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# Wave functions from an off-energy-shell generalisation of the Gordon's method

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**Abstract.** An ansatz is introduced in the Gordon's method for nonlocal separable potentials to construct expressions for off-shell wave functions associated with the physical, regular and standing wave boundary conditions. This method has certain calculational advantages and is particularly suitable for dealing with potentials of higher rank. Results obtained for the Mongan case IV potential agree with those derived by the complicated techniques.

**Keywords.** Nuclear reactions; scattering theory; Gordon's method; off-shell scattering; wave functions.

## 1. Introduction

In a recent paper (Talukdar *et al* 1979a) (referred to as paper I hereafter) we derived an off-shell generalisation of the Gordon's method (Gordon 1970) and constructed analytical expressions for off-shell Jost solutions and Jost functions for non-local potentials of the separable class. Admittedly, in possession of appropriate Jost solutions one could solve off-shell wave functions for the radial part of the physical ($\psi^P$), regular ($\phi$) and principal value ($\psi^S$) by using relations given by Fuda and Whiting (1973), Warke and Srivastava (1977) and Talukdar *et al* (1977). Before such an attempt is made it will be judicious to look into the physics of the problem more closely.

The studies made by the above workers relate to scattering by local potentials and are based on inward extrapolation of appropriate asymptotic waves. One could ask: Are these results useful even when the potential is nonlocal? However, the following should be noted. A nonlocal potential couples the wave function at a point with its values at other positions; this in turn could obscure the intuitive meaning of the extrapolation procedure. It will be desirable to construct expressions for $\psi^P$, $\phi$ and $\psi^S$ for non-local potentials in terms of a more transparent formulation and clarify the conceptual problem. Gordon's method provides a convenient framework to deal with the situation.

## 2. Ansatz for wave functions

Using Gordon's method we obtained without much difficulty the generalised Jost solutions (Paper I) presumably because the Jost boundary condition is part of the associated differential equation through the inhomogeneity term $(k^2-q^2)\exp(iqr)$. But the situation is a little complicated with other boundary conditions when we need an additional ansatz. We elucidate this point by dealing with the more simple Yamaguchi (1954) potential and generalise our method to obtain $\psi^P$, $\phi$ and $\psi^S$ for a rank $n$ separable potential. For simplicity, all our results relate to the $s$-wave case with the subscript $l=0$ omitted; the generalisation to higher partial waves is trivial. We work in units in which $\hbar^2/2m$ is unity.

For a rank $n$ separable potential, the off-shell Jost solution $f(k, q, r)$ satisfies the inhomogeneous differential equation

$$\left(\frac{d^2}{dr^2}+k^2\right)f(k,q,r)-\sum_{i=1}^{n}\lambda_i v^{(i)}(r)\int_0^\infty v^{(i)}(s)f(k,q,s)ds=(k^2-q^2)\exp(iqr), \quad (1)$$

where $k, q, \lambda_i$ and $v^{(i)}$ have the same meaning as used in paper I. In contrast, $\psi^P$, $\phi$ and $\psi^S$ satisfy

$$\left(\frac{d^2}{dr^2}+k^2\right)R(k,q,r)-\sum_{i=1}^{n}\lambda_i v^{(i)}(r)\int_0^\infty v^{(i)}(s)R(k,q,s)ds=\frac{k^2-q^2}{q^\mu}\sin qr \quad (2)$$

with $\mu=0$ for $\psi^P$ and $\psi^S$ and $\mu=1$ for $\phi$.

The Yamaguchi potential represents a rank one potential with the form factor $v(r)=\exp(-\alpha r)$. For this potential equations (1) and (2) read

$$\left(\frac{d^2}{dr^2}+k^2\right)f(k,q,r)-\lambda\exp(-\alpha r)\int_0^\infty \exp(-\alpha s)f(k,q,s)\,ds$$
$$=(k^2-q^2)\exp(iqr), \quad (1a)$$

and

$$\left(\frac{d^2}{dr^2}+k^2\right)R(k,q,r)-\lambda\exp(-\alpha r)\int_0^\infty \exp(-\alpha s)R(k,q,s)\,ds$$
$$=\frac{k^2-q^2}{q^\mu}\sin qr. \quad (2a)$$

In Gordon's method one solves these equations by assuming that the integrals

$$\int_0^\infty \exp(-\alpha s)f(k,q,s)\,ds \quad\text{and}\quad \int_0^\infty \exp(-\alpha s)R(k,q,s)\,ds$$

are simply constants. The unknown constants which appear are determined by substituting the solutions back into respective differential equations and matching desired boundary conditions.

