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Mixing of meson isosinglets

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Abstract. The mixing angles for the vector and pseudoscalar meson isosinglets are obtained in a non-relativistic quark model. Schwinger-type mass relations are also obtained for SU(4) and SU(5). Quark contents of different meson isosinglets are computed which agree well with similar estimation of Maki and co-workers and Boal.

Keywords. Mixing angles; Isosinglets; SU(4); meson.

1. Introduction

The non-relativistic quark model calculations (Lipkin 1973) have been very successful in obtaining mass relations, sum rules, scattering amplitudes and computing the decay widths. The discovery of ψ (3·1 GeV) and Υ (9·4 GeV) necessitated the introduction of new quantum numbers and hence new quarks. The fourth quark beyond the Gellmann-Zweig quarks u, d, s is called the 'charmed' quark c, and was introduced prior to the discovery of ψ to suppress the strangeness—changing neutral current (Glashow *et al* 1970). The fifth quark is generally termed ' bottom ' quark b.

To explain the experimentally observed Okubo-Zweig-Iizuka (OZI) rule-violating processes, mixing of meson isosinglets, both vector and pseudoscalar, is necessary. In the present model, assuming that SU(3) symmetry is not broken in the quark-antiquark annihilation channel, we derive the well-known SU(3) Schwinger's mass formula (Schwinger 1964), which justifies our assumption. With a similar assumption for SU(4), we derive the Schwinger-type mass relation and observe that the mass relations are better satisfied for linear masses in the case of vector mesons and square masses in the case of pseudoscalar mesons. We then proceed to compute the SU(4) mixing angles for vector isosinglets using linear masses and pseudoscalar isosinglets using the square masses. Extending the model to SU(5), we have obtained a Schwinger type mass relation for SU(5).

2. Model

The quark-antiquark potential parametrised earlier (Sastry and Misra 1970) can be extended to include the new quarks as

$$V|u_+ \bar{d}_-\rangle = V_{dd}^{NN}|u_+ \bar{d}_-\rangle + V_{ed}^{NN}\left[|u_+ \bar{d}_-\rangle - |u_- \bar{d}_+\rangle\right],$$

$$V|u_+ \bar{s}_-\rangle = V_{dd}^{Ns}|u_+ \bar{s}_-\rangle + V_{ed}^{Ns}\left[\,|u_+\bar{s}_-\rangle - |u_- \bar{s}_+\rangle\right],$$

$$V|u_+ \bar{c}_-\rangle = V_{dd}^{Nc}|u_+ \bar{c}_-\rangle + V_{ed}^{Nc}\left[\,|u_+\bar{c}_-\rangle - |u_- \bar{c}_+\rangle\right],$$

$$V|u_+ \bar{b}_-\rangle = V_{dd}^{Nb}|u_+ \bar{b}_-\rangle + V_{ed}^{Nb}\left[\,|u_+\bar{b}_-\rangle - |u_- \bar{b}_+\rangle\right], \quad (1)$$

and

$$\begin{aligned} V|u_+ \bar{u}_-\rangle = {} & V_{dd}^{NN}|u_+ \bar{u}_-\rangle + V_{ed}^{NN}\left[\,|u_+ \bar{u}_-\rangle - |u_- \bar{u}_+\rangle\right] \\ & + V_{de}^{NN}\left[\,|u_+ \bar{u}_-\rangle + |d_+ \bar{d}_-\rangle\right] + V_{de}^{Ns}|s_+ \bar{s}_-\rangle \\ & + V_{de}^{Nc}|c_+ \bar{c}_-\rangle + V_{de}^{Nb}|b_+ \bar{b}_-\rangle \\ & + V_{ee}^{NN}\left[\,|u_+ \bar{u}_-\rangle + |d_+ \bar{d}_-\rangle - |u_- \bar{u}_+\rangle - |d_- \bar{d}_+\rangle\right] \\ & + V_{ee}^{Ns}\left[\,|s_+ \bar{s}_-\rangle - |s_- \bar{s}_+\rangle\right] + V_{ee}^{Nc}\left[\,|c_+ \bar{c}_-\rangle - |c_- \bar{c}_+\rangle\right] \\ & + V_{ee}^{Nb}\left[\,|b_+ \bar{b}_-\rangle - |b_- \bar{b}_+\rangle\right]. \end{aligned} \quad (2)$$

