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Reduction and second quantisation of generalised electromagnetic field for zero mass system

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Abstract. Reduction and second quantisation of generalised electromagnetic fields in the presence of massless spin-½ particles carrying both electric and magnetic charges have been carried out in terms of Lomont-Moses realisation of irreducible representations of Poincare group and the expression for field Hamiltonian has been derived.

Keywords. Generalised field; second quantisation; four-vector; annihilation operators; electromagnetic field; zero mass system; creation operator.

1. Introduction

The possible development of the consistent theories (Rabl 1968; Schwinger 1966a; Rohrlich 1966; Gamblin 1967) of Dirac's monopole (Dirac 1931, 1948) led Schwinger (1969) to put forward a hypothesis concerning a magnetic model of matter in terms of hadrons and dyons so as to understand the nature of hypercharge and the possible time-reversal mechanism. Schwinger (1966a, b, c) formulated the quantum field theory of spin-½ magnetic charge which was extended by Zwanziger (1968, 1971) to particles carrying both electric and magnetic charges. In order to avoid the arbitrary (Peres 1968) string variables introduced by Zwanziger (1971), studies on reduction in purely relativistic group theoretical manner (Parkash and Rajput 1976), second quantisation (Rajput and Parkash 1979a) and interaction (Rajput and Parkash 1978) of generalised electromagnetic fields associated with a massive spin-1 particle carrying electric and magnetic charges by introducing two four-vector potentials have been undertaken.

Assuming the generalised charge of a massless spin-½ particle as a complex quantity with electric and magnetic charges as its real and imaginary parts, studies have been conducted on the reduction and second quantisation of generalised electromagnetic field in the presence of generalised charges in terms of the Lomont-Moses (1967) realisation of irreducible representations of Poincare group. It is shown that the longitudinal part of wave function transforming as generalised field is non-vanishing in general and in the absence of electric or magnetic charge sources the corresponding longitudinal field vanishes.

The second quantisation of the generalised fields has been done in purely relativistic group theoretical Lorentz covariant manner on replacing the amplitudes and their complex conjugates in the reduced expansions by annihilation and creation operators respectively. The commutation rules for the transverse and longitudinal field operators and vector potential operators have been derived and it has been shown that all commutation rules for transverse fields and potential operators are local ones. Choosing a suitable Lagrangian density and using the second quantised reduced expansions for the field operators, the expression for total Hamiltonian has been derived. It has also been shown that this Hamiltonian consists of a free electromagnetic field Hamiltonian, the Dirac field Hamiltonian and the interaction Hamiltonian.

2. Reduction of generalised fields in the presence of spin-½ particles

We consider here spin-½ massless particle carrying generalised charge q given by

$$q = e - ig, \tag{1}$$

where e and g are respectively the electric and magnetic charges. The generalised charge introduced by Parkash and Rajput (1974) as a complex quantity has the advantage over the generalized charge vector quantity (Zwanziger 1968), in that the charge quantisation condition is better explained and the coupling parameters can be derived more convincingly.

The wave function which transforms as generalised electromagnetic field in the presence of spin-½ massless particles carrying generalised charge q may be considered as consisting of a transverse part ϕ^T and a longitudinal part ϕ^L in the following manner:

$$\phi(\mathbf{x}, t) = \mathbf{E} - i\,\mathbf{H} = \phi^T(\mathbf{x}, t) + \phi^L(\mathbf{x}, t), \tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are generalised electromagnetic fields. Reduced expansions of ϕ^T in terms of irreducible representations of proper, orthochronous, inhomogeneous Lorentz group have the following form (Rajput 1970):

$$\phi^T(\mathbf{x}, t) = i(8\pi^2)^{-1/2} \sum_{\lambda=\pm 1} \int d\mathbf{p}\, [(1-\lambda)\, g(\mathbf{p}, \lambda)\, \sigma(\mathbf{p}, \lambda) \exp\{i(\mathbf{p}\cdot\mathbf{x} - pt)\}$$

$$- (1+\lambda) g^*(\mathbf{p}, \lambda)\, \sigma^*(\mathbf{p}, \lambda) \exp\{-i(\mathbf{p}\cdot\mathbf{x} - pt)\}], \tag{3}$$

where summation is taken over the helicity λ, the integral is taken over all the real positive values of momentum $\mathbf{p}$, $g(\mathbf{p}, \lambda)$ is the representation of wave functions of transverse photons and the vector $\sigma(\mathbf{p}, \lambda)$ is given (Moses 1968) as follows in terms of the components of $\mathbf{p}$:

$$\begin{aligned} \sigma(\mathbf{p}, \lambda) = \; & \{[p_1(p_1 + i\lambda p_2)]/[p(p+p_3)]\} - 1, \\ & \{[p_2(p_1 + i\lambda p_2)]/[p(p+p_3)]\} - i\lambda, \\ & (p_1 + i\lambda p_2)/p. \end{aligned} \tag{4}$$

Transverse part $\phi^T(\mathbf{x}, t)$ satisfies the ordinary Maxwell's equations for free electromagnetic fields.

