive 6*j* symbols are obtained for D_5 using the orthogonality and Racah back coupling relations. All the nontrivial inequivalent 6*j* symbols are computed by a recursive method with the help of primitives.

For any group, the 9*j* symbol can be expressed as a sum of products of three 6*j* symbols (equation 10.3 of Butler 1975). But the expression including all the phases and permutational matrices was not given by Butler. Hence we rederive it in § 3. Using this expression and 6*j* symbols of D_5 we calculate the 9*j* symbols of the double group D_5. We have also derived some general expressions satisfied by the 9*j* symbols, similar to equations 8.15 to 8.18 of Griffith (1962), using the most general phases of Butler (1975).

In § 4, all the 2*jm* factors for the group chain $D_5 \supset C_5$ are calculated. The set of primitive 3*jm* factors is derived from the orthogonality relations and symmetry properties of 3*jm* factors. The complete set of nontrivial inequivalent 3*jm* factors for $D_5 \supset C_5$ are then calculated using a recursion relation. The notation and terminology

†Golding and Newmarch claim that their identification of Γ in the exponent with the *J* values of the Ket vector defining $|\Gamma a\rangle$ automatically gives the correct transformation properties for $\bar{V}$ coefficients under permutation i.e. if the coupling is symmetric the sum of the *J* values is even and if antisymmetric it is odd. Unfortunately this claim was found to be failing in the case of the double group D_5.

used in this paper is mostly that of Butler (1975) and Butler and Wybourne (1976a). The character table for D_5 is taken from Herzberg (1966).

2. 6j symbols for D_5 double group

The double point group D_5 is a non-simply reducible group. The multiplication table* and the symmetric, antisymmetric terms of the kronecker square $\Gamma_i^{\times 2}$ are given in tables 1 and 2 respectively. Herzberg's notation for indicating the irreducible representations is given in the parentheses of the first column of table 1. The irreducible representations (irreps) of a finite group are classified (Butler and King 1974) into orthogonal, symplectic or complex by the evaluation of Frobenius-Schur invariant (Hamermesh 1962) C_Γ and we assign the $2j$ symbol ϕ_Γ the value of C_Γ in the first two cases. In the third case, when the irreps are complex, the $2j$ symbol ϕ_Γ is determined using the relation $\phi_{\Gamma_1} \phi_{\Gamma_2} \phi_{\Gamma_3} = 1$ (equation (8.10) of Butler 1975) and we name the irreps as quasi-orthogonal or quasi-symplectic according as $\phi_\Gamma = \pm 1$.

The permutational symmetries of the $3jm$ symbol are given by

$$(\lambda_a \; \lambda_b \; \lambda_c)_{s i_a i_b i_c} = \sum_r \{(\pi) \; \lambda_1 \; \lambda_2 \; \lambda_3\}_{sr} \; (\lambda_1 \; \lambda_2 \; \lambda_3)_{r i_1 i_2 i_3}$$

where π is a permutation of 1, 2, 3. Using the phase convention of Butler (1975), when λ_1, λ_2 and λ_3 are all distinct and none of them is a scalar representation we have

$$\text{(i)} \quad \{(\pi) \; \lambda_1 \; \lambda_2 \; \lambda_3\}_{rs} = \delta_{rs} \text{ for } \pi \text{ even,}$$

$$= \theta(\lambda_1 \; \lambda_2 \; \lambda_3 \; r) \, \delta_{rs} \text{ for } \pi \text{ odd,}$$

where $$\theta(\lambda_1 \; \lambda_2 \; \lambda_3 \; r) = \pm 1 \qquad (1)$$

or $$\text{(ii)} \quad \{(\pi) \; \lambda_1 \; \lambda_2 \; \lambda_3\}_{rs} = \delta_{rs} \text{ for } \pi \text{ odd or even.}$$

Table 1. Multiplication table for the double group D_5

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
$\Gamma_1(A_1)$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
$\Gamma_2(A_2)$	—	Γ_1	Γ_3	Γ_4	Γ_6	Γ_5	Γ_7	Γ_8
$\Gamma_3(E_1)$			$\Gamma_1+\Gamma_2+\Gamma_4$	$\Gamma_3+\Gamma_4$	Γ_8	Γ_8	$\Gamma_7+\Gamma_8$	$\Gamma_5+\Gamma_6+\Gamma_7$
$\Gamma_4(E_2)$				$\Gamma_1+\Gamma_2+\Gamma_3$	Γ_7	Γ_7	$\Gamma_5+\Gamma_6+\Gamma_8$	$\Gamma_7+\Gamma_8$
Γ_5 } $(E_{5/2})$ Γ_6 }	—	—	—	—	Γ_2	Γ_1 Γ_2	Γ_4 Γ_4	Γ_3 Γ_3
$\Gamma_7(E_{1/2})$	—	—	—	—	—	—	$\Gamma_1+\Gamma_2+\Gamma_3$	$\Gamma_3+\Gamma_4$
$\Gamma_8(E_{3/2})$								$\Gamma_1+\Gamma_2+\Gamma_4$

*There are some misprints in the multiplication table of Herzberg (1966).

Table 2. Reduction of the kronecker squares, classification and powers of IRs of the double group D_5

Γ_i	Symmetric	Antisymmetric	ϕ_c	Type	Power
Γ_1	Γ_1	—	+1	Orthogonal	2
Γ_2	Γ_1	—	+1	,,	2
Γ_3	$\Gamma_1+\Gamma_4$	Γ_2	1	,,	2
Γ_4	$\Gamma_1+\Gamma_3$	Γ_2	+1	,,	4
Γ_5	Γ_2	—	−1	Quasi-symplectic	5
Γ_6	Γ_2	—	−1	,,	5
Γ_7	$\Gamma_2+\Gamma_3$	Γ_1	−1	Symplectic	1
Γ_8	$\Gamma_2+\Gamma_4$	Γ_1	−1	,,	3

If one of them is a scalar representation then

$$\{(\pi)\ \lambda\ \lambda^*\ 1\}_{11} = \theta\,(\lambda\ \lambda^*\ 1,\ 1) = \phi_\lambda = \pm\ 1.$$