The plus and minus subscripts to the quark symbols indicate 'spin-up' and 'spin-down'. The superscripts in the V's correspond to the quark symbols (N for u or d quarks, s for strange quark, etc). The subscripts d and e in the V's do not have any special significance. But it may be observed that V_{ed}, V_{de} and V_{ee} correspond to the projection of the initial $|q_+ \bar{q}_-\rangle$ state to the spin singlet, isospin singlet and spin-isospin singlets respectively. Further V_{de} and V_{ee} correspond to the annihilation processes (such as $u\bar{u} \to s\bar{s}$, etc.) and hence contribute to the OZI rule (Okubo 1963; Zweig 1964; Iizuka 1966) violating processes.

Now we can write down the masses of the unmixed states of both vector and pseudoscalar mesons as

$$\begin{aligned} \rho &= V_{dd}^{NN}, & \pi &= V_{dd}^{NN} + 2V_{ed}^{NN}, \\ K^* &= V_{dd}^{Ns}, & K &= V_{dd}^{Ns} + 2V_{ed}^{Ns}, \\ D^* &= V_{dd}^{Nc}, & D &= V_{dd}^{Nc} + 2V_{ed}^{Nc}, \\ F^* &= V_{dd}^{sc}, & F &= V_{dd}^{sc} + 2V_{ed}^{sc}, \\ L^* &= V_{dd}^{Nb}, & L &= V_{dd}^{Nb} + 2V_{ed}^{Nb}, \\ M^* &= V_{dd}^{sb}, & M &= V_{dd}^{sb} + 2V_{ed}^{sb}, \\ N^* &= V_{dd}^{cb}, & N &= V_{dd}^{cb} + 2V_{ed}^{cb}. \end{aligned} \quad (3)$$

The particle symbols here stand for linear masses for vector mesons and squared masses for pseudoscalar mesons. Maki *et al* (1976) and Boal (1978) in their estimations for quark contents of meson isosinglets have also used linear mass for vector mesons and squared mass for pseudoscalar mesons. L, M, N are bottom mesons with the quark structures $u\bar{b}$ or $d\bar{b}$, $s\bar{b}$ and $c\bar{b}$ respectively. The quark masses in equation (3) are not mentioned explicitly as they can be conveniently absorbed in the V_{dd}'s.

The usual additivity assumption for the quark-antiquark amplitudes can be written in the form

$$2\,(V_{ij}^{\alpha\beta} + V_{ij}^{\gamma\delta}) = (V_{ij}^{\alpha\gamma} + V_{ij}^{\beta\delta} + V_{ij}^{\alpha\delta} + V_{ij}^{\beta\gamma}), \tag{4}$$

where i, j stand for d or e and α, β, γ, δ stand for N, s, c or b. The following mass relations immediately follow for the 24-plet of vector mesons

$$\begin{aligned} F^* + \rho &= D^* + K^*,\\ M^* + \rho &= L^* + K^*,\\ N^* + \rho &= L^* + D^*. \end{aligned} \tag{5}$$

The above relations could as well be obtained by simply counting the quarks with the same quantum numbers on both sides. Similar relations also hold good for pseudoscalar mesons. The first relation of equation (5) can be used to predict the mass of F^*. With $D^*=2{\cdot}01$ GeV F^* comes out to be 2·132 GeV whereas the experimental value is 2·14 GeV (Ono 1978). The corresponding relation for the pseudoscalar mesons predicts the mass of F to be 1·986 GeV with the experimental value at 2·03 GeV (Ono 1978).