Let us first solve for the Jost solutions $f(k, q, r)$. From equation (1a) $f(k, q, r)$ is obtained in the form

$$f(k, q, r) = \exp(iqr) + \frac{C(k, q)}{\alpha^2 + k^2} \exp(-\alpha r), \tag{3}$$

where $$C(k, q) = \lambda \int_0^\infty \exp(-\alpha s) f(k, q, s) ds. \tag{4}$$

As stated above the constants $C(k, q)$ will be determined by combining (1a), (3) and (4). Meanwhile, the Jost boundary condition has already been incorporated in (3) since as $r \to \infty$, $f(k, q, r)$ goes to $\exp(iqr)$. Thus we should no longer worry about the boundary condition. The constant $C(k, q)$ is obtained as

$$C(k, q) = \frac{\lambda}{D(k)} \frac{1}{\alpha^2 + k^2} \frac{1}{\alpha - iq}, \tag{5}$$

with the Fredholm determinant

$$D(k) = 1 - \frac{\lambda}{2\alpha(\alpha^2 + k^2)}. \tag{6}$$

In a similar manner, the solution of (2a) is obtained in the form

$$R(k, q, r) = \frac{\sin qr}{q^\mu} + \frac{d(k, q)}{\alpha^2 + k^2} \exp - (\alpha r), \tag{7}$$

where $$d(k, q) = \lambda \int_0^\infty \exp(-\alpha s) R(k, q, s) ds. \tag{8}$$

It is important to note that equation (7) does not involve any of the boundary conditions like the physical, regular or standing wave. We attempt to include the boundary condition by the following ansatz. We write

$$R(k, q, r) = \frac{\sin qr}{q^\mu} + \frac{d(k, q)}{\alpha^2 + k^2} [A \cos kr + B \sin kr + \exp(-\alpha r)], \tag{9}$$

where $A$ and $B$ are constants to be determined by the actual boundary condition. Combining (7), (8) and (9), we get

$$d(k, q) = \frac{\lambda q^{1-\mu}}{(\alpha^2 + q^2) D^\pi(k)}, \tag{10}$$

where $$D^\pi(k) = 1 - \frac{\lambda}{2\alpha(\alpha^2 + k^2)} - \frac{\lambda(A\alpha + Bk)}{(\alpha^2 + k^2)^2}. \tag{11}$$

The superscript $\pi$ on $D$ relate to the boundary condition to be imposed. More specifically, $\pi$ will be substituted by $P$, $R$ and $S$ for physical, regular and standing wave boundary conditions respectively.

### 2.1. *Physical solution* (*outgoing wave boundary condition*)

In this case $\mu = 0$. The wave function

$$R(k, q, o) = \psi^P (k, q, o) = 0, \tag{12}$$

and

$$\psi^P(k, q, r) \xrightarrow[r\to\infty]{} \sin qr - \frac{d(k, q)}{\alpha^2 + k^2} \exp (ikr). \tag{13}$$

In view of (12) and (13) the coefficients $A$ and $B$ become

$$A = -1, \quad B = -i. \tag{14}$$

Thus the physical wave function is obtained as

$$\psi^P(k, q, r) = \sin qr + \frac{\lambda q}{D^P(k)\,(\alpha^2+k^2)\,(\alpha^2+q^2)} [\exp(-\alpha r) - \exp(ikr)], \tag{15}$$

where

$$D^P(k) = 1 - \frac{\lambda}{2\alpha(\alpha^2+k^2)} + \frac{\lambda(\alpha+ik)}{(\alpha^2+k^2)^2}. \tag{16}$$

