Pramāna, Vol. 13, No 5, November 1979, pp. 513-523, © printd in India

Mixing of meson isosinglets in SU(5) and an extension to SU(N)

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MS received 14 June 1979; revised 22 August 1979

Abstract. The mixing angles of meson isosinglets belonging to the 24-dimensional and singlet representations of SU(5) are calculated under specific assumptions in the non-relativistic quark model. The procedure to extend the scheme to SU(N) has been outlined. The results have been compared with other earlier estimates.

Keywords. SU(5); SU(N); mixing angles; quark model; meson isosinglets.

1. Introduction

The discovery of Ψ(3·1 GeV) (Aubert *et al* 1974) and γ (9·4 GeV) (Herb *et al* 1977) introduced new quark c (charm) and b (bottom) to the previously known Gell-Mann Zweig quarks u, d, s. Experimental evidence of OZI rule (Okubo 1963; Zweig 1964; Iizuka 1966) violating decays of the isosinglet mesons (both vector and pseudoscalar) shows that they are not *ideally* mixed. In an earlier paper (Mishra and Sastry 1979) we have calculated the SU(4) mixing angles and computed the quark contents in a non-relativistic quark model assuming the SU(4) invariance in the quark-antiquark annihilation channel. We extend the model to SU(5) with similar assumptions and the mixing angles are calculated. With the knowledge of mixing angles the quark contents are computed. We use linear masses for vector mesons (Maki *et al* 1976; Boal 1978).

With the above assumptions of invariance of quark-antiquark amplitudes in annihilation channels, which worked well in SU(3), SU(4) and SU(5) the mixing scheme has been extended to SU(N).

2. Model

The quark antiquark interaction can be parametrised as (Sastry and Misra 1970; Mishra and Sastry 1979)

$$V\,|u_+\bar{d}_-\rangle = V_{dd}^{NN}\,|u_+\bar{d}_-\rangle + V_{ed}^{NN}\,[\,|u_+\bar{d}_-\rangle - |u_-\bar{d}_+\rangle],$$

$$V\,|u_+\bar{s}_-\rangle = V_{dd}^{Ns}\,|u_+\bar{s}_-\rangle + V_{ed}^{Ns}\,[\,|u_+\bar{s}_-\rangle - |u_-\bar{s}_+\rangle],$$

$$V\,|u_+\bar{c}_-\rangle = V_{dd}^{Nc}\,|u_+\bar{c}_-\rangle + V_{ed}^{Nc}\,[\,|u_+\bar{c}_-\rangle - |u_-\bar{c}_+\rangle],$$

$$V\,|u_+\bar{b}_-\rangle = V^{Nb}_{dd}\,|u_+\bar{b}_-\rangle + V^{Nb}_{ed}\,[\,|u_+\bar{b}_-\rangle - |u_-\bar{b}_+\rangle],$$

$$V\,|u_+\bar{u}_-\rangle = V^{NN}_{dd}\,|u_+\bar{u}_-\rangle + V^{NN}_{ed}\,[\,|u_+\bar{u}_-\rangle - |u_-\bar{u}_+\rangle]$$

$$+ V^{NN}_{de}[|u_+\bar{u}_-\rangle + |d_+\bar{d}_-\rangle] + V^{Ns}_{de}|s_+\bar{s}_-\rangle + V^{Nc}_{de}|c_+\bar{c}_-\rangle + V^{Nb}_{de}|b_+\bar{b}_-\rangle$$

$$+ V^{NN}_{ee}[|u_+\bar{u}_-\rangle + |d_+\bar{d}_-\rangle - |u_-\bar{u}_+\rangle - |d_-\bar{d}_+\rangle] + V^{Ns}_{ee}[|s_+\bar{s}_-\rangle - |s_-\bar{s}_+\rangle]$$

$$V + ^{Nc}_{ee}[|c_+\bar{c}_-\rangle - |c_-\bar{c}_+\rangle] + V^{Nb}_{ee}[|b_+\bar{b}_-\rangle - |b_-\bar{b}_+\rangle]. \tag{1}$$

The $+$ and $-$ subscripts to the quark symbols indicate 'spin-up' and 'spin-down'. The superscripts to the V's correspond to the quark symbols (N for u and d quarks, s for strange quarks etc.). V_{ed}, V_{de}, and V_{ee} correspond to the projection of the initial $|q_+\bar{q}_-\rangle$ state to the spin-singlet, isospin-singlet and spin-isospin singlet states respectively.