2.1. *SU*(3)

Coming to the isosinglets and considering only SU(3) for the present we have for the vector mesons

$$V\begin{pmatrix} |\,\omega_8\rangle \\ |\,\omega_0\rangle \end{pmatrix} = \begin{pmatrix} \frac{4}{3}K^* - \frac{1}{3}\rho, & \frac{4}{3\sqrt{2}}(\rho - K^*) + \sqrt{2}\,(V_{de}^{NN} - V_{de}^{Ns}) \\ \frac{4}{3\sqrt{2}}(\rho - K^*) + \sqrt{2}(V_{de}^{NN} - V_{de}^{Ns}), & \frac{2}{3}K^* + \frac{1}{3}\rho + 2V_{de}^{Ns} + V_{de}^{NN} \end{pmatrix} \begin{pmatrix} |\,\omega_8\rangle \\ |\,\omega_0\rangle \end{pmatrix}, \tag{6}$$

where $|\,\omega_8\rangle$ and $|\,\omega_0\rangle$ are the ${}^3S_1$ quark-antiquark states corresponding to the λ_8 and λ_0 of the octet and singlet representation of SU(3). If we now make the assumption that $V_{de}^{NN} = V_{de}^{Ns}$, which amounts to assuming that SU(3) is not broken in the annihilation channels, we obtain

$$\omega + \phi = 2K^* + 3V_{de}^{NN}, \tag{7}$$

$$\omega\phi = (\tfrac{4}{3} K^* - \tfrac{1}{3}\rho)(\tfrac{2}{3} K^* + \tfrac{1}{3}\rho + 3V_{de}^{NN}) - \tfrac{8}{9}(\rho - K^*)^2, \tag{8}$$

where ω and ϕ stand for masses of the physical states. With the help of (7), equation (8) can be rewritten as

$$(\omega - \omega_8)(\phi - \omega_8) = -\tfrac{8}{9}(\rho - K^*)^2, \tag{9}$$

which is Schwinger's mass formula (Schwinger 1964). In equation (9)

$$\omega_8 = \tfrac{4}{3} K^* - \tfrac{1}{3}\rho. \tag{10}$$

The corresponding equation for pseudoscalars is

$$(\eta - \eta_8)(\eta' - \eta_8) = -\tfrac{8}{9}(\pi - K)^2. \tag{11}$$

The above Schwinger's mass formula i.e. equation (9) for vector mesons is better satisfied in linear masses and similarly equation (11) for pseudoscalars is better satisfied in square masses. Hence in our calculations we have used linear masses for vector mesons and squared masses for pseudoscalar mesons.

It may be noted that the mass relations which follow from Okubo's ansatz (Okubo 1963)

i.e. $\omega = \rho,$

$$\omega + \phi = 2K^*, \tag{12}$$

could be obtained only if $V_{de}^{NN} = V_{de}^{Ns} = 0$. This means that the OZI rule is exact.

If we write the $|\phi\rangle$ and $|\omega\rangle$ states as

$$|\phi\rangle = \cos\theta\,|\omega_8\rangle - \sin\theta\,|\omega_0\rangle,$$

$$|\omega\rangle = \sin\theta\,|\omega_8\rangle + \cos\theta\,|\omega_0\rangle, \tag{13}$$

then $\tan\theta$ satisfies the quadratic equation

$$(\omega - \omega_8)\tan^2\theta - \frac{8}{3\sqrt{2}}(\rho - K^*)\tan\theta + (\omega - \omega_0) = 0. \tag{14}$$

Equation (14) is a perfect square because of the Schwinger's mass formula (equation (9)) and $\tan\theta$ becomes without any ambiguity of sign

$$\tan\theta = \frac{4(\rho - K^*)}{3\sqrt{2}(\omega - \omega_8)}. \tag{15}$$

This gives $\theta=37{\cdot}54°$ (with linear masses), which is close to the ideal mixing angle (no OZI rule violation).

For the pseudoscalar isosinglets, if we define the physical $|\eta\rangle$ and $|\eta'\rangle$ states as

$$|\eta\rangle = \cos\theta\,|\eta_8\rangle - \sin\theta\,|\eta_0\rangle,$$

$$|\eta'\rangle = \sin\theta\,|\eta_8\rangle + \cos\theta\,|\eta_0\rangle, \tag{16}$$

we get $$\tan\theta = \frac{4\,(\pi - K),}{3\sqrt{2}\,(\eta' - \eta_8)}, \tag{17}$$

provided we assume that SU(3) violation is absent in the annihilation channels V_{de} and V_{ee}. This corresponds to $\theta=-19{\cdot}25°$ (with square masses).

