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A universal scaling law for hadronic reactions

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Abstract. A master scaling law is proposed for arbitrary distributions in arbitrary hadronic processes of which all experimentally established scaling laws (and a host of others, easily deduced as occasion demands) are special cases.

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1. Introduction

Purely hadronic processes display a number of scaling laws, the best examples being:

(i) elastic scaling (Singh and Roy 1970a, 1970b) (for an up-to-date picture of the experimental situation, see Divakaran and Gangal 1976)

$$\frac{d\sigma_{\text{el}}}{dt} \Big/ \frac{d\sigma_{\text{el}}}{dt}\,(t=0) = \phi\left[t\,\frac{d\sigma_{\text{el}}}{dt}\,(t=0)/\sigma_{\text{el}}\right];$$

(ii) 2-body inelastic scaling (Divakaran 1978)

$$\frac{d\sigma_i}{dt} \Big/ \left[\frac{d\sigma_i}{dt}\right]_{\max} = \phi\left[t\left(\frac{d\sigma_i}{dt}\right)_{\max} \Big/ \sigma_i\right],$$

where i stands for a specific inelastic channel and max for the maximum value in t;

(iii) KNO scaling (Slattery 1972)

$$\langle n\rangle\, \sigma_n/\sigma_{\text{inel}} = \phi\,(n/\langle n\rangle),$$

where σ_n is the n-particle cross-section and $\langle n\rangle$ is the average multiplicity;

(iv) scaling 'in the mean' (see Dao *et al* 1974; for more recent data see Amaglobeli *et al* 1977 and references quoted therein)

$$(\langle x\rangle_n/n\sigma_n)\,(d\sigma_n/dx) = \phi\,(x/\langle x\rangle_n),$$

where $E\,d\sigma_n/dx$ is the invariant n-particle semi-inclusive differential cross-section, $x=p_L$ or p_T of the detected particle and $\langle\ \rangle_n$ is the mean value for a fixed n. The scaling function ϕ (which is of course not necessarily the same for different processes and in different distributions) is independent of s over a very large range and, in the case of scaling in the mean, also of n.

As understood at present, these scaling phenomena have little in common, either in our theoretical understanding or even in the form in which they are expressed. Thus, to take models first, an approximate form of elastic (geometrical) scaling was recognised (Dias de Deus 1973) to be a property of a specific model of diffraction scattering; KNO scaling was proposed by Koba *et al* (1972) on the basis of Feynman scaling which has also been used, together with some other assumptions, to justify (Svensson and Sollin 1975; Yaes 1976) scaling in the mean. At a more basic theoretical level, there have been a number of investigations on the model-independent foundations of elastic scaling (see Divakaran 1978 for a brief summary and complete references). These investigations have remained inconclusive, but have nevertheless given valuable insights.

The purpose of the present paper is to formulate a universal scaling law, of which the known scaling laws (i) to (iv) are special cases. This has the obvious advantage of recognising a hitherto unsuspected common formulation applicable to virtually all known scaling laws and of suggesting an unlimited number of new ones. But, more important from the theoretical point of view, insights gained in the model-independent work on the foundations of elastic and inelastic scaling now become immediately relevant to all scaling phenomena, as explained in the last section of this paper.

The object of interest is the simultaneous differential distribution $f(x_1, \ldots x_k, s)$ in the variables $x_1, \ldots, x_k$ at the centre-of-mass energy $\sqrt{s}$ in the relevant hadronic process. In the next section we discuss the elementary case $k=1$, applicable to (i), (ii) and (iii) above. Generalisation to the case $k=2$ is taken up in § 3 and applied to scaling in the mean. Extension to $k>2$ is trivial and will be pointed out at the appropriate point. The notational convention of writing $f(x_1, s)$ for $\int f(x_1, x_2, s)\, dx_2$ (and similarly for $f(x_2, s)$) and $f(s)$ for $\int f(x_1, x_2, s)\, dx_1\, dx_2$ $(= \int f(x_1, s)\, dx_1)$ is economical and will be adopted. All integrals with the limits not explicitly indicated are over the entire physical range of the relevant variables.