From the above interaction potential the following SU(5) mass relations follow:

$$L^* + K^* = M^* + \rho,$$
$$L^* + D^* = N^* + \rho. \tag{2}$$

The above relations could as well be obtained by simply counting the quarks with the same quantum numbers on both sides. Similar relations also hold good for pseudoscalar measons.

$^3S_1$ quark-antiquark states corresponding to the λ_8, λ_{15}, λ_{24} and λ_0 of the 24-dimensional and singlet representations of SU(5) denoted by $|\omega_8\rangle$, $|\omega_{15}\rangle$, $|\omega_{24}\rangle$ and $|\omega_0\rangle$ are not eigenstates of the quark-antiquark interaction potential, we have chosen. The physical states $|\omega\rangle$, $|\phi\rangle$, $|\psi\rangle$ and $|\gamma\rangle$ are mutually orthonormal linear combinations of $|\omega_8\rangle$, $|\omega_{15}\rangle$, $|\omega_{24}\rangle$, $|\omega_0\rangle$. Extending the assumption made in our earlier paper (Mishra and Sastry 1979) i.e. assuming the SU(5) invariance of quark-antiquark amplitudes in annihilation channel, we get

$$V\begin{pmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_{24}\rangle\\|\omega_0\rangle\end{pmatrix} = \begin{pmatrix}\omega_8 & A & B & C\\ A & \omega_{15} & D & E\\ B & D & \omega_{24} & F\\ C & E & F & \omega_0\end{pmatrix}\begin{pmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_{24}\rangle\\|\omega_0\rangle\end{pmatrix}, \tag{3}$$

where $A = \dfrac{4}{6\sqrt{2}}(\rho - K^*)$, $D = \dfrac{1}{2\sqrt{15}}(2\rho + K^* - 3D^*)$,

$B = \dfrac{2}{\sqrt{30}}(\rho - K^*)$, $E = \dfrac{1}{\sqrt{15}}(2\rho + K^* - 3D^*)$,

$$C = \frac{4}{\sqrt{30}}(\rho - K^*), \quad F = \frac{1}{5}(2\rho + K^* + D^* - 4L^*),$$

$$\omega_8 = \frac{4}{3}K^* - \frac{1}{3}\rho,$$

$$\omega_{15} = \frac{3}{2}D^* + \frac{K^*}{6} - \frac{2}{3}\rho,$$

$$\omega_{24} = \frac{1}{10}(16L^* + K^* + D^* - 8\rho),$$

and
$$\omega_0 = \frac{1}{5}(2K^* + 2D + 2L^* - \rho) + 5V_{de}^{NN}.$$

Diagonalisation of the mass matrix in equation (3) gives the eigenvalues i.e. the masses of the physical states. The characteristic equation obtained from the mass matrix and the eigenvalues lead to the following sum rules

$$\omega + \phi + \psi + \gamma = \omega_8 + \omega_{15} + \omega_{24} + \omega_0, \tag{4}$$

$$\omega\phi + \phi\psi + \omega\psi + \gamma(\omega + \phi + \psi) = \omega_8\omega_{15} + \omega_{15}\omega_{24} + \omega_8\omega_{24} + \omega_0(\omega_8 + \omega_{15} + \omega_{24}) - A^2 - B^2 - C^2 - D^2 - E^2 - F^2, \tag{5}$$