In order to derive the reductions of the longitudinal part ϕ^L we define it in terms of the generalised scalar potential $V_0(\mathbf{x}, t)$ as

$$\phi^L(\mathbf{x}, t) = -\nabla V_0(\mathbf{x}, t) = -\nabla V_0(\mathbf{x}) \exp(\pm ipt). \tag{5}$$

This scalar potential may be written in terms of four-component Dirac field $\psi_{1/2}$ associated with massless spin-$\frac{1}{2}$ particles carrying generalised charge q as follows:

$$V_0(\mathbf{x}) = q \int D(\mathbf{x} - \mathbf{x}') \bar{\psi}_{1/2}(\mathbf{x}') \gamma_4 \psi_{1/2}(\mathbf{x}') \, d\mathbf{x}', \tag{6}$$

where γ_4 is the fourth Dirac matrix, the bar denotes adjoint and the function $D(\mathbf{x}-\mathbf{x}')$ is given by

$$1/[4\pi(\mathbf{x} - \mathbf{x}')],$$

$$= \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}}{p^2} \exp\{i\mathbf{p}(\mathbf{x} - \mathbf{x}'). \tag{7}$$

Using the reduced expansion for $\psi_{1/2}(\mathbf{x})$ as derived by Moses (1968) in terms of irreducible representations of Poincare group, the generalised scalar field becomes

$$V_0(\mathbf{x}, t) = \frac{q}{8\pi^3} \sum_{\lambda' = \pm\frac{1}{2}} \int \frac{d\mathbf{p}'}{p'^3} [f(\mathbf{p}', \lambda') \exp\{i(\mathbf{p}' \cdot \mathbf{x} - p' t)\}$$

$$+ k^*(\mathbf{p}', \lambda') \exp\{-i(\mathbf{p}' \cdot \mathbf{x} - p' t)\}], \tag{8}$$

where $\mathbf{p}'$ is the momentum vector of spin-$\frac{1}{2}$ massless particle and $f(\mathbf{p}', \lambda')$ and $k(\mathbf{p}', \lambda')$ are respectively the wave functions of particles with positive and negative energies and helicity λ'. Substituting this result into equation (5), we get the reduced expansion:

$$\phi^L(\mathbf{x}, t) = -\frac{iq}{8\pi^3} \sum_{\lambda' = \pm\frac{1}{2}} \int \frac{d\mathbf{p}'}{p'^3} \mathbf{p}' [f(\mathbf{p}', \lambda') \exp\{i(\mathbf{p}' \cdot \mathbf{x} - p' t)\}$$

$$- k^*(\mathbf{p}', \lambda') \exp\{-i(\mathbf{p}' \cdot \mathbf{x} - p' t)\}]. \tag{9}$$

The transverse field may also be written in the following form in terms of the generalised transverse vector potential $\mathbf{V}^T(\mathbf{x}, t)$:

$$\phi^T(\mathbf{x}, t) = -i\nabla \times \mathbf{V}^T(\mathbf{x}, t). \tag{10}$$

The generalised field $\phi(\mathbf{x}, t)$ may therefore be described by the generalized four-potential,

$$\{V_\mu\} = (\mathbf{V}^T, i V_0),$$

with $V_0 = A_0 - i B_0$

and $\mathbf{V}^T = \mathbf{A}^T - i\,\mathbf{B}^T.$ (11)

$\mathbf{A}^T$ and $\mathbf{B}^T$ are two vector potentials and A_0 and B_0 are the scalar potentials associated with electric and magnetic charges. The reduced expansion of $\mathbf{V}^T(\mathbf{x}, t)$ may be written as

$$(8\,\pi^2)^{-1/2} \sum_{\lambda=\pm 1} \int \frac{d\mathbf{p}}{p} [(1-\lambda)\, g(\mathbf{p}, \lambda)\, \sigma(\mathbf{p}, \lambda) \exp\{i(\mathbf{p}\cdot\mathbf{x} - p\,t)\}$$

$$+ (1+\lambda)\, g^*(\mathbf{p}, \lambda)\, \sigma^*(\mathbf{p}, \lambda) \exp\{-i(\mathbf{p}\cdot\mathbf{x} - p\,t)\}], \quad (12)$$

and the reduction of $V_0(\mathbf{x}, t)$ has already been obtained in equation (8).