2. Simple scaling

Our starting point is the observation that the elastic and inelastic scaling laws (examples (i) and (ii)) are of the general form

$$f(x, s) = f^0(s)\, \phi(\xi), \tag{1}$$

where $$\xi = x f^0(s)/f(s); \quad f^0(s) = \max_x f(x, s); \tag{2}$$

i.e., all the s-dependence of $\phi(\xi, s) \equiv f(x, s)/f^0(s)$ is through the coefficient $f^0(s)/f(s)$ of x in the definition of the scaling variable. Scaling is the statement that

$$\phi(\xi) = \lim_{s\to\infty} \phi(\xi, s)$$

exists (in practice, that $\phi(\xi, s)$ has a very weak s dependence over a large range of sufficiently large s). Both elastic and inelastic scaling are special cases with $x=t$, $f(x, s) = (d\sigma/dt)\,(t, s)$; if in particular the scattering is diffractive (e.g., elastic), $f^0(s) = (d\sigma/dt)\,(t=0, s)$ either from observation or from unitarity.

Note that if $f(x, s)$ scales, so does $g(s)f(x, s)$ for arbitrary g; normalisations are therefore unimportant.

To bring KNO scaling within the scope of (1), note that if $f(x, s)$ *scales* (i.e., has the property of (1) and (2)), it also *scales in the mean*:

$$f(x, s) \langle x \rangle (s) = f(s) \psi (x / \langle x \rangle (s)). \tag{3}$$

This follows trivially from (1) and (2) and the definition of the (s-dependent) mean of x,

$$\langle x \rangle (s) = \int f(x, s)\, x dx \Big/ \int f(x, s)\, dx$$

$$= (f(s)/f^0(s)) \int \phi(\xi)\, \xi d\xi \Big/ \int \phi(\xi)\, d\xi = \lambda f(s)/f^0(s),$$

where λ is a non-trivial constant. A special case of (3) covers KNO scaling:† take $x=n$ and $f(x, s)=\sigma_n(s)$ so that $f(s)=\sigma_{\text{inel}}(s)$.

Thus, elastic, 2-body inelastic and KNO scaling are all special cases of (1).

3. The master scaling law

We now seek a generalisation of the simple scaling law (1), to 2 variables. There is an obvious constraint to be satisfied: $f(x_1, x_2, s)$ when integrated over one of x_1 or x_2 must satisfy simple scaling in the other variable, e.g.

$$f(x_1, s) = f_1^0(s)\, \phi_1(\xi_1), \tag{4}$$

where

$$\xi_1 = x_1 f_1^0(s)/f(s),\ f_1^0(s) = \max_{x_1} f(x_1, s), \tag{5}$$

and (as a reminder of our notational convention),

$$f(x_1, s) = \int f(x_1, x_2, s)\, dx_2.$$

This constraint *uniquely* determines the master scaling law:

$$f(s) f(x_1, x_2, s) = f_1^0(s) f_2^0(s) \phi(\xi_1, \xi_2). \tag{6}$$

[The generalisation for $k > 2$ is

$$(f(s))^{k-1} f(x_1, \ldots, x_k, s) = f_1^0(s)\, f_2^0(s) \ldots f_k^0(s)\, \phi(\xi_1, \ldots, \xi_k). \tag{7}$$

†This has also been noted by K V L Sarma, private communication. Alternatively, if we take $x=n(n-1)$, an experimentally better variant of KNO scaling results (Rao and Sarma 1973).

with $\xi_i = x_i f_i^0(s)/f(s)$ and

$$f_i^0(s) = \max_{x_i} \int f(x_1, \ldots, x_k, s)\, dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_k].$$

As in the case of simple scaling, (6) can be written in a number of equivalent ways in terms of various means. First define the *complete mean* of x_i:

$$\langle x_i \rangle (s) = \int f(x_1, x_2, s)\, x_i\, dx_1\, dx_2 / f(s). \tag{8}$$

It is easily verified that $f(x_1, x_2, s)$ *scales in the complete mean:*

$$f(x_1, x_2, s) \langle x_1 \rangle (s) \langle x_2 \rangle (s) = f(s)\, \psi\, (x_1/\langle x_1 \rangle (s), x_2/\langle x_2 \rangle (s)). \tag{9}$$

We may also define *partial means*, e.g.,

$$\langle x_1 \rangle (x_2, s) = \int f(x_1, x_2, s)\, x_1\, dx_1 / f(x_2, s). \tag{10}$$

A simple calculation using (6) gives

$$x_1/(\langle x_1 \rangle (x_2, s)) = \xi_1 \lambda_1(\xi_2), \tag{11}$$

where

$$\lambda_1^{-1}(\xi_2) = \int \phi\, (\xi_1, \xi_2)\, \xi_1\, d\xi_1 \Big/ \int \phi\, (\xi_1, \xi_2)\, d\xi_1 \tag{12}$$

and a similar equation with $1 \longleftrightarrow 2$. The solution of these equations for ξ_1 and ξ_2 is of the form

$$\xi_i = \rho_i\, (x_1/\langle x_1 \rangle (x_2, s), x_2/\langle x_2 \rangle (x_1, s)). \tag{13}$$