If two of the three irreps are equal, the value of the $3j$ symbol $\theta\,(\lambda\,\lambda\,\lambda',\,r)$ is equal to $\pm$ 1 according as the rth term of λ'^* occurs in the symmetric or antisymmetric part of the product $\lambda \times \lambda$ respectively. If we can select a set of $1j$ symbols $(-1)^\lambda$ such that

$$\phi_\lambda = (-1)^{2\lambda} \tag{2}$$

$$\theta\,(\lambda_1\,\lambda_2\,\lambda_3\,r) = (-1)^{\lambda_1+\lambda_2+\lambda_3+r-1} \tag{3}$$

then the arbitrary choice of phase in (1) is removed. But in the case of D_5 we cannot select a set of $1j$ symbols $(-1)^\lambda$ satisfying the equations (2) and (3). Therefore we use Butler's second choice of phase namely

$$\{(\pi)\ \lambda_1\ \lambda_2\ \lambda_3\}_{rs} = \delta_{rs} \text{ for } \pi \text{ odd or even,}$$

in the case of λ_1, λ_2 and λ_3 are all distinct and none of them is a scalar representation.

The spin representation Γ_7 of D_5 is a faithful representation and may be chosen as primitive irrep (Butler and Wybourne 1976a). Power of an irrep λ is defined as the minimum positive integer n such that

$$\epsilon^{\times n} \supset \lambda \text{ or } (\epsilon^*)^{\times n} \supset \lambda$$

where ϵ is the primitive irrep. The power of all the irreps of D_5 are given in table 2. A primitive $6j$ symbol has the primitive irrep at least once but do not contain the scalar representation. The trivial $6j$ symbol being proportional to the $3j$ symbol are known from equation (17)** of Butler and Wybourne (1976a).

Out of 51 non-trivial inequivalent $6j$ symbols of D_5, 31 are primitives. Using the orthogonality and Racah back-coupling relations (equations (25) and (26) of Butler

**Equation (17) of Butler and Wybourne (1976a) should be read as

$$\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_2^* & \lambda_1 & 1\end{Bmatrix}_{11rs} = |\,\lambda_1\,\lambda_2\,|^{-1/2}\ \ominus\ (\lambda_1\lambda_2\lambda_3,\,r)\delta_{r\,s},$$

and Wybourne 1976a) and systematically increasing the power of the largest irrep, all the $6j$ primitives are calculated. The free phases of the primitive $6j$s are fixed by a subset of primitives known as fundamentals. The phases of all the fundamentals for the double group D_5 are chosen to be $+1$ and the fundamentals are selected in such that the $6j$ symbols for the true representations of D_5 coincide with those of Griffith (1962).

Once the set of primitive $6j$ symbols are obtained the remaining 20 $6j$ symbols are calculated recursively using the modified form of the generalised Biedenharn-Elliott sum rule.†† (equation (27) of Butler and Wybourne 1976a). At this state no phase freedom exists. The complete set of non-trivial inequivalent $6j$ symbols is listed in table 3.

Table 3. Non-trivial inequivalent $6j$ symbols for the group D_5

(2 3 3 / 2 3 3)	$= +\frac{1}{2}$	(2 3 3 / 3 4 4)	$= -\frac{1}{2}$
(2 3 3 / 4 3 3)	$= +\frac{1}{2}$	(2 3 3 / 4 4 4)	$= +\frac{1}{2}$
(2 3 3 / 5 8 8)	$= +\frac{1}{2}$	*(2 3 3 / 7 7 7)	$= +\frac{1}{2}$
(2 3 3 / 7 8 8)	$= -\frac{1}{2}$	(2 3 3 / 8 5 6)	$= -1/\sqrt{2}$
(2 3 3 / 8 7 7)	$= -\frac{1}{2}$	(2 4 4 / 2 4 4)	$= +\frac{1}{2}$
(2 4 4 / 3 4 4)	$= +\frac{1}{2}$	(2 4 4 / 5 7 7)	$= -\frac{1}{2}$
(2 4 4 / 7 5 6)	$= -1/\sqrt{2}$	(2 4 4 / 7 8 8)	$= -\frac{1}{2}$
*(2 4 4 / 8 7 7)	$= +\frac{1}{2}$	(2 4 4 / 8 8 8)	$= +\frac{1}{2}$
(2 5 5 / 2 5 6)	$= -1$	(2 5 5 / 3 8 8)	$= -1/\sqrt{2}$
*(2 5 5 / 4 7 7)	$= +1/\sqrt{2}$	(2 7 7 / 2 7 7)	$= -\frac{1}{2}$
(2 7 7 / 3 7 7)	$= +\frac{1}{2}$	*(2 7 7 / 3 8 8)	$= +\frac{1}{2}$
(2 7 7 / 4 8 8)	$= -\frac{1}{2}$	(2 8 8 / 2 8 8)	$= -\frac{1}{2}$
(2 8 8 / 4 8 8)	$= +\frac{1}{2}$	(3 3 4 / 3 3 4)	$= 0$
(3 3 4 / 3 4 4)	$= +\frac{1}{2}$	(3 3 4 / 5 7 8)	$= +i/2$
*(3 3 4 / 7 8 7)	$= +\frac{1}{2}$	(3 3 4 / 8 8 5)	$= +i/2$
(3 3 4 / 8 8 7)	$= 0$	(3 4 4 / 3 4 4)	$= 0$
*(3 3 4 / 5 7 7)	$= +i/2$	*(3 4 4 / 7 5 8)	$= +\frac{1}{2}$
(3 4 4 / 8 7 7)	$= 0$	*(3 4 4 / 8 8 7)	$= +\frac{1}{2}$
(3 5 8 / 3 5 8)	$= -\frac{1}{2}$	(3 5 8 / 3 6 8)	$= +\frac{1}{2}$
(3 5 8 / 3 7 8)	$= -\frac{1}{2}$	(3 5 8 / 4 7 7)	$= +i/2$
(3 5 8 / 4 8 7)	$= +\frac{1}{2}$	(3 7 7 / 3 7 7)	$= 0$
(3 7 7 / 3 7 8)	$= -\frac{1}{2}$	(3 7 8 / 3 7 8)	$= 0$
(3 7 8 / 4 7 8)	$= +\frac{1}{2}$	(4 5 7 / 4 5 7)	$= -\frac{1}{2}$
(4 5 7 / 4 6 7)	$= +\frac{1}{2}$	(4 5 7 / 4 8 7)	$= -\frac{1}{2}$
(4 7 8 / 4 7 8)	$= 0$	(4 7 8 / 4 8 8)	$= -\frac{1}{2}$
(4 8 8 / 4 8 8)	$= 0$		

Note: The $6j$ symbols having star on the left side are fundamentals.

$$(ijk/qrs)_{r_1r_2r_3r_4} = \begin{Bmatrix} \Gamma_i & \Gamma_j & \Gamma_k \\ \Gamma_q & \Gamma_r & \Gamma_s \end{Bmatrix}_{r_1r_2r_3r_4}$$

For the group under consideration the kronecker multiplicities are 1 and hence r_1, r_2, r_3 and r_4 are suppressed.