The components of generalised current four-vector $\{J_\mu\}$ associated with spin-$\frac{1}{2}$ particles carrying generalised charge q may be written as

$$J_\mu(\mathbf{x}, t) = j_\mu(\mathbf{x}, t) - i\,k_\mu(\mathbf{x}, t) = J_\mu(\mathbf{x}) \exp(\pm\, i\,p\,t), \quad (13)$$

where $j_\mu(\mathbf{x}, t)$ and $k_\mu(\mathbf{x}, t)$ are the corresponding components of electric and magnetic current four-vectors and $J_\mu(\mathbf{x})$ is given by

$$J_\mu(\mathbf{x}) = q\, \bar{\psi}_{1/2}(\mathbf{x})\, \gamma_\mu\, \psi_{1/2}(\mathbf{x}). \quad (14)$$

Using the reduction of $\psi_{1/2}(\mathbf{x})$ in this equation and substituting it into equation (13) we get the following reduced expansions for the generalised charge and current source densities:

$$J_0(\mathbf{x}, t) = \frac{q}{8\pi^3} \sum_{\lambda=\pm\frac{1}{2}} \int \frac{d\mathbf{p}'}{p'} [f(\mathbf{p}', \lambda') \exp\{\, i(\mathbf{p}'\cdot\mathbf{x} - p'\,t)\}$$

$$+ k^*(\mathbf{p}', \lambda') \exp\{-\,i(\mathbf{p}'\cdot\mathbf{x} - p'\,t)\}] \quad (15)$$

and
$$\mathbf{J}(\mathbf{x}, t) = \frac{q}{8\pi^3} \sum_{\lambda'=\pm\frac{1}{2}} \int \frac{d\mathbf{p}'}{p'^2}\, \mathbf{p}' [f(\mathbf{p}', \lambda') \exp\{i(\mathbf{p}'\cdot\mathbf{x} - p'\,t)\}$$

$$+ k^*(\mathbf{p}', \lambda') \exp\{-\,i(\mathbf{p}'\cdot\mathbf{x} - p'\,t)\}], \quad (16)$$

from which we may verify the equation of continuity

$$\partial_\mu J_\mu = 0 \quad (17)$$

From equations (15), (16) and (13) the relationship of electric charge and current source densities with magnetic charge and current source densities in terms of the ratio of, electric and magnetic, fundamental charges may be obtained.

Maxwell's field equations for the generalized electric and magnetic fields in the presence of generalized four-current $\{J_\mu\}$ may be written in the following forms:

$$\nabla \, \phi = J_0 \tag{18}$$

and

$$\nabla \times \phi = -i \frac{\partial \phi}{\partial t} - i\mathbf{J}, \tag{19}$$

It may also be written in terms of the components of generalised four-vector $\{ V_\mu \}$

$$J_\mu = \partial_\mu \, \partial_\nu \, V_\mu - \partial_\nu \, \partial_\nu \, V_\mu. \tag{20}$$

3. Second quantisation

Replacing the functions $g(\mathbf{p}, \lambda)$ and $g^*(\mathbf{p}, \lambda)$ in the reduced expansions (3) and (12) by annihilation and creation operators of photons, we get the second quantised reduced expansions for the transverse operators of fields and potentials. Similarly the second quantised reduced expansions for longitudinal field and potential operators are obtained by replacing the wave functions ($f(\mathbf{p}', \lambda')$ and $k(\mathbf{p}', \lambda')$ in equations (9) and (8) by annihilation operators and their complex conjugates by creation operators of positive and negative energy spin-$\frac{1}{2}$ particle carrying the generalised charge.

Using these second quantised reduced expansions, the following commutation rules for the field operators may be derived by assuming the Bose statistics for annihilation and creation operators of photons and Fermi-Dirac statistics for the annihilation and creation operators of massless spin-$\frac{1}{2}$ particles:

$$[\hat{V}_i^T(\mathbf{x}), \hat{V}_j^{T*}(\mathbf{x}')]_- = 4\pi i \left[\left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right) \triangle(\mathbf{x} - \mathbf{x}') + \epsilon_{ijk} \nabla_k \operatorname{Im} D(\mathbf{x} - \mathbf{x}') \right], \tag{21}$$

$$[\hat{\psi}_i^T(\mathbf{x}), \hat{\psi}_j^{T*}(\mathbf{x}')]_- = 4\pi i \, [(\nabla_i \nabla_j - \nabla^2 \delta_{ij}) \triangle(\mathbf{x} - \mathbf{x}') + \epsilon_{ijk} \nabla_k \operatorname{Im} D(\mathbf{x} - \mathbf{x}')], \tag{22}$$