From (6), (11) and (13), f also *scales in the partial mean:*

$$f(x_1, x_2, s) \langle x_1 \rangle (x_2, s) \langle x_2 \rangle (x_1, s)$$

$$= \chi(x_1/\langle x_1 \rangle (x_2, s), x_2/\langle x_2 \rangle (x_1, s)). \tag{14}$$

Finally, yet another useful form of the master scaling law results on converting ξ_1, say, into $x_1/\langle x_1 \rangle (x_2, s)$ through (11):

$$f(x_1, x_2, s) \langle x_1 \rangle (x_2, s) = f_2^0(s)\, \omega(x_1/\langle x_1 \rangle (x_2, s), \xi_2). \tag{15}$$

It is this last form (15) which leads most directly to the scaling of Dao *et al* (1974). Take $x_1 = p_L$ (or p_T), $x_2 = n$ and $f(x_1, x_2, s) = (n\sigma_n)^{-1} d\sigma_n/dp_L$, the normalised, invariant, semi-inclusive P_L distribution, with $\int (d\sigma_n/dp_L)\, dp_L = n\sigma_n$. Then

$$f(s) = \sum_n (n\sigma_n(s))^{-1} \int \frac{d\sigma_n}{dp_L} dp_L = \sum_n 1 = N, \tag{16}$$

where $N \propto s^{1/2}$ is the maximum value of the multiplicity; and

$$f_2^0(s) = \max_n (n\sigma_n)^{-1} \int \frac{d\sigma_n}{dp_L} dp_L = 1. \tag{17}$$

Substituting in (15), we have

$$(n\sigma_n)^{-1} \frac{d\sigma_n}{dp_L} (p_L, s) \langle p_L \rangle_n (s) = \omega(p_L/\langle p_L \rangle_n (s), n/N). \tag{18}$$

In the scaling limit (in which $N \to \infty$), the dependence of ω on n disappears and we have precisely the scaling of Dao *et al* (1974), (it may be recalled that even for the lowest energy (13 GeV/c) data used by Dao *et al*, $N \simeq 35$ while $n=4$).

4. Theoretical basis

What equations (6) and (7) express is a universal scaling law in several scaling variables which has the feature that the validity of scaling in all k variables implies its validity in any subset of them when the others are integrated over. This is a good guiding principle in the search for scaling laws in general and indeed works very well—it subsumes all hitherto known scaling behaviours in hadronic reactions, as we have seen. It is therefore very desirable that it should be tested in as many experimental situations and for as wide a choice of scaling variables as possible.

But as mentioned in § 1, a greater theoretical gain is that this unification extends to all (multiparticle) distributions the model independent understanding we have gained of elastic and inelastic 2-body scaling. A proof of scaling would consist in showing that the function

$$\phi(\xi_1, \ldots, \xi_k; s) \equiv [f_1^0(s), \ldots, f_k^0(s)]^{-1} (f(s))^{k-1} f(x_1, \ldots, x_k, s),$$

has a limit $\phi(\xi_1,\ldots,\xi_k)$ as $s \to \infty$ with $\xi_1,\ldots,\xi_k$ fixed. Such a proof from general principles is nowhere near available, even for elastic scattering. The strongest statement that can be made without unjustified assumptions is that the following *necessary* condition for the validity of scaling is true (Divakaran and Gangal 1976): every sequence ϕ_a $(\xi_1,\ldots,\xi_k, s_n)$, with $s_n \to \infty$ as $n \to \infty$ has a convergent subsequence, where the subscript a signifies the average of ϕ over the scaling variables in infinitesimally small ' bins ' of widths $a_1,\ldots,a_k$. This result is a consequence of the Ascoli-Arzelà theorem (first used in this context by Cornille and Martin 1976). It is trivial to verify that the reasoning used to check its validity for the general 2 body$\to$2 body process (Divakaran 1978) goes through for the general scaling formula, equation (7). Thus the master scaling law satisfies at least a necessary zeroth condition which the ' correct ' scaling law must meet.

The obstacle to a completely satisfactory proof is that the limit scaling function has to be shown to be unique, i.e., independent of the sequences used to define it (Cornille and Martin 1976). Our unified formulation of scaling makes it possible to hope that a solution of this problem for the simplest elastic angular distributions will also solve it for the general case. The unification also makes it very unlikely that diffraction has anything much to do with scaling so that we may freely search for other general properties shared by all hadronic distributions as a key to the understanding of scaling.

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