### 2.2. *Regular solution*

For the regular solution $\mu=1$ and the boundary conditions are

$$R(k, q, o) = \phi(k, q, o) = 0, \tag{17a}$$

and

$$\left.\frac{d\phi(k, q, r)}{dr}\right|_{r=0} = 1. \tag{17b}$$

Using the boundary conditions (17), the coefficients $A$ and $B$ in (9) are found as

$$A = -1, \; B = \alpha/k. \tag{18}$$

Thus the regular wave function $\phi(k, q, r)$ is obtained in the form

$$\phi(k,q,r) = \frac{\sin qr}{q} + \frac{\lambda}{D^R(k)(\alpha^2+k^2)(\alpha^2+q^2)} \left[\frac{\alpha}{k}\sin kr - \cos kr + \exp(-\alpha r)\right], \tag{19}$$

Where $D^R(k)$ equals the Fredham determinant $D(k)$ associated with the Jost solution since Yamaguchi potential is symmetric in $r$ and $s$.

2.3. *Standing wave solution (principal value wave function)*

The boundary conditions for this solution are

$$R(k, q, o) = \psi^S(k, q, o) = 0, \tag{20a}$$

and $$\psi^S(k, q, r) \xrightarrow[r\to\infty]{} \sin qr - \frac{d(k, q)}{\alpha^2+k^2}\cos kr, \tag{20b}$$

with $\mu=0$. In this case

$$A = -1 \text{ and } B = 0. \tag{21}$$

Therefore $$\psi^S(k, q, r) = \sin qr + \frac{\lambda q}{D^S(k)(\alpha^2+k^2)(\alpha^2+q^2)}[\exp(-\alpha r)-\cos kr], \tag{22}$$

where $$D^S(k) = 1 - \frac{\lambda}{2\alpha(\alpha^2+k^2)} + \frac{\lambda\alpha}{(\alpha^2+k^2)^2}. \tag{23}$$

## 3. Rank n separable potential

Results to be presented for a rank $n$ separable potential are now in order. In paper I we discussed in some detail the method for obtaining the Jost solution. We therefore concentrate our attention to other solutions. Denoting the integral $\lambda_i \int_0^\infty v^{(i)}(s)\, R(k, q, s)\, ds$ by $d_i(k, q)$ the general solution of equation (2) is written as

$$R(k, q, r) = \frac{\sin qr}{q^\mu} + \sum_{i=1}^{n}[G_i(k, r)+A_i \cos kr + B_i \sin kr]\, d_i(k, q), \tag{24}$$

where $G_i(k, r)$ is the particular solution of

$$\left(\frac{d^2}{dr^2} + k^2\right) G_i(k, r) = v^{(i)}(r). \tag{25}$$

Clearly, the constants $A_i$ and $B_i$ in (24) will be determined by matching $R(k, q, r)$ to the desired boundary condition. The treatment will be similar to that used for the Yamaguchi potential. Substituting (24) in (2) we see that $d_i$'s are determined from the matrix equation

$$\sum_{j=1}^{n}\left[\delta_{ij} - \lambda_i \int_0^\infty ds\, v^{(i)}(s)\left\{G_j(k, s)+A_j \cos ks + B_j \sin ks\right\}\right] d_j(k, q)$$

$$= \frac{\lambda_i}{q^\mu}\int_0^\infty \sin qs\, v^{(i)}(s)\, ds \tag{26}$$

## 4. Mongan case IV potential

Specialising equations (24) to (26) to the rank two case, the results for $\psi^P$, $\phi$ and $\psi^S$ can be obtained rather straightforwardly for the Mongan potential (Mongan 1968, 1969)

$$V(r, r') = \lambda_1 \exp[-a_1(r+r')] + \lambda_2 \exp[-a_2(r+r')]. \tag{27}$$

We now obtain the following physical, regular and standing wave solutions in (28) to (30) respectively.

$$\psi^P(k, q, r) = \sin qr + \frac{1}{D^P(k)} [Q_1(k, q) \exp(-a_1 r) + Q_2(k, q) \exp(-a_2 r) - Q_3(k, q) \exp(ikr)]. \tag{28}$$

$$\phi(k, q, r) = \frac{\sin qr}{q} + \frac{1}{kD^R(k)} [R_1(k, q) \sin kr - R_2(k, q) \cos kr + R_3(k, q) \exp(-a_1 r) + R_4(k, q) \exp(-a_2 r)]. \tag{29}$$

$$\psi^S(k, q, r) = \sin qr - \frac{1}{D^S(k)} [W_3(k, q) \cos kr - W_1(k, q) \exp(-a_1 r) - W_2(k, q) \exp(-a_2 r)]. \tag{30}$$