$$\omega\phi\psi + \gamma(\omega\phi + \phi\psi + \omega\psi) = \omega_8\omega_{15}\omega_{24} + \omega_0(\omega_8\omega_{15} + \omega_{15}\omega_{24} + \omega_8\,\omega_{24}) - \omega_0(A^2 + B^2 + D^2) - \omega_8(D^2 + E^2 + F^2) - \omega_{15}(B^2 + C^2 + F^2) - \omega_{24}(A^2 + C^2 + E^2) + 2[ABC + ACE + BCF + DEF], \tag{6}$$

$$\omega\phi\psi\gamma = \omega_0\omega_8\omega_{15}\omega_{24} + 2DEF\omega_8 + 2ABD\omega_0 + 2ACE\omega_{24} + 2BCF\omega_{15} + A^2F^2 + B^2E^2 + C^2D^2 - 2ACDF - 2ABEF - 2BCDE - F^2\omega_8\omega_{15} - D^2\omega_8\omega_0 - E^2\omega_8\omega_{24} - A^2\omega_0\omega_{24} - B^2\omega_0\omega_{15} - C^2\omega_{15}\omega_{24}. \tag{7}$$

ω_0, in the mass matrix, which contained an unknown constant can be evaluated using the first sum rule, i.e. equation (4)

$$\omega_0 = \omega + \phi + \psi + \gamma - \omega_8 - \omega_{15} - \omega_{24}. \tag{8}$$

Equation (5) is nothing but an extension to SU(5) of the mass formula originally derived by Schwinger (1964) for SU(3) and could be rewritten with the help of equation (8) as (Mishra and Sastry 1979)

$$(\omega-\omega_8)(\phi+\psi+\gamma-\omega_8-\omega_{15}-\omega_{24})+(\phi-\omega_{15})(\psi+\gamma-\omega_{15}-\omega_{24})$$

$$+(\psi-\omega_{24})(\gamma-\omega_{24}) = -\frac{8}{9}(\rho-K^*)^2 - \frac{1}{12}(2\rho+K^*-3D^*)^2$$

$$-\frac{1}{25}(2\rho+K^*+D^*-4L^*)^2. \tag{9}$$

With $\gamma=9{\cdot}4$ GeV and ω_0 as evaluated from equation (8), equation (9) can be solved for L^* which turns out to be 5·08 GeV. Some of the other theoretical estimates of L^* are 5·290 GeV (Ono 1978) and 5·340 GeV (Boal 1978). Equations (6) and (7) are satisfied to within 5% with this value of L^* (5·08 GeV).

3. Mixing angles

Let us represent the physical states $|\omega\rangle, |\phi\rangle, |\psi\rangle, |\gamma\rangle$ as

$$\begin{bmatrix} |\omega\rangle \\ |\phi\rangle \\ |\psi\rangle \\ |\gamma\rangle \end{bmatrix} = X(x)Y(y)Z(z)X'(x')Y'(y')Z'(z') \begin{bmatrix} |\omega_8\rangle \\ |\omega_{15}\rangle \\ |\omega_{24}\rangle \\ |\omega_0\rangle \end{bmatrix} \tag{10}$$

with
$$X(x) = \begin{bmatrix} \cos x & -\sin x & 0 & 0 \\ \sin x & \cos x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad X'(x') = \begin{bmatrix} \cos x' & 0 & 0 & -\sin x' \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \sin x' & 0 & 0 & \cos x' \end{bmatrix}$$

$$Y(y) = \begin{bmatrix} \cos y & 0 & -\sin y & 0 \\ 0 & 1 & 0 & 0 \\ \sin y & 0 & \cos y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Y'(y') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos y' & 0 & -\sin y' \\ 0 & 0 & 1 & 0 \\ 0 & \sin y' & 0 & \cos y' \end{bmatrix}$$

$$Z(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos z & -\sin z & 0 \\ 0 & \sin z & \cos z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z'(z') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos z' & -\sin z' \\ 0 & 0 & \sin z' & \cos z' \end{bmatrix}$$