$$[\hat{\psi}_i^T(\mathbf{x}), \hat{V}_j^{T*}(\mathbf{x}')]_- = 4\pi i \, [(\nabla_i \nabla_j - \nabla^2 \delta_{ij}) \operatorname{Re} D(\mathbf{x} - \mathbf{x}') + \epsilon_{ijk} \nabla_k \triangle^1(\mathbf{x} - \mathbf{x}')], \tag{23}$$

where Re and Im denote real and imaginary parts and

$$\triangle(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}}{p} \sin \mathbf{p}(\mathbf{x} - \mathbf{x}'),$$

$$\triangle^1(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}}{p} \exp\{-i\mathbf{p}(\mathbf{x} - \mathbf{x}')\},$$

The other symbols have their usual meanings. The following commutation rules for the transverse electric and magnetic fields and potential operators may be derived by using the corresponding reduced expansions:

$$[\hat{E}_i^T(\mathbf{x}), \hat{H}_j^{T*}(\mathbf{x}')]_- = \pi\, \epsilon_{ijk}\, \nabla_k \operatorname{Im} \delta(\mathbf{x}-\mathbf{x}'), \tag{24}$$

$$[\hat{A}_i^T(\mathbf{x}), \hat{B}_j^{T*}(\mathbf{x}')]_- = [\hat{A}_i^T(\mathbf{x}), \hat{B}_j^{T*}(\mathbf{x}')]_- = -\pi\, i\, \epsilon_{ijk}\, \nabla_k \operatorname{Im} D(\mathbf{x}-\mathbf{x}'), \tag{25}$$

$$[\hat{E}_i^T(\mathbf{x}), \hat{B}_j^{T*}(\mathbf{x}')]_- = [\hat{E}_i^T(\mathbf{x}), \hat{B}_j^{T*}(\mathbf{x}')]_- = [\hat{H}_i^T(\mathbf{x}), \hat{A}_j^T(\mathbf{x}')]_-$$

$$= [\hat{H}_i^T(\mathbf{x}), \hat{A}_j^{T*}(\mathbf{x}')]_- = \pi\, \epsilon_{ijk}\, \nabla_k \operatorname{Re} D(\mathbf{x}-\mathbf{x}'); \tag{26}$$

all of which are the usual local commutation rules contrary to those derived by Zwanziger (1968) who introduced the extended fields in terms of the controversial string variables in order to make the commutation rules as local ones.

The Hamiltonian density $H(\mathbf{x})$ of generalised electromagnetic field may be considered as consisting of three terms as follows:

$$H(\mathbf{x}) = H_\gamma(\mathbf{x}) + H_M(\mathbf{x}) + H_I(\mathbf{x}), \tag{27}$$

where $H_\gamma(\mathbf{x})$ corresponds to free electromagnetic transverse field, $H_M(\mathbf{x})$ is the Hamiltonian density of free Dirac field and $H_I(\mathbf{x})$ denotes the interaction Hamiltonian density (Parkash and Rajput 1974). Thus we have (Zwanziger 1968, Rajput and Parkash 1979b);

$$H_\gamma(\mathbf{x}) = \tfrac{1}{2}\, \phi^{T*}(\mathbf{x}) \cdot \phi^T(\mathbf{x}), \tag{28}$$

$$H_M(\mathbf{x}) = \bar{\psi}_{1/2}(\mathbf{x})\,(-i\gamma \cdot \nabla)\,\psi_{1/2}(\mathbf{x}), \tag{29}$$

$$H_I(\mathbf{x}) = J_\mu(\mathbf{x})\, V_\mu^*(\mathbf{x}). \tag{30}$$

Substituting the second quantised reduced expansions of the field operators in these equations and carrying out the volume integrations, we get the following value for the total Hamiltonian:

$$\mathcal{H} = \mathcal{H}_\gamma + \mathcal{H}_M + \mathcal{H}_I, \tag{31}$$

where

$$\mathcal{H}_\gamma = 4\pi \int d\mathbf{p}\, [g^*(\mathbf{p},-1)\, g(\mathbf{p},-1) + g(\mathbf{p},+1)\, g^*(\mathbf{p},+1)], \tag{32}$$

$$\mathcal{H}_M = \sum_{\lambda'=\pm\frac{1}{2}} \int d\mathbf{p}'\, [f^*(\mathbf{p}',\lambda') f(\mathbf{p}',\lambda') - k(\mathbf{p}',\lambda')\, k^*(\mathbf{p}',\lambda')] \tag{33}$$

and

$$\mathcal{H}_I = \frac{q q^*}{8\pi^3} \sum_{\lambda'=\pm\frac{1}{2}} \int \frac{d\mathbf{p}'}{p'^4}\, [f(\mathbf{p}',\lambda') f^*(\mathbf{p}',\lambda') + k^*(\mathbf{p}',\lambda')\, k(\mathbf{p}',\lambda')]. \tag{34}$$