2.2. *SU*(4)

For SU(4) the mass matrix corresponding to the isosinglet vector mesons can be written as (assuming once again that SU(4) symmetry is not broken in the annihilation process)

$$V\begin{bmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_0\rangle\end{bmatrix} = \begin{bmatrix}\omega_8 & A & B\\ A & \omega_{15} & C\\ B & C & \omega_0\end{bmatrix}\begin{bmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_0\rangle\end{bmatrix} \tag{18}$$

where $$\omega_8 = \frac{4}{3}K^* - \frac{1}{3}\rho,$$

$$\omega_{15} = \frac{3}{2}D^* + \frac{1}{6}K^* - (2/3)\,\rho,$$

$$\omega_0 = \frac{1}{2}(K^* + D^*) + 4V_{de}^{NN},$$

$$A = \frac{2}{3\sqrt{2}}(\rho - K^*),$$

$$B = \frac{2}{\sqrt{6}}(\rho - K^*),$$

$$C = \frac{1}{2\sqrt{3}}(2\rho + K^* - 3D^*). \tag{19}$$

Here $|\omega_{15}\rangle$, $|\omega_8\rangle$, $|\omega_0\rangle$ are the ${}^3S_1$ quark-antiquark states corresponding to λ_{15}, λ_8 and λ_0 of 15-plet and the singlet representation of SU(4). If we denote the physical states by $|\omega\rangle$, $|\phi\rangle$ and $|\psi\rangle$ corresponding to the mass matrix, we obtain two Schwinger-type mass formulae for SU(4):

$$(\omega-\omega_8)(\phi+\psi-\omega_8-\omega_{15})+(\phi-\omega_{15})(\psi-\omega_{15})$$
$$= -\frac{8}{9}(\rho-K^*)^2-\frac{1}{12}(2\rho+K^*-3D^*)^2, \tag{20}$$

$$\omega\,\phi\,\psi = \omega_0\omega_8\omega_{15}+2ABC-A^2\omega_0-B^2\omega_{15}-C^2\omega_8, \tag{21}$$

where $\omega_0 = \omega+\phi+\psi-\omega_8-\omega_{15}$. (22)

The above formulae also hold good for pseudoscalar mesons with appropriate change of meson symbols such as $\rho\to\pi$, $K^*\to K$, etc. Putting the physical mass values we have verified that the above mass relations are satisfied within 10% if we use linear masses for vector mesons and squared masses for pseudoscalar mesons. But they do not agree at all if we use square masses for vector mesons.

Let us now define the physical states corresponding to the diagonalisation of the mass matrix as (Kandaswamy *et al* 1978)

$$\begin{bmatrix} |\omega\rangle \\ |\phi\rangle \\ |\psi\rangle \end{bmatrix} = X(x)\;Y(y)\;Z(z)\begin{bmatrix} |\omega_8\rangle \\ |\omega_{15}\rangle \\ |\omega_0\rangle \end{bmatrix}, \tag{23}$$

where
$$X(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Y(y) = \begin{bmatrix} \cos y & 0 & -\sin y \\ 0 & 1 & 0 \\ \sin y & 0 & \cos y \end{bmatrix},$$

and
$$Z(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos z & -\sin z \\ 0 & \sin z & \cos z \end{bmatrix}.$$

Explicitly

$$|\psi\rangle = \sin y\,|\omega_8\rangle + \cos y\,\sin z\,|\omega_{15}\rangle + \cos y\,\cos z\,|\omega_0\rangle, \tag{24}$$

and the mass of ψ is given by

$$\psi = \omega_8\sin^2 y+\omega_{15}\cos^2 y\sin^2 z+\omega_0\cos^2 y\cos^2 z$$
$$+2A\sin y\cos y\sin z+2B\sin y\cos y\cos z$$
$$+2C\cos^2 y\sin z\cos z. \tag{25}$$