where x', y', z' are the mixing angles of γ with ω, ϕ, and ψ respectively; y, z are the mixing angles of ψ with ω and ϕ; and x is the mixing angle of $\omega-\phi$ system.
From equation (10)

$$|\gamma\rangle = \sin x'|\omega_8\rangle+\cos x' \sin y'|\omega_{15}\rangle+\cos x' \cos y' \sin z'|\omega_{24}\rangle$$

$$+\cos x' \cos y' \cos z'|\omega_0\rangle. \tag{11}$$

The mass of γ comes out to be

$$\begin{aligned}\gamma=&\omega_8 \sin^2 x'+\omega_{15}\cos^2 x' \sin^2 y'+\omega_{24}\cos^2 x' \cos^2 y' \sin^2 z'\\&+\omega_0 \cos^2 x' \cos^2 y' \cos^2 z'+2A \sin x' \cos x' \sin y'\\&+2B \sin x' \cos x' \cos y' \sin z'+2C \sin x' \cos x' \cos y' \cos z'\\&+2D \cos^2 x' \sin y' \cos y' \sin z'+2E \cos^2 x' \sin y' \cos y' \sin z'\\&+2F \cos^2 x' \cos^2 y' \sin z' \cos z'.\end{aligned} \tag{12}$$

Writing this as a quadratic equation in $\tan x'$ we have

$$\begin{aligned}&\tan^2 x' (\omega_8-\gamma)+2 \tan x' (A \sin y'+B \cos y' \sin z'+C \cos y' \cos z')\\&+2D \sin y' \cos y' \sin z'+2E \sin y' \cos y' \cos z'\\&+2F\cos^2 y' \sin z' \cos z'+\omega_{15} \sin^2 y'+\omega_{24} \cos^2 y' \sin^2 z'\\&+\omega_0 \cos^2 y' \cos^2 z'=0\end{aligned} \tag{13}$$

For real value of x', we should have

$$\begin{aligned}&\tan^2 y' [A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)]+2 \tan y' [\sin z' \{AB-D(\omega_8-\gamma)\}\\&+\cos z' \{AC-E(\omega_8-\gamma)\}]+\sin^2 z' \{B^2-(\omega_8-\gamma)(\omega_{24}-\gamma)\}\\&+2 \sin z' \cos z' \{BC-F(\omega_8-\gamma)\}\\&+\cos^2 z' \{C^2-(\omega_8-\gamma)(\omega_0-\gamma)\}\geqslant 0\end{aligned} \tag{14}$$

Let the left hand side of expression (14) be equal to $X\geqslant 0$. Therefore for real values of y' we should have

$$\begin{aligned}&[\sin z' \{AB-D(\omega_8-\gamma)\}+\cos z' \{AC-E(\omega_8-\gamma)\}]^2\\&-[A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)] [\sin^2 z' \{B^2-(\omega_8-\gamma)(\omega_{24}-\gamma)\}\\&+2 \sin z' \cos z' \{BC-F(\omega_8-\gamma)\}\\&+\cos^2 z' \{C^2-(\omega_8-\gamma)(\omega_0-\gamma)\}-X]\geqslant 0\end{aligned} \tag{15}$$

or

$$\begin{aligned}&\tan^2 z' [\{AB-D(\omega_8-\gamma)\}^2-\{A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)\}\\&\{B^2-(\omega_8-\gamma)(\omega_{24}-\gamma)\}]+2 \tan z' [\{AB-D(\omega_8-\gamma)\}\\&\{(AC-E(\omega_8-\gamma)\} - \{A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)\} \{BC-F(\omega_8-\gamma)\}]\end{aligned}$$

$$+[\{AC-E(\omega_8-\gamma)\}^2-\{A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)\}\{C^2-(\omega_8-\gamma)(\omega_0-\gamma)\}]$$