Here $\mathcal{H}_\gamma$ and $\mathcal{H}_M$ are the usual expressions for Hamiltonians of free electromagnetic field and free Dirac field respectively. It follows from equation (34) that the interaction Hamiltonian depends on the square of the modulus of generalised charge which vanishes only when $e = g = 0$.

4. Discussion

In equation (2) the total wave function has been considered to consist of transverse and longitudunal parts. The transverse wave function ϕ^T consists of $\mathbf{E}^T$ and $\mathbf{H}^T$ while $\mathbf{E}^L$ and $\mathbf{H}^L$ make ϕ^L. Equations (3) and (12) give the reductions of transverse wave functions and the generalised vector potentials in terms of photon wave functions and equations (9) and (8) give the reductions of longitudinal part of the wave function and generalised scalar potential associated with spin-$\frac{1}{2}$ massless particles. The introduction of generalised potential simplifies the use of two-vector potentials and the results have been found in a more compact form.

Equation (9) shows that the asymmetry between electric and magnetic fields has been eliminated here and both the fields are non-zero. Any longitudinal field vanishes only when the corresponding charge is zero. Equations (9) and (2) predict the proportionality of longitudinal parts E^L and H^L of the electromagnetic field with e and g respectively. The symmetric behaviour of four-current density can be observed from equations (15) and (16). From equations (15), (16) and (13) it can be observed that $j_\mu/k_\mu = e/g$. A particular case of this equation is the proportionality of j_μ with k_μ, when the electric and magnetic charges associated with different particles have the same ratio. Under this condition ($e_i/g_i = e_j/g_j$), if a suitable choice of charge axis is made, the generalised charge of a particle may be identified either as purely electric or purely magnetic. The latter may otherwise need the string variable but does not require it in our theory.

Commutation relations (24), (25) and (26) for transverse electric and magnetic field and potential operators are local ones contrary to those derived by Zwanziger (1968), who introduced the controversial string variables to maintain the commutation relations as local ones. These local commutation relations are consistent with those derived in Rajput and Parkash (1979a) for the generalised electromagnetic fields associated with spin-1 massive particles, where the local character could be achieved by making the mass as vanishing.

The interaction Hamiltonian is given by equation (34) and equations (32) and (33) give the free field Hamiltonians. The appearance of $q\,q^*$ in the interaction Hamiltonian indicates the coupling between generalised charges. The dependence of coupling $Q = q_i\, q_j^*$ on electric and magnetic coupling parameters has earlier been discussed (Parkash and Rajput 1974). The constancy condition ($e_i/g_i = e_j/g_j =$ constant) corresponds to zero value of the magnetic coupling parameter.

References

Dirac P A M 1931 *Proc. R. Soc. (London)* **A133** 60
Dirac P A M 1948 *Phys. Rev.* **74** 817
Gamblin R L 1967 *J. Math. Phys.* **10** 46

Lomont J S and Moses H E 1967 *J. Math. Phys.* **8** 837
Moses H E 1968 *J. Math. Phys.* **9** 16
Parkash Om and Rajput B S 1974 *Indian J. Phys.* **48** 152
Parkash Om and Rajput B S 1976 *Indian J. Phys.* **50** 929
Peres A 1968 *Phys. Rev.* **167** 1449
Rabl A 1968 *Phys. Rev.* **179** 1363
Rajput B S 1970 *Nuovo Cimento.* **A66** 517
Rajput B S and Parkash Om 1978 *Indian J. Pure Appl. Phys.* **16** 593
Rajput B S and Parkash Om 1979a *Indian J. Phys.* (In press)
Rajput B S and Parkash Om 1979b *Phys. Rev.* (In press)
Rohrlich F 1966 *Phys. Rev.* **150** 1104
Schwinger J 1966a *Phys. Rev.* **144** 1087
Schwinger J 1966b *Phys. Rev.* **151** 1048
Schwinger J 1966c *Phys. Rev.* **151** 1055
Schwinger J 1969 *Science* **165** 757
Zwanziger D 1968 *Phys. Rev.* **176** 1489
Zwanziger D 1971 *Phys. Rev.* **D3** 880