Written as a quadratic equation in $\tan y$ this becomes

$$(\psi-\omega_8)\tan^2 y-2(A\sin z+B\cos z)\tan y$$
$$-\omega_{15}\sin^2 z-\omega_0\cos^2 z-2C\sin z\cos z+\psi=0. \tag{26}$$

For real values of $\tan y$, we should have

$$(A \sin z + B \cos z)^2 + (\psi - \omega_8)(\omega_{15} \sin^2 z + \omega_0 \cos^2 z + 2C \sin z \cos z - \psi) \geqslant 0,$$

or

$$\tan^2 z\,[A^2 - (\psi-\omega_8)(\psi-\omega_{15})] + 2 \tan z\,[AB + C(\psi-\omega_8)] + B^2 - (\psi-\omega_8)(\psi-\omega_0) \geqslant 0. \tag{27}$$

But with the help of equations (20) and (21) we can show that the above expression is a perfect square is of the form $-(\alpha \tan z - \beta)^2$, which implies that the inequality (27) must indeed be an equality. Hence

$$\tan z = \frac{-[AB + C(\psi-\omega_8)]}{A^2 - (\psi-\omega_8)(\psi-\omega_{15})}. \tag{28}$$

From equation (26) we have

$$\tan y = \frac{A \sin z + B \cos z}{\psi - \omega_8}. \tag{29}$$

Knowing the values of y and z, we can solve for x from the equation

$$\tan^2 x\,(\omega - a) - 2b \tan x + (a - \phi) = 0, \tag{30}$$

where $a = \omega_{15} \cos^2 z + \omega_0 \sin^2 z - 2C \sin z \cos z$,

and $b = \sin y \sin z \cos z\,(\omega_{15} - \omega_0) - A \cos y \cos z + B \cos y \sin z + C \cos 2z \sin y$.

The above relations for x, y, z also hold good for pseudoscalars with appropriate changes of symbols.

The values of the angles for vector mesons (using linear masses) turn out to be

$$x = \left.\begin{matrix} +88{\cdot}5^\circ \\ -48{\cdot}1^\circ \end{matrix}\right\}; \quad y = 0{\cdot}326^\circ, \quad z = -66{\cdot}17^\circ. \tag{31}$$

For pseudoscalar mesons the angles (using squared masses) are

$$x = \left.\begin{matrix} +38{\cdot}4^\circ \\ -43{\cdot}6^\circ \end{matrix}\right\}, \quad y = -0{\cdot}34^\circ, \quad z = -47{\cdot}28^\circ. \tag{32}$$

With the values of x, y, z from equations (31) and (32) (negative values of x are used) the quark contents of vector and pseudoscalar meson isosinglets are given in table 1. Positive values of x have been discarded (Okubo 1977) because ω and ϕ are experi-

Table 1. Quark contents of vector and pseudoscalar meson isosinglets.

	$\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$	$s\bar{s}$	$c\bar{c}$
ω	0·9899	−0·1178	0·0760
ϕ	0·1150	0·9928	0·0690
ψ	−0·0844	−0·0667	0·9940
η	0·9680	−0·201	−0·148
η'	0·1768	0·9700	0·1592
η_c	0·1762	0·1319	0·9752

It is understood that the quark states are in spin triplet states for vector mesons and spin singlet states for pseudoscalar mesons.

mentally known to mix almost ideally and a positive values of x gives a large departure from ideal mixing which is physically untenable. In the case of pseudoscalar mesons, positive values of x can be safely ruled out on the basis of the processes $\eta\to 2\gamma$ and $\eta'\to 2\gamma$ (see Okubo 1977 for a detailed discussion).

Similar estimates for the quark contents of vector and pseudoscalar meson isosinglets have been made earlier by Boal (1978), Hackman *et al* (1978) and Maki *et al* (1976). Our estimates of the quark contents of the physical vector mesons indicate an appreciable departure from ideal mixing compared to the earlier estimates.