$$\geqslant X(1+\tan^2 z'). \tag{16}$$

But the left hand side of (16) could be shown to be of the form $-(\alpha \tan z'+\beta)^2$ and therefore the inequality in (16) could not be satisfied unless both sides are separately equal to zero. Therefore we get

$$\tan z' = -\frac{\{AB-D(\omega_8-\gamma)\}\{AC-E(\omega_8-\gamma)\}-\{A^2-(\omega_8-\gamma)\}\{\omega_{15}-\gamma)\}\{BC-F(\omega_8-\gamma\}}{\{AB-D(\omega_8-\gamma)\}^2-\{A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)\}\{B^2(\omega_8-\gamma)(\omega_{24}-\gamma)\}} \tag{17}$$

$$\tan y' = -\frac{\sin z'\{AB-D(\omega_8-\gamma\}+\cos z'\{AC-E(\omega_8-\gamma)\}}{A^2-(\omega_8-\gamma)(\omega_{15}-\gamma)}, \tag{18}$$

$$\tan x' = -\frac{A\sin y'+B\cos y'\sin z'+C\cos y'\cos z'}{(\omega_8-\gamma)}. \tag{19}$$

To obtain x, y, z we construct the states $|\Omega_8\rangle$, $|\Omega_{15}\rangle$ and $|\Omega_0\rangle$ as

$$\begin{bmatrix}|\Omega_8\rangle\\|\Omega_{15}\rangle\\|\Omega_0\rangle\\|\gamma\rangle\end{bmatrix} = X'(x')Y'(y')Z'(z')\begin{bmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_{24}\rangle\\|\omega_0\rangle\end{bmatrix}, \tag{20}$$

Now V

$$\begin{bmatrix}|\Omega_8\rangle\\|\Omega_{15}\rangle\\|\Omega_0\rangle\end{bmatrix} = \begin{bmatrix}\Omega_8 & P & Q\\P & \Omega_{15} & R\\Q & R & \Omega_0\end{bmatrix}\begin{bmatrix}|\Omega_8\rangle\\|\Omega_{15}\rangle\\|\Omega_0\rangle\end{bmatrix}, \tag{21}$$

where $\Omega_8 = \langle\Omega_8|V|\Omega_8\rangle = \omega_8\cos^2 x'+(\omega_{15}\sin^2 y'+\omega_{24}\cos^2 y'\sin^2 z'$

$$+\omega_0\cos^2 y'\cos^2 z')\sin^2 x'-2\sin x'\cos x'(A\sin y'$$

$$+B\cos y'\sin z'+C\cos y'\cos z')+2\sin^2 x'\sin y'\cos y'$$

$$(D\sin z'+E\cos z')+2F\sin^2 x'\cos^2 y'\sin z'\cos z',$$

$$\Omega_{15} = \langle\Omega_{15}|V|\Omega_{15}\rangle = \omega_{15}\cos^2 y'+\sin^2 y'(\omega_{24}\sin^2 z'+\omega_0\cos^2 z')$$

$$-2\sin y'\cos y'(D\sin z'+E\cos z')+2F\sin^2 y'\sin z'\cos z',$$

$$\Omega_0 = \langle \Omega_0 \mid V \mid \Omega_0 \rangle = \omega_{24} \cos^2 z' + \omega_0 \sin^2 z' - 2F \sin z' \cos z'$$

$$\begin{aligned} P = \langle \Omega_8 \mid V \mid \Omega_{15} \rangle &= \sin x' \sin y' \cos y' (-\omega_{15} + \omega_{24} \sin^2 z' + \omega_0 \cos^2 z') \\ &+ \cos x' \{A \cos y' - \sin y' (B \sin z' + C \cos z')\} - \sin x' \cos 2y' \\ &\times (D \sin z' + E \cos z') + 2F \sin x' \sin y' \cos y' \sin z' \cos z' \end{aligned}$$