††There are some misprints in equation (20) of Butler and Wybourne (1976a). The $6j$ symbol in left hand side i.e.

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}_{s_1 s_2 s_3 r}$$

should be $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}^*_{s_1 s_2 s_3 r}$

and the $3j$ symbols

$$\{(132)\ \nu_1\ \lambda_2\ \nu_3^*\}_{s_1\, s_1'},\ \{(123)\ \lambda_1\ \nu_2^*\ \nu_3\}\ {s_2\, s_2'}$$

should be read as

$$\{(132)\ \nu_1\ \lambda_2\ \nu_3^*\}_{s_2\, s_2'},\ \{(123)\ \lambda_1\ \nu_2^*\ \nu_3\}_{s_1\, s_1'}$$

3. Special formulae for 9j symbol

The 9j symbol defined by Butler (1975) is

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \\ s_1 & s_2 & s_2 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} = \Sigma\,(i\,i'\,i''\,j\,j'\,j''\,k\,k'\,k'')\,(\lambda_1\,\lambda_2\,\lambda_3)_{r_1\,i\,i'\,i''} \times (\mu_1\,\mu_2\,\mu_3)_{r_2\,j\,j'\,j''}\,(\nu_1\,\nu_2\,\nu_3)_{r_3\,k\,k'\,k''} \times (\lambda_1\,\mu_1\,\nu_1)^{i\,j\,k}_{s_1}\,(\lambda_2\,\mu_2\,\nu_2)^{i'\,j'\,k'}_{s_2}\,(\lambda_3\,\mu_3\,\nu_3)^{i''\,j''\,k''}_{s_3} \tag{4}$$

Throughout this section the summation labels are given inside parenthesis immediately next to Σ. Equation (9.12) of Butler (1975) can be written as

$$\Sigma\,(r_3\,r_4\,\lambda_3\,i'\,i''\,j_1')\,|\lambda_3|\,\phi_{\lambda_2}\,\phi_{\mu_1}\,(\mu_2)^{j_1 j'}\,(\lambda_2)^{i' i_1'}$$

$$\times\,(\lambda_1\,\lambda_2\,\lambda_3)_{r_4\,i\,i'\,i''}\,(\mu_1^*\,\mu_2\,\lambda_3)^{j_2 j_1' i''}_{r_3} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$$

$$= \Sigma\,(j'')\,(\lambda_1\,\mu_2^*\,\mu_3)_{r_1\,i\,j'\,j''}\,(\mu_1^*\,\lambda_2^*\,\mu_3)^{j_2\,i_1'\,j''}_{r_2}\,. \tag{5}$$

Applying equation (5) to the product $(\mu_1\,\mu_2\,\mu_3)_{r_2\,j\,j'\,j''}\,(\lambda_2\,\mu_2\,\nu_2)^{i'\,j'\,k'}_{s_2}$ in equation (4) we get on the r.h.s. of equation (4)

$$\Sigma\,(\sigma\,t_1\,t_2\,r_2'\,s_2'\,i\,i'\,i''\,i_1'\,j\,j_1\,j''\,k\,k'\,k'')\,|\sigma|\,\{(123)\,\mu_3\,\mu_1\,\mu_2\}_{r_2 r_2'}$$

$$\times\,(123)\,\nu_2\,\lambda_2\,\mu_2\}_{s_2\,s_2'}\,\phi_{\lambda_2}\,\phi_{\nu_2}\,(\mu_1^*)_{j_1 j}\,(\lambda_2^*)^{i_1' i'}\,(\mu_3\,\lambda_2^*\,\sigma)_{t_2\,j''\,i_1'\,m}$$

$$\times\,(\nu_2\,\mu_1^*\,\sigma)^{k'\,j_1\,m}_{t_1} \begin{Bmatrix} \mu_3 & \lambda_2^* & \sigma \\ \nu_2^* & \mu_1^* & \mu_2 \end{Bmatrix}_{r_2'\,s_2'\,t_1\,t_2} (\lambda_1\,\lambda_2\,\lambda_3)_{r_1\,i\,i'\,i''}$$

$$\times\,(\nu_1\,\nu_2\,\nu_3)_{r_3\,k\,k'\,k''}\,(\lambda_1\,\mu_1\,\nu_1)^{i\,j\,k}_{s_1}\,(\lambda_3\,\mu_3\,\nu_3)^{i''\,j''\,k''}_{s_3}$$

Suitably raising and lowering the indices and using equation (9.12) of Butler (1975) wice we get

$$\Sigma\,(\sigma t_1\,t_2\,t_3\,t_4\,t_2'\,r_1\,r_2'\,r_3'\,s_2'\,s_3'\,i\,i_1\,k''\,m\,m'\,k_1'')\,|\sigma|$$

$$\times\,\phi_{\lambda_2}\,\phi_{\mu_3}\,\phi_{\nu_1}\,\phi_{\nu_2}\,\{(123)\,\mu_3\,\mu_1\,\mu_2\}_{r_2\,r_2'}\,\{(123)\,\nu_2\,\lambda_2\,\mu_2\}_{s_2\,s_2'}$$

$$\times\,\{(132)\,\nu_2\,\nu_3\,\nu_1\}_{r_3\,r_3'}\,\{(123)\,\nu_3\,\lambda_3\,\mu_3\}_{s_3\,s_3'}\,\{(132)\,\lambda_2^*\,\sigma\,\mu_3\}_{t_2\,t_2'}$$