2.3. $SU(5)$

In SU(5) with a similar assumption about the quark-antiquark annihilation, the mass matrix of the isosinglets is

$$V\begin{bmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_{24}\rangle\\|\omega_0\rangle\end{bmatrix}=\begin{bmatrix}\omega_8 & A & B & C\\ A & \omega_{15} & D & E\\ B & D & \omega_{24} & F\\ C & E & F & \omega_0\end{bmatrix}\begin{bmatrix}|\omega_8\rangle\\|\omega_{15}\rangle\\|\omega_{24}\rangle\\|\omega_0\rangle\end{bmatrix} \tag{33}$$

where $$A=\frac{4}{6\sqrt{2}}(\rho-K^*);\ B=\frac{2}{\sqrt{30}}(\rho-K^*)\ ; \tag{34}$$

$$C=\frac{4}{\sqrt{30}}(\rho-K^*);\ D=\frac{1}{2\sqrt{15}}(2\rho+K^*-3D^*);$$

$$E=\frac{1}{\sqrt{15}}(2\rho+K^*-3D^*);\quad F=\frac{1}{5}(2\rho+K^*+D^*-4L^*)$$

and $$\omega_0=\omega+\phi+\psi+\Upsilon-\omega_8-\omega_{15}-\omega_{24}$$

We can now write three mass relations for the vector mesons belonging to the 24-dimensional representation of SU(5). The simplest one corresponding to the Schwinger relation of SU(3) is

$$(\omega - \omega_8)(\phi + \psi + \Upsilon - \omega_8 - \omega_{15} - \omega_{24}) + (\phi - \omega_{15})(\psi + \gamma - \omega_{15} - \omega_{24})$$

$$+(\psi - \omega_{24})(\gamma - \omega_{24}) = -\frac{8}{9}(\rho - K^*)^2 - \frac{1}{12}(2\rho + K^* - 3D^*)^2$$

$$-\frac{1}{25}(2\rho + K^* + D^* - 4L^*)^2. \quad (35)$$

In the above equation, taking $\Upsilon = 9{\cdot}4$ GeV and other known SU(4) masses, L^* comes out to be 5·08 GeV. Some of the other theoretical estimates of L^* are 5·290 GeV (Ono 1978) and 5·340 GeV (Boal 1978).

With L^* at the above predicted value of 5·08 GeV, it follows from equation (5) that $M^*=5{\cdot}202$ GeV and $N^*=6{\cdot}320$ GeV.

It is hoped that equation (35) and other two mass relations which will come out from the mass matrix would simplify the calculation of mixing angles in SU(5).

3. Conclusions

The assumption that SU(3) symmetry is not broken in the annihilation channels gave the well known Schwinger's mass formulae and when the assumption was extended to SU(4) and SU(5) we get similar type of mass formulae. We observed that these mass formulae are well satisfied if we use linear masses for the vector mesons and squared masses for the pseudoscalar mesons. As a check on above mixing angles we calculated the ratio $\Gamma(\psi \to \eta'\gamma/\Gamma(\psi \to \eta\gamma)$ whicht urns out to be 2·25 compared to the experimental value of $2{\cdot}2 \pm 1{\cdot}0$. The OZI-violating decay processes are being worked out and will be published elsewhere.

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References

Boal D H 1978 *Phys. Rev.* **D18** 3446
Glashow S L, Iliopoulos J and Maiani L 1970 *Phys. Rev.* **D2** 1285
Hackman R H *et al* 1978 *Phys. Rev.* **D18** 2537
Iizuka J 1966 *Prog. Theor. Phys. Suppl.* **37-38** 21
Kandaswamy J, Schechter J and Singer M 1978 *Phys. Rev.* **D17** 1430
Lipkin H J 1973 *Phys. Rep.* **C8** 173

Maki Z, Teshima T and Umemura I 1976 RIFP Preprint No. 255
Okubo S 1963 *Phys. Lett.* **5** 165
Okubo S 1977 *Phys. Rev.* **D16** 2336
Ono S 1978 *Phys. Rev.* **D17** 888
Sastry C V and Misra S P 1970 *Phys. Rev.* **D1** 166
Schwinger J 1964 *Phys. Rev. Lett.* **12** 237
Zweig G 1964 CERN Report Nos. Th 401 and Th 412 (unpublished)