$$\begin{aligned} Q = \langle \Omega_8 \mid V \mid \Omega_0 \rangle &= (\omega_0 - \omega_{24}) \sin x' \cos y' \sin z' \cos z' \\ &+ \cos x' (B \cos z' - C \sin z') - \sin x' \sin y' (D \cos z' - E \sin z') \\ &+ F \sin x' \cos y' \cos 2z', \end{aligned}$$

$$\begin{aligned} R = \langle \Omega_{15} \mid V \mid \Omega_0 \rangle &= (\omega_0 - \omega_{24}) \sin y' \sin z' \cos z' \\ &+ \cos y' (D \cos z' - E \sin z') - F \sin y' \cos 2z'. \end{aligned}$$

Now the problem of mixing of ω, ϕ, ψ reduces to the SU(4) problem (Mishra and Sastry 1979) and hence

$$\tan z = -\frac{PQ - R(\Omega_8 - \psi)}{P^2 - (\Omega_8 - \psi)(\Omega_{15} - \psi)}, \tag{22}$$

$$\tan y = -\frac{P \sin z + Q \cos z}{\Omega_8 - \psi}, \tag{23}$$

and tan x satisfies the following quadratic equation

$$\tan^2 x (\omega - a) + 2b \tan x + (a - \phi) = 0, \tag{24}$$

where $a = \Omega_{15} \cos^2 z + \Omega_0 \sin^2 z - 2R \sin z \cos z,$

$$\begin{aligned} b &= (\Omega_0 - \Omega_{15}) \sin y \sin z \cos z + P \cos y \cos z \\ &- Q \cos y \sin z - R \sin y \cos 2z, \end{aligned}$$

Numerical values of the mixing angles have been calculated from the equations (17), (18), (19), (22), (23) and (24) using the following input masses.

ρ	= 0·77 GeV,	ϕ	= 1·020 GeV,
ω	0·783 GeV,	ψ	= 3·1 GeV,
K^*	= 0·892 GeV,	γ	= 9·4 GeV,
D^*	= 2·005 GeV,	L^*	= 5·08 GeV.

The angles are

$$x' = -0{\cdot}0016^\circ;\ y' = -0{\cdot}0234^\circ;\ z' = -63{\cdot}282^\circ;\ z = -66{\cdot}167^\circ;$$

$$y = +0{\cdot}326^\circ;\ x = -45{\cdot}87^\circ,\ -83{\cdot}97^\circ. \tag{25}$$

$x = -83{\cdot}97^\circ$ leads to a large departure from ideal mixing for the $\omega - \phi$ system, which is experimentally untenable and hence discarded. It is interesting to note that in the SU(4) limit ($\gamma \to \infty$; $L^* \to \infty$ such that $\gamma/L^* = 2$) we get $\tan z' = -2$ ($z' = -63{\cdot}43^\circ$) $x' = 0$ and $y' = 0$ and γ becomes a pure $b\bar{b}$ state; x, y, z coincide with the corresponding angles of SU(4).

The exact quark content of the physical states is given in table 1. Numerical values of the quark contents of the meson isosinglets as given in table 1, depend upon the value of L^* (5·08 GeV according to our estimation) which is yet to be confirmed. Further, variation of the quark contents with a small variation of input masses appears to be very small. As for example, taking $\gamma = 9{\cdot}46$ GeV instead of $\gamma = 9{\cdot}4$ GeV and the corresponding value of L^* as determined from equation (9) (5·111 GeV) it has been seen that changes in the values of the mixing angles x', y', z' are very little. Hence the quark contents of the meson isosinglets, which depend upon the mixing angles are not very sensitive to the small changes in the input masses.

Although the same algebraic formulae hold good for the mixing of the pseudoscalar isosinglets calculation of the numerical values is difficult since the pseudoscalar state corresponding to γ is not yet identified.