$$\times\,\{(132)\,\lambda_2\,\lambda_3\,\lambda_1\}_{r_1\,r_1'}\,(\lambda_1)^{i\,i_1}\,(\sigma)^{mm'}\,(\nu_3)\,k''k_1''$$

$$\times\,(\nu_3^*\,\sigma\,\lambda_1)_{t_4\,k_1''\,m\,i}\,(\lambda_1^*\,\nu_3\,\sigma^*)^{t_3\,i_1\,k''\,m'}$$

$$\times \begin{Bmatrix} \nu_3^* & \sigma & \lambda_1 \\ \lambda_2^* & \lambda_3 & \mu_3^* \end{Bmatrix}_{s_3'\,t_2'\,r_1'\,t_4} \begin{Bmatrix} \mu_3 & \lambda_2^* & \sigma \\ \nu_2^* & \mu_1^* & \mu_2 \end{Bmatrix}_{r_2'\,s_2'\,t_1\,t_2} \begin{Bmatrix} \lambda_1^* & \nu_3 & \sigma^* \\ \nu_2 & \mu_1 & \nu_1^* \end{Bmatrix}_{s_1\,r_3'\,t_1\,t_3}$$

changing the $3jm$ symbol with complex conjugate representations $(\lambda_1^* \; \nu_3 \; \sigma^*)^{t_3 i_1 k'' m'}$ to the complex conjugated $3jm$ symbol $(\lambda_1 \; \nu_3^* \; \sigma)^{ik_1'' \, m}_{t_3}$ and using orthogonality relation, we arrive at

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop {\begin{matrix} s_1 & s_2 & s_3 \end{matrix}} = \begin{aligned} &\Sigma \, (\sigma t_1 \; t_2 \; t_3 \; t_4 \; r_1' \; r_2' \; r_3' \; s_2' \; s_3' \; t_2' \; t_3') \, |\sigma| \\ &\times \phi_{\lambda_2} \phi_{\mu_2} \phi_{\nu_1} \phi_{\nu_2} \phi_{\nu_3} \{(123) \; \mu_3 \; \mu_1 \; \mu_2\}_{r_2 r_2'} \\ &\times \{(123) \; \nu_2 \; \lambda_2 \; \mu_2\}_{s_2 s_2'} \{(132) \; \nu_2 \; \nu_3 \; \nu_1\}_{r_3 r_3'} \\ &\times \{(123) \; \nu_3 \; \lambda_3 \; \mu_3\}_{s_3 s_3'} \{(132) \; \lambda_2^* \; \sigma \; \mu_3\}_{t_2 t_2'} \; \{(132) \; \lambda_2 \; \lambda_3 \; \lambda_1\}_{r_1 r_1'} \\ &\times \{(123) \; \lambda_1 \; \nu_3^* \; \sigma\}_{t_4 t_3'} \, A(\lambda_1 \; \nu_3^* \; \sigma)_{t_3' t_3} \\ &\quad \begin{Bmatrix} \nu_3^* & \sigma & \lambda_1 \\ \lambda_2 & \lambda_3 & \mu_3^* \end{Bmatrix}_{r_3' t_2' r_1' t_4} \begin{Bmatrix} \mu_3 & \lambda_2^* & \sigma \\ \nu_2^* & \mu_1^* & \mu_2^* \end{Bmatrix}_{r_2' s_2' t_1 t_2} \begin{Bmatrix} \lambda_1^* & \nu_3 & \sigma^* \\ \nu_2 & \mu_1 & \nu_1^* \end{Bmatrix}_{s_1 r_3' t_1 t_3} \end{aligned} \tag{6}$$

Butler (1975) proved that the multiplicity metric tensor A_{rs} can always be chosen to be the unit matrix for all those groups for which equation (8.10) of Butler (1975) holds. Using the above results and phase convention of § 2 equation (6) reduces to

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop {\begin{matrix} s_1 & s_2 & s_3 \end{matrix}} = \begin{aligned} &\Sigma \, (\sigma \; t_1 \; t_2 \; t_3) \, |\sigma| \, \phi_{\lambda_2} \phi_{\mu_3} \begin{Bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \nu_2^* & \nu_3 & \sigma \end{Bmatrix}_{t_2 t_1 r_3 s_1} \\ &\times \begin{Bmatrix} \lambda_2 & \mu_2 & \nu_2 \\ \mu_1 & \sigma & \mu_3^* \end{Bmatrix}_{t_3 r_2 t_1 s_2} \begin{Bmatrix} \lambda_3 & \mu_3 & \nu_3 \\ \sigma & \lambda_1^* & \lambda_2 \end{Bmatrix}_{r_1 t_3 t_2 s_3} \end{aligned} \tag{7}$$

The $9j$ symbol, when one of its entries is the identity representation 1 (trivial $9j$ symbol), reduces to a single $6j$ symbol.

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \mu \\ \lambda_3 & \lambda_4 & \mu^* \\ \nu & \nu^* & 1 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ 1 \end{matrix} \atop {\begin{matrix} s_1 & s_2 & 1 \end{matrix}}$$

$$= |\mu, \nu|^{-1/2} \, \phi_{\lambda_2} \phi_{\nu} \, \theta \, (\lambda_2 \; \lambda_4 \; \nu^* \; s_2) \; \theta \, (\lambda_3 \; \lambda_4 \; \mu^* \; r_2) \begin{Bmatrix} \lambda_1 & \lambda_3 & \nu \\ \lambda_4 & \lambda_2^* & \mu \end{Bmatrix}_{r_1 r_2 s_2 s_1} . \tag{8}$$

We note that there are some misprints in equations (10.3) and (10.4) of Butler (1975) and they are rectified in equation (7) and (8) of this paper. The symmetry properties of the $9j$ symbol are as follows:

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop {\begin{matrix} s_1 & s_2 & s_3 \end{matrix}} = \begin{Bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{Bmatrix}^* \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} \atop {\begin{matrix} r_1 & r_2 & r_3 \end{matrix}} \tag{9}$$

$$= \begin{Bmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* \\ \mu_1^* & \mu_2^* & \mu_3^* \\ \nu_1^* & \nu_2^* & \nu_3^* \end{Bmatrix}^{*} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop \begin{matrix} s_1 & s_2 & s_3 \end{matrix} \tag{10}$$

$$= \Sigma\,(r_1'\,r_2'\,r_3')\ \{(\pi),\ \lambda_a\ \lambda_b\ \lambda_c\}_{r_1 r_1'}\ \{(\pi)\ \mu_a\ \mu_b\ \mu_c\}_{r_2 r_2'}$$

$$\times\ \{(\pi),\ \nu_a\ \nu_b\ \nu_c\}_{r_3 r_3'} \begin{Bmatrix} \lambda_a & \lambda_b & \lambda_c \\ \mu_a & \mu_b & \mu_c \\ \nu_a & \nu_b & \nu_c \end{Bmatrix} \begin{matrix} r_1' \\ r_2' \\ r_3' \end{matrix}, \atop \begin{matrix} s_a & s_b & s_c \end{matrix} \tag{11}$$

where a, b, c is any permutation π, of 1, 2, 3 (permutation of columns).

$$= \Sigma\,(s_1'\,s_2'\,s_3)\ \{(\pi),\ \pi\,(\lambda_1)\ \pi\,(\mu_1)\ \pi\,(\nu_1)\}_{s_1 s_1'}\ \{(\pi),\ \pi\,(\lambda_2)\ \pi\,(\mu_2)\ \pi\,(\nu_2)\}_{s_2 s_2'}$$

$$\times\ \{(\pi),\ \pi\,(\lambda_3)\ \pi\,(\mu_3)\ \pi(\nu_3)\}_{s_3 s_3'} \begin{Bmatrix} \pi\,(\lambda_1) & \pi\,(\lambda_2) & \pi\,(\lambda_3) \\ \pi\,(\mu_1) & \pi\,(\mu_2) & \pi\,(\mu_3) \\ \pi\,(\nu_1) & \pi\,(\nu_2) & \pi\,(\nu_3) \end{Bmatrix} \begin{matrix} \pi\,(r_1) \\ \pi\,(r_2). \\ \pi\,(r_3) \end{matrix} \atop \begin{matrix} s_1' & s_2' & s_3' \end{matrix} \tag{12}$$

where π is any permutation of λ, μ, ν (permutation of rows).

We have derived some general equations satisfied by the $9j$ symbols, similar to equations (8.15) to (8.18) of Griffith (1962), using the most general phases of Butler (1975), and the final equations are as follows:

$$\Sigma\,(r_3\ s_3\ \nu_3\ k'')\ |\ \nu_3\ |\ (\nu_1\ \nu_2\ \nu_3)_{r_3}^{k\,k'\,k''}\ (\lambda_3\ \mu_3\ \nu_3)_{s_3 i'' j'' k''} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop \begin{matrix} s_1 & s_2 & s_3 \end{matrix}$$

$$= \Sigma\,(i\ i'\ j\ j')\ (\lambda_1\ \lambda_2\ \lambda_3)_{r_1\,i\,i'\,i''}\ (\mu_1\ \mu_2\ \mu_3)_{r_2\,j\,j'\,j''}$$

$$\times\ (\lambda_1\ \mu_1\ \nu_1)_{s_1}^{ijk}\ (\lambda_2\ \mu_2\ \nu_2)_{s_2}^{i'j'k'} \tag{13}$$

$$\Sigma\,(r_3\ s_2\ s_3\ \nu_2\ \nu_3\ k'\ k'')\ |\ \nu_2,\ \nu_3\ |\ (\nu_1\ \nu_2\ \nu_3)_{r_3}^{kk'k''}$$

$$\times\ (\lambda_3\ \mu_3\ \nu_3)_{s_3\ i''\ j''\ k''}\ (\lambda_2\ \mu_2\ \nu_2)_{s_2\ i'\ j'\ k'} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop \begin{matrix} s_1 & s_2 & s_3 \end{matrix}$$

$$= \Sigma\,(i\,j)\ (\lambda_1\ \lambda_2\ \lambda_3)_{r_1\ ii'\ i''}\ (\lambda_1\ \mu_1\ \nu_1)_{s_1}^{i\ j\ k}\ (\mu_1\ \mu_2\ \mu_3)_{r_2\ jj'\ j''} \tag{14}$$

$$\Sigma\,(r_3\ s_2\ s_1\ \nu_1\ \nu_2)\ |\ \nu_1,\ \nu_2\ | \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \atop \begin{matrix} s_1 & s_2 & s_3 \end{matrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3' \\ \mu_1 & \mu_2 & \mu_3' \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}^{*} \begin{matrix} r_1' \\ r_2' \\ r_3 \end{matrix} \atop \begin{matrix} s_1 & s_2 & s_3 \end{matrix}$$

$$= |\ \lambda_3,\ \mu_3\ |^{-1}\ \delta_{\lambda_3\,\lambda_3'}\ \delta_{\mu_3\,\mu_3'}\ \delta_{r_1\,r_1'}\ \delta_{r_2\,r_2'}\ \delta_{s_3 s_3'}\ \delta\,(\lambda_1\lambda_2\lambda_3)$$

$$\times\ \delta\,(\mu_1\ \mu_2\ \mu_3)\ \delta\,(\lambda_3\ \mu_3\ \nu_3). \tag{15}$$

$$\Sigma\,(\nu_1\,\nu_2\,r_3\,s_1\,s_2)\,|\,\nu_1,\,\nu_2\,|\,\theta\,(\mu_1\,\mu_2\,\mu_3,\,r_2)\,\theta\,(\mu_2\,\lambda_2\,\nu_2\,s_2)$$

$$\times\,\theta\,(\mu_1\,\lambda_2\sigma_2,\,t_2\,\begin{array}{c}\begin{Bmatrix}\lambda_1 & \mu_1 & \nu_1\\ \lambda_2 & \mu_2 & \nu_2\\ \lambda_3 & \mu_3 & \nu_3\end{Bmatrix}\\ r_1\;r_2\;r_3\end{array}\begin{array}{c}{}^{*}s_1\\ s_2\\ s_3\\ \end{array}\begin{array}{c}\begin{Bmatrix}\lambda_1 & \mu_1 & \nu_1\\ \mu_2 & \lambda_2 & \nu_2\\ \sigma_1 & \sigma_2 & \nu_3\end{Bmatrix}\\ t_1\;t_2\;r_3\end{array}\begin{array}{c}s_1\\ s_2\\ t_3\\ \end{array} = \begin{array}{c}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\\ \sigma_1 & \sigma_2 & \nu_3\end{Bmatrix}\\ t_1\;t_2\;s_3\end{array}\begin{array}{c}r_1\\ r_2\\ t_3\\ \end{array} \qquad (16)$$

The 9*j* symbol vanishes unless there is a representation σ satisfying the triangular condition simultaneously for the triples $(\lambda_1\,\nu_3^*\,\sigma)$, $(\nu_2\,\mu_1^*\,\sigma)$ and $(\lambda_2^*\,\mu_3\,\sigma)$. The non-trivial inequivalent 9*j* symbols of the double group D_5 are calculated using (7). As the 9*j* symbols involving only the true representations coincide with those of Griffith (1962), those involving the spin representations are tabulated in table 4. The non-trivial 9*j* symbols other than those that can be obtained by the application of equations (9), (10), (11) and (12) on the tabulated ones are zero.