In equations (28) to (30) $Q_i$, $R_i$ and $W_i$ are given by

$$Q_1(k, q) = \frac{\lambda_1 q}{(a_1^2+k^2)(a_1^2+q^2)} \left[1 + \frac{\lambda_2(a_2^2-k^2)}{2a_2(a_2^2+k^2)^2} + \frac{i\lambda_2 k}{(a_2^2+k^2)^2}\right] - \frac{\lambda_1\lambda_2 q}{(a_1^2+k^2)^2(a_2^2+k^2)(a_2^2+q^2)} \left[\frac{a_1 a_2 - k^2}{a_1+a_2} + ik\right], \tag{31a}$$

$$Q_2(k, q) = \frac{\lambda_2 q}{(a_2^2+k^2)(a_2^2+q^2)} \left[1 + \frac{\lambda_1(a_1^2-k^2)}{2a_1(a_1^2+k^2)^2} + \frac{i\lambda_1 k}{(a_1^2+k^2)^2}\right] - \frac{\lambda_1\lambda_2 q}{(a_1^2+k^2)(a_1^2+q^2)(a_2^2+k^2)} \left[\frac{a_1 a_2 - k^2}{a_1+a_2} + ik\right], \tag{31b}$$

and $$Q_3(k, q) = Q_1(k, q) + Q_2(k, q), \tag{31c}$$

$$R_1(k, q) = \frac{a_1}{k} R_3(k, q + \frac{a_2}{k} R_4(k, q), \tag{32a}$$

$$R_2(k, q) = R_3(k, q) + R_4(k, q), \tag{32b}$$

with $$R_3(k, q) = \frac{\lambda_1 k}{a_1^2+k^2}\left[\frac{1}{a_1^2+q^2} - \frac{\lambda_2}{2a_2(a_1^2+q^2)(a_2^2+k^2)} + \frac{\lambda_2}{(a_1+a_2)(a_1^2+k^2)(a_2^2+q^2)}\right], \tag{32c}$$

$$R_4(k, q) = \frac{\lambda_2 k}{a_2^2+k^2}\left[\frac{1}{a_2^2+q^2} - \frac{\lambda_1}{2a_1(a_1^2+k^2)(a_2^2+q^2)} + \frac{\lambda_1}{(a_1+a_2)(a_1^2+q^2)(a_2^2+k^2)}\right]. \tag{32d}$$

$$W_1(k, q) = \frac{\lambda_1 q}{(a_1^2+k^2)(a_1^2+q^2)}\left[1 + \frac{\lambda_2(a_2^2-k^2)}{2a_2(a_2^2+k^2)^2}\right] - \frac{\lambda_1\lambda_2 q\,(a_1a_2-k^2)}{(a_1+a_2)(a_1^2+k^2)^2(a_2^2+k^2)(a_2^2+q^2)}, \tag{33a}$$

$$W_2(k, q) = \frac{\lambda_2 q}{(a_2^2+k^2)(a_2^2+q^2)}\left[1 + \frac{\lambda_1(a_1^2-k^2)}{2a_1(a_1^2+k^2)^2}\right] - \frac{\lambda_1\lambda_2 q(a_1a_2-k^2)}{(a_1+a_2)(a_1^2+k^2)(a_1^2+q^2)(a_2^2+k^2)^2}, \tag{33b}$$

and $$W_3(k, q) = W_1(k, q) + W_2(k, q). \tag{33c}$$

The Fredholm determinant $D^R(k)$ associated with the regular solution is equal to the Fredholm determinant $D(k)$ for the Jost solution given in paper I. Expressions for $D^P(k)$ and $D^S(k)$ are given by