4. Extension to SU(N)

It was pointed out by Kobayashi and Maskawa (1973) that a minimum number of six quarks would be necessary to successfully incorporate CP violation in an unified theory of weak and electromagnetic intractions. The sixth quark-antiquark state in the e^+e^- annihilation is a possibility in the very near future. We therefore generalise the above mixing scheme to SU(N). With N quarks and N antiquarks mesons will belong to the singlet and N^2-1 representation obtained in the reduction of

$$N \times \bar{N} = \underline{1} + (\underline{N^2-1}). \tag{26}$$

Table 1. Quark contents of meson isosinglets (vector). It is understood that the quark states are in spin-triplet states.

meson	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$	$s\bar{s}$	$c\bar{c}$	$b\bar{b}$
ω	0·9850	−0·1562	0·0731	−0·0016
ϕ	0·1504	0·9854	0·0788	−0·0016
ψ	−0·0844	−0·0666	0·9942	−0·0014
γ	0·0017	0·0013	0·0017	0·9999

There will be $N-1$ isosinglets corresponding to the extended Gell-Mann matrices $\lambda_0, \lambda_8, \lambda_{15} \ldots \lambda_{N^2-1}$. If we denote these states by $|\omega_0\rangle, |\omega_8\rangle, |\omega_{15}\rangle \ldots |\omega_{N^2-1}\rangle$,

$$V \begin{bmatrix} |\omega_8\rangle \\ |\omega_{15}\rangle \\ \vdots \\ |\omega_{N^2-1}\rangle \\ |\omega_0\rangle \end{bmatrix} = M \begin{bmatrix} |\omega_8\rangle \\ |\omega_{15}\rangle \\ \vdots \\ |\omega_{N^2-1}\rangle \\ |\omega_0\rangle \end{bmatrix}, \tag{27}$$

where M is $(N-1) \times (N-1)$ matrix

$$M = \begin{bmatrix} \omega_8 & A_1 & A_2 & \ldots & A_{N-2} \\ A_1 & \omega_{15} & B_2 & \ldots & B_{N-2} \\ A_2 & B_2 & \omega_{24} & \ldots\ldots & \\ \cdot & \cdot & \cdot & \ldots\ldots & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \omega_{N^2-1} & \cdot\cdot \\ A_{N-2} & B_{N-2} & \cdot & \ldots & \omega_0 \end{bmatrix}. \tag{28}$$

Let the physical states be denoted by $|\omega\rangle, |\phi\rangle, |\Psi\rangle \ldots |\Gamma\rangle, |\Theta\rangle$. Then

$$\begin{bmatrix} |\omega\rangle \\ |\phi\rangle \\ |\Psi\rangle \\ \cdot \\ \cdot \\ |\Gamma\rangle \\ |\Theta\rangle \end{bmatrix} = X(x_1)\ldots Y(y_1)\ldots Y(y_{N-3})\, Z(z_1)\ldots Z(z_{N-2}) \begin{bmatrix} |\omega_8\rangle \\ |\omega_{15}\rangle \\ |\omega_{24}\rangle \\ \cdot \\ \cdot \\ |\omega_{N^2-1}\rangle \\ |\omega_0\rangle \end{bmatrix} \tag{29}$$

where z , $z_2 \ldots z_{N-2}$ are the mixing angles of Θ with the other states and so on. The mixing matrices $Z(z_1), Z(z_2)\ldots Z(z_{N-2})$ are given by

$$Z(z_1) = \begin{bmatrix} \cos z_1 & 0 & 0 & \cdot & -\sin z_1 \\ 0 & 1 & 0 & \cdot\cdot & 0 \\ 0 & 0 & 1 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 0 \\ \sin z_1 & \cdot & \cdot & 0 & \cos z_1 \end{bmatrix}$$

and so on with

$$Z(z_{N-2}) = \begin{bmatrix} 1 & 0 & . & .. & & 0 \\ 0 & 1 & . & .. & & 0 \\ . & . & . & .. & & . \\ . & . & . & .. & & . \\ . & . & . & . & \cos z_{N-2} & -\sin z_{N-2} \\ 0 & 0 & 0 & 0 & \sin z_{N-2} & \cos z_{N-2} \end{bmatrix}$$