Table 4. Non-trivial inequivalent 9*j* symbols for the double group D_5

(2 3 3 / 3 5 8 / 3 8 5)	= −1/4	(2 3 3 / 3 5 8 / 3 8 6)	= +1/4
(2 3 3 / 3 5 8 / 3 8 7)	= +1/4	(2 3 3 / 3 7 7 / 3 7 8)	= +1/4
(2 3 3 / 4 5 7 / 4 8 7)	= −*i*/4	(2 3 3 / 4 5 7 / 4 8 8)	= +1/4
(2 3 3 / 4 7 8 / 4 8 7)	= +1/4	(2 3 3 / 5 2 5 / 5 3 8)	= −1/2
(2 3 3 / 5 3 8 / 5 4 7)	= $+i/2\sqrt{2}$	(2 3 3 / 5 4 7 / 5 4 7)	= $-i/2\sqrt{2}$
(2 3 3 / 7 2 7 / 7 3 7)	= −1/4	(2 3 3 / 7 2 7 / 7 3 8)	= −1/4
(2 3 3 / 7 3 7 / 7 4 8)	= −1/4	(2 3 3 / 7 3 8 / 7 4 5)	= −*i*/4
(2 3 3 / 7 4 5 / 7 4 8)	= −1/4	(2 3 3 / 8 2 8 / 8 3 5)	= −1/4
(2 3 3 / 8 2 8 / 8 3 7)	= −1/4	(2 3 3 / 8 3 5 / 8 4 8)	= +*i*/4
(2 3 3 / 8 3 7 / 8 4 7)	= −1/4	(2 3 3 / 8 4 7 / 8 4 8)	= +1/4
(2 4 4 / 4 5 7 / 4 7 5)	= −1/4	(2 4 4 / 4 5 7 / 4 7 6)	= +1/4
(2 4 4 / 4 5 7 / 4 7 8)	= +1/4	(2 4 4 / 4 7 6 / 4 8 8)	= +1/4
(2 4 4 / 5 2 5 / 5 4 7)	= −1/2	(2 4 4 / 5 3 8 / 5 3 8)	= $+i/2\sqrt{2}$
(2 4 4 / 5 3 8 / 5 4 7)	= $+1/2\sqrt{2}$	(2 4 4 / 7 2 7 / 7 4 5)	= −1/4
(2 4 4 / 7 2 7 / 7 4 8)	= −1/4	(2 4 4 / 7 3 7 / 7 3 8)	= −1/4
(2 4 4 / 7 3 7 / 7 4 5)	= −*i*/4	(2 4 4 / 7 3 8 / 7 4 8)	= +1/4
(2 4 4 / 8 2 8 / 8 4 7)	= −1/4	(2 4 4 / 8 2 8 / 8 4 8)	= −1/4
(2 4 4 / 8 3 5 / 8 3 7)	= −*i*/4	(2 4 4 / 8 3 7 / 8 4 8)	= +1/4
(2 5 5 / 5 2 5 / 5 5 2)	= +1	(2 5 5 / 5 3 8 / 5 8 3)	= +1/2
(2 5 5 / 5 4 7 / 5 7 4)	= +1/2	(2 5 5 / 6 3 8 / 6 8 3)	= −1/2
(2 5 5 / 6 4 7 / 6 7 4)	= −1/2	(2 5 5 / 7 2 7 / 7 5 4)	= +1/2
(2 5 5 / 7 3 7 / 7 8 4)	= $-i/2\sqrt{2}$	(2 5 5 / 7 3 8 / 7 8 3)	= $-1/2\sqrt{2}$
(2 5 5 / 8 2 8 / 8 5 3)	= +1/2	(2 5 5 / 8 3 7 / 8 8 4)	= $-1/2\sqrt{2}$
(2 5 5 / 8 4 7 / 8 7 4)	= $-1/2\sqrt{2}$	(2 7 7 / 7 2 7 / 7 7 2)	= +1/4
(2 7 7 / 7 2 7 / 7 7 3)	= +1/4	(2 7 7 / 7 3 7 / 7 8 3)	= +1/4
(2 7 7 / 7 3 8 / 7 8 4)	= +1/4	(2 7 7 / 7 4 5 / 7 5 4)	= +1/4
(2 7 7 / 7 4 5 / 7 6 4)	= −1/4	(2 7 7 / 7 4 5 / 7 8 4)	= +1/4
(2 7 7 / 8 2 8 / 8 7 3)	= +1/4	(2 7 7 / 8 2 5 / 8 7 4)	= +1/4
(2 7 7 / 8 3 5 / 8 7 4)	= −*i*/4	(2 7 7 / 8 3 5 / 8 8 4)	= −1/4
(2 7 7 / 8 3 7 / 8 7 3)	= +1/4	(2 7 7 / 8 4 8 / 8 8 4)	= −1/4
(2 8 8 / 8 2 8 / 8 8 2)	= +1/4	(2 8 8 / 8 2 8 / 8 8 4)	= +1/4
(2 8 8 / 8 3 5 / 8 5 3)	= +1/4	(2 8 8 / 8 3 5 / 8 6 3)	= −1/4
(2 8 8 / 8 3 5 / 8 7 3)	= +1/4	(2 8 8 / 8 3 7 / 8 7 4)	= +1/4
(2 8 8 / 8 4 7 / 8 8 4)	= +1/4	(3 3 4 / 3 5 8 / 4 8 7)	= +1/4
(3 3 4 / 3 7 7 / 4 7 5)	= −1/4	(3 3 4 / 3 7 8 / 4 8 8)	= +1/4
(3 3 4 / 3 8 5 / 4 5 7)	= +1/4	(3 3 4 / 3 8 5 / 4 6 7)	= +1/4
(3 3 4 / 3 8 7 / 4 7 8)	= +1/4	(3 3 4 / 4 5 7 / 4 8 8)	= −*i*/4
(3 3 4 / 4 7 5 / 4 8 7)	= +*i*/4	(3 3 4 / 4 7 8 / 4 7 8)	= −1/4