$$D^P(k) = D(k) + R(k) + iI(k), \tag{34a}$$

with $$R(k) = \frac{\lambda_1 a_1}{(a_1^2+k^2)^2} + \frac{\lambda_2 a_2}{(a_2^2+k^2)^2} - \frac{\lambda_1\lambda_2 k^2(a_1-a_2)^2}{2a_1a_2(a_1^2+k^2)^2(a_2^2+k^2)^2}, \tag{34b}$$

and $$I(k) = \frac{\lambda_1 k}{(a_1^2+k^2)^2} + \frac{\lambda_2 k}{(a_2^2+k^2)^2} + \frac{\lambda_1\lambda_2 k(a_1-a_2)^2(a_1a_2-k^2)}{2a_1a_2(a_1+a_2)(a_1^2+k^2)^2(a_2^2+k^2)^2}. \tag{34c}$$

$$D^S(k) = D(k) + R(k). \tag{35}$$

Based on Gordon's method we have presented in this paper a straightforward approach to calculate wave functions which occur in the theory of $T$ and $K$ matrices. These wave functions could also be obtained by using standard Green's function techniques (Talukdar *et al* 1979b). In this case there would appear Volterra integrals which tend to complicate the calculation by at least an order of magnitude. Because of its simplicity, the method presented here will be particularly suitable to deal with separable potential of rank higher than two, which are often used to describe the nucleon-nucleon potential more realistically.

In the theory of local potentials $\psi^P$, $\phi$ and $\psi^S$ are expressed in terms of Jost solutions as

$$\psi^P(k, q, r) = \frac{1}{2i}[f(k, q, r) - f(k, -q, r)] - \frac{1}{2i}\frac{f(k, q) - f(k, -q)}{f(k)} f(k, r), \tag{36}$$

$$\phi(k, q, r) = \frac{1}{2iq}\left[\{f(k, q, r) - f(k, -q, r)\} + \left\{f(-k) - 1 + \frac{f(k, -q) - f(-k)}{f(k)}\right\} f(k, r) - \left\{f(k) - 1 + \frac{f(k, q) - f(k)}{f(-k)}\right\} f(-k, r)\right], \tag{37}$$

and

$$\psi^S(k, q, r) = \frac{1}{2i}[f(k, q, r) - f(k, -q, r)] - \frac{1}{2i}\frac{f(k, q) - f(k, -q)}{f(k) + f(-k)}[f(k, r) + f(-k, r)]. \tag{38}$$

In these equations, functions appearing in the right hand side have been obtained in paper I for the Mongan Case IV potential. Using the appropriate Jost solutions and Jost functions in equations (36) to (38) we find that the results agree with those in (28) to (30). Thus we venture to suggest that, despite nonlocal effects tend to obscure the intuitive meaning of the extrapolation procedure, the results of Fuda and Whiting (1973), Warke and Srivastava (1977) and of Talukdar *et al* (1977) are valid in general.

## 5. Conclusion

Based on the original conjecture of Gordon (1970) and its subsequent generalisation by Talukdar *et al* (1979a), we have derived a straightforward method for constructing off-shell solutions useful in three-particle scattering. For example, relations for the wave functions can be used to derive expressions for off-shell $T$ and $K$ matrices (Talukdar *et al* 1977) in terms of appropriate Jost functions, Fredholm determinants and transforms of the form factors of the potential. The results for most of the separable nonlocal potential cited in the literature will be characterised by elementary transcendental functions.

There has been a resurgence of interest in the studies of scattering by nonlocal potentials because of the following. An awkward analytical constraint associated with the short-range local potential is that the phase shift $\delta_l(k)$ is a continuous function of momentum $k$. There exist situations where relaxation of the constraint is necessary in order to accommodate experimental results. For example, in the vicinity of an isolated compound resonance, the phase shift for the resonant partial wave develops a jump of magnitude $\pi$. The change in phase becomes discontinuous as the width of the resonance approaches zero. Recently, it has been emphasized by Mulligan *et al* (1976) that this constraint can be relaxed in going from a local to a nonlocal potential. The nonlocal potential is thus effective in treating a much wider variety of phenomena than that encompassed with a short-range local potential. One of the tasks in developing the description of physical processes characteristic of a nonlocal potential must be the analysis of off-shell effects due to such a potential.

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