If $M' = M - \Theta I$, (30)

then

$$\tan z_1 = -\frac{M'_{12} \sin z_2 + M'_{13} \cos z_2 \sin z_3 + \ldots + M'_{1,N-2} \cos z_1 \cos z_2 \ldots \cos z_{N-2}}{M'_{11}} \quad (31)$$

To obtain $\tan z_2$, we construct the $(N-2) \times (N-2)$ matrix M'' from the matrix M' such that

$$M''_{ij} = M'_{1,\, i+1} M'_{1,\, j+1} - M'_{11} M'_{i+1,\, j+1}. \quad (32)$$

Then

$$\tan z_2 = \frac{-M''_{12} \sin z_3 + M''_{13} \cos z_3 \sin z_4 + \ldots + M''_{1,N-3} \cos z_3 \cos z_4 \ldots \cos z_{N-2}}{M''_{11}} \quad (33)$$

The same procedure is repeated to obtain $\tan z_3$, $\tan z_4$, .., $\tan z_{N-2}$. Tan z_{N-2} will depend only on the known masses belonging to the (N^2-1) representation of SU(N). In the SU$(N-1)$ limit, $\tan z_{N-2} = -\sqrt{N-1}$ and $z_{N-3} = \ldots = z_1 = 0$, in which case Θ will turn out to be a pure state of Nth quark and Nth anti-quark.

In order to get the mixing angles of $|\Gamma\rangle$ with the other states, $y_1, y_2 \ldots y_{N-3}$ we construct the following states $|\Omega_8\rangle, |\Omega_{15}\rangle, \ldots |\Omega_0\rangle$:

$$\begin{bmatrix} |\Omega_8\rangle \\ |\Omega_{15}\rangle \\ . \\ . \\ . \\ |\Omega_0\rangle \\ |\Theta\rangle \end{bmatrix} = Z(z_1)\, Z(z_2) \ldots Z(z_{N-2}) \begin{bmatrix} |\omega_8\rangle \\ |\omega_{15}\rangle \\ . \\ . \\ . \\ . \\ |\omega_0\rangle \end{bmatrix} \quad (34)$$

$$\text{Then } V\begin{bmatrix} |\Omega_8\rangle \\ |\Omega_{15}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ |\Omega_0\rangle \end{bmatrix} = \begin{bmatrix} \Omega_8 & P_1 & P_2 & \cdots & P_{N-3} \\ P_1 & \Omega_{15} & Q_2 & \cdots & Q_{N-3} \\ P_2 & Q_2 & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ P_{N-3} & Q_{N-3} & \cdot & \cdots & \Omega_0 \end{bmatrix} \begin{bmatrix} |\Omega_8\rangle \\ |\Omega_{15}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ |\Omega_0\rangle \end{bmatrix} \tag{35}$$

The calculation procedure outlined above for mixing angles $z_1, z_2 \ldots z_{N-2}$ can be adopted to obtain the angles $y_1, y_2 \ldots y_{N-3}$ and by following the same procedure the angles corresponding to the mixing of the other physical states can also be computed.

Conclusion

The quark contents of the physical isosinglet states of SU(5) i.e. ω, ϕ, ψ and γ estimated by Boal (1978), differ with our results in that the $b\bar{b}$ contents of ω, ϕ and ψ are an order of magnitude less in our estimate. For example the $b\bar{b}$ content of the ψ-meson in his estimates is 0·049 as against 0·0014 in our studies. Again the non-bottom quark contents in γ, in our estimate is also an order of magnitude smaller than those of Boal, indicating γ to be closer to being a pure $b\bar{b}$ state in our model.

Mixing angles in SU(3), SU(4) and SU(5) can be seen to be particular cases of our general mixing scheme for SU(N). Moreover SU(6) angles have been explicitly calculated otherwise and checked to agree with the general mixing scheme for $N=6$.

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