(3 3 4 / 5 7 4 / 8 7 3)	= +1/4	(3 4 4 / 5 8 3 / 8 5 3)	= +1/4
(3 3 4 / 5 8 3 / 8 6 3)	= +1/4	(3 3 4 / 5 8 3 / 8 7 4)	= −*i*/4
(3 3 4 / 7 7 3 / 7 7 3)	= +1/4	(3 3 4 / 7 7 3 / 8 8 4)	= +1/4
(3 4 4 / 4 5 7 / 4 7 6)	= +1/4	(3 4 4 / 4 7 8 / 4 8 7)	= +1/4
(3 3 4 / 7 8 3 / 8 7 3)	= +1/4	(3 4 4 / 4 5 7 / 4 7 5)	= +1/4
(3 3 4 / 4 5 7 / 4 7 6)	= +1/4	(3 4 4 / 4 7 8 / 4 8 7)	= +1/4
(3 4 4 / 4 8 8 / 4 8 8)	= +1/4	(3 4 4 / 5 3 8 / 8 4 8)	= +1/4
(3 4 4 / 5 4 7 / 8 3 5)	= +1/4	(3 4 4 / 5 4 7 / 8 3 6)	= −1/4
(3 4 4 / 7 3 7 / 7 4 8)	= +1/4	(3 4 4 / 7 3 8 / 8 4 7)	= +1/4
(3 4 4 / 7 4 5 / 8 3 7)	= +1/4	(3 5 8 / 5 3 8 / 8 8 4)	= +1/4
(3 5 8 / 5 4 7 / 8 7 3)	= +i/4	(3 5 8 / 5 4 7 / 8 7 4)	= +1/4
(3 5 8 / 5 7 4 / 8 4 7)	= +1/4	(3 5 8 / 5 7 4 / 8 4 8)	= +1/4
(3 5 8 / 5 8 3 / 8 3 5)	= +1/4	(3 5 8 / 5 8 3 / 8 3 6)	= +1/4
(3 5 8 / 5 8 3 / 8 3 7)	= +1/4	(3 5 8 / 6 3 8 / 8 8 4)	= −1/4
(3 5 8 / 6 4 7 / 8 7 3)	= +1/4	(3 5 8 / 6 4 7 / 8 7 4)	= −1/4
(3 5 8 / 6 7 4 / 8 4 7)	= +1/4	(3 5 8 / 6 7 4 / 8 4 8)	= −1/4
(3 5 8 / 6 8 3 / 8 3 7)	= −1/4	(3 5 8 / 7 3 7 / 7 8 3)	= −*i*/4
(3 5 8 / 7 3 8 / 7 8 4)	= +*i*/4	(3 5 8 / 7 4 5 / 7 7 3)	= +1/4
(3 5 8 / 7 4 6 / 7 7 3)	= −1/4	(3 5 8 / 7 4 5 / 8 7 3)	= +1/4
(3 5 8 / 7 4 8 / 8 7 4)	= +1/4	(3 5 8 / 7 7 3 / 8 4 7)	= −*i*/4
(3 5 8 / 7 8 4 / 8 3 7)	= +1/4	(3 7 7 / 7 3 7 / 7 7 3)	= +1/4
(3 7 7 / 7 3 8 / 7 8 3)	= +1/4	(3 7 7 / 7 4 5 / 7 5 4)	= +1/4
(3 7 7 / 7 4 5 / 7 6 4)	= +1/4	(3 7 7 / 7 4 8 / 7 8 4)	= +1/4
(3 7 7 / 7 3 8 / 8 7 4)	= +1/4	(3 7 7 / 7 4 5 / 8 8 4)	= −*i*/4
(3 7 8 / 7 3 8 / 8 8 4)	= +1/4	(3 7 8 / 7 5 4 / 8 4 7)	= +1/4
(3 7 8 / 7 8 3 / 8 3 7)	= +1/4	(3 7 8 / 7 8 4 / 8 4 8)	= +1/4
(4 5 7 / 5 7 4 / 7 4 5)	= +1/4	(4 5 7 / 5 7 4 / 7 4 6)	= +1/4
(4 5 7 / 5 7 4 / 7 4 8)	= +1/4	(4 5 7 / 6 7 4 / 7 4 8)	= −1/4
(4 5 7 / 8 4 8 / 8 7 4)	= +1/4	(4 7 8 / 7 8 4 / 8 4 7)	= +1/4
(4 7 8 / 7 4 8 / 8 8 4)	= +1/4	(4 8 8 / 8 4 8 / 8 8 4)	= +1/4
(3 5 8 / 7 4 6 / 8 7 3)	= +1/4		

Note:
$$(i\,j\,k\,/\,l\,m\,n\,/\,q\,s\,t)_{r_1r_2r_3\,s_1s_2s_3} = \begin{Bmatrix} \Gamma_i & \Gamma_j & \Gamma_k \\ \Gamma_l & \Gamma_m & \Gamma_n \\ \Gamma_g & \Gamma_s & \Gamma_t \end{Bmatrix}\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}$$
$$\qquad\qquad s_1 \quad s_2 \quad s_3$$

The group under consideration is kronecker multiplicity free and hence r_1, r_2, r_3 and s_1, s_2, s_3 are suppressed.

4. 3jm factors for $D_5 \supset C_5$

The branching rules for $D_5 \to C_5$, character table and multiplication table for C_5 double group are given in tables 5, 6 and 7 respectively.

The first step in calculating the $3jm$ factors of $D_5 \supset C_5$ is to fix the $2jm$ factors. Choosing

$$(\Gamma_1)_{\gamma_1\gamma_1} = (\Gamma_2)_{\gamma_1\gamma_1} = (\Gamma_3)_{\gamma_3\gamma_4} = (\Gamma_4)_{\gamma_2\gamma_5}$$
$$= (\Gamma_5)_{\gamma_6\gamma_6} = (\Gamma_7)_{\gamma_8\gamma_9} = (\Gamma_8)_{\gamma_7\gamma_{10}} = +1,$$

equation (31) of Butler and Wybourne (1976a) gives

$$(\Gamma_3)_{\gamma_4\gamma_3} = (\Gamma_4)_{\gamma_5\gamma_2} = +1, \text{ and } (\Gamma_6)_{\gamma_6\gamma_6} = (\Gamma_7)_{\gamma_9\gamma_8} = (\Gamma_8)_{\gamma_{10}\gamma_7} = -1,$$

where we have made use of the $2j$ symbols of the groups D_5 and C_5. For Abelian groups all irreps are orthogonal or quasi-orthogonal, thereby giving $\phi_{\gamma k} = +1$. The trivial $3jm$ factors follow immediately (equation (29) of Butler and Wybourne 1976a) from the equation

$$\begin{pmatrix} \lambda & \lambda^* & 1 \\ a\sigma & a'\sigma' & 1 \end{pmatrix} = \langle 1 \mid \lambda a\sigma; \lambda^* a'\sigma' \rangle = \mid \sigma \mid^{1/2} \mid \lambda \mid^{-1/2} (\lambda)_{a\sigma;\, a'\sigma'}.$$

Table 5. Branching rules for $D_5 \to C_5$

D_5	C_5
Γ_1	γ_1
Γ_2	γ_1
Γ_3	$\gamma_3+\gamma_4$
Γ_4	$\gamma_2+\gamma_5$
Γ_5	γ_6
Γ_6	γ_6
Γ_7	$\gamma_8+\gamma_9$
Γ_8	$\gamma_7+\gamma_{10}$

Table 6. Character table for C_5

	E	C_5	C_5^2	C_5^3	C_5^4	$\bar{E}$	$\bar{C}_5$	$\bar{C}_5^2$	$\bar{C}_5^3$	$\bar{C}_5^4$
γ_1	1	1	1	1	1	1	1	1	1	1
γ_2	1	w^2	w^4	$-w$	$-w^3$	1	w^2	w^4	$-w$	$-w^3$
γ_3	1	w^4	$-w^3$	w^2	$-w$	1	w^4	$-w^3$	w^2	$-w$
γ_4	1	$-w$	w^2	$-w^3$	w^4	1	$-w$	w^2	$-w^3$	w^4
γ_5	1	$-w^3$	$-w$	w^4	w^2	1	$-w^3$	$-w$	w^4	w^2
γ_6	1	-1	1	-1	1	-1	1	-1	1	-1
γ_7	1	w	w^2	w^3	w^4	-1	$-w$	$-w^2$	$-w^3$	$-w^4$
γ_8	1	w^3	$-w$	$-w^4$	w^2	-1	$-w^3$	w	w^4	$-w^2$
γ_9	1	$-w^2$	w^4	w	$-w^3$	-1	w^2	$-w^4$	$-w$	w^3
γ_{10}	1	$-w^4$	$-w^3$	$-w^2$	$-w$	-1	w^4	w^3	w^2	w

$w = \exp(i\pi/5)$

Table 7. Multiplication table for C_5

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}
γ_1	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}
γ_2		γ_3	γ_4	γ_5	γ_1	γ_9	γ_8	γ_6	γ_{10}	γ_7
γ_3			γ_5	γ_1	γ_2	γ_{10}	γ_6	γ_9	γ_7	γ_8
γ_4				γ_2	γ_3	γ_7	γ_9	γ_{10}	γ_8	γ_6
γ_5					γ_4	γ_8	γ_{10}	γ_7	γ_6	γ_9
γ_6						γ_1	γ_4	γ_5	γ_2	γ_3
γ_7							γ_2	γ_3	γ_5	γ_1
γ_8								γ_4	γ_1	γ_2
γ_9									γ_3	γ_4
γ_{10}										γ_5

Table 8. Non-trivial inequivalent $3jm$ factors for $D_5 \supset C_5$

(2 3 3 / 1 3 4)	=	$+i/\sqrt{2}$
(2 4 4 / 1 2 5)	=	$-i/\sqrt{2}$
(2 5 5 / 1 6 6)	=	$-i$
(2 7 7 / 1 8 9)	=	$+i/\sqrt{2}$
(2 8 8 / 1 7 10)	=	$+i/\sqrt{2}$
(3 3 4 / 3 3 2)	=	$+1/\sqrt{2}$
(3 4 4 / 4 4 2)	=	$+i/\sqrt{2}$
(3 5 8 / 3 6 7)	=	$-i/\sqrt{2}$
(3 6 8 / 3 6 7)	=	$-i/\sqrt{2}$
(3 7 7 / 3 8 8)	=	$+1/\sqrt{2}$
(3 7 8 / 3 9 10)	=	$+1/\sqrt{2}$
(4 5 7 / 2 6 8)	=	$+1/\sqrt{2}$
(4 6 7 / 2 6 8)	=	$+1/\sqrt{2}$
(4 7 8 / 2 9 7)	=	$+1/\sqrt{2}$
(4 8 8 / 2 10 10)	=	$+i/\sqrt{2}$

Note:

$$(ijk/lmn)^r_s = \begin{Bmatrix} \Gamma_i & \Gamma_j & \Gamma_k \\ \gamma_l & \gamma_m & \gamma_n \end{Bmatrix}^r_s$$

The magnitudes of the primitive $3jm$ factors are obtained using the orthogonality relations (equation (35) and (36) of Butler and Wybourne 1976a). Choosing the relative phases from the orthogonality relations and systematically increasing the power of the largest irrep, we obtain five independent and one dependent primitive $3jm$ factors of $D_5 \supset C_5$. The non-trivial non-primitive inequivalent $3jm$ factors for $D_5 \supset C_5$ are calculated recursively using the $3jm$ primitives and the $6j$ symbols of D_5 in equation (41) of Butler and Wybourne (1976a). The $6j$ symbols of C_5 may be taken to be $+1$. The complete set of non-trivial inequivalent $3jm$ factors for $D_5 \supset C_5$ are listed in table